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Monotone iterative technique for weighted fractional semilinear evolution equations in Banach spaces

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Abstract. This paper discusses the existence of mild solutions for Riemann-Liouville fractional semilinear evolution equations in an ordered Banach space. Under some monotonicity conditions and noncompactness measure method in the weighted space of continuous functions, we prove that the functional sequences are convergent and that their limits are maximal and minimal mild solutions of the considered problem. An example to illustrate the applications of the main results is given.

1. Introduction

Recently, fractional differential equations have attracted considerable interest in both mathematics and applications, since they have been proved to be valuable tools in modeling many physical phenomena. There has been significant development in fractional differential equations in recent years, see the monographs of Benchohra et al.[9, 10], Samko et al.[34], Kilbas et al.[25], Miller and Ross [30], Podlubny [33], and the references therein. The definitions of Riemann-Liouville fractional derivatives or integrals initial conditions play an important role in some practical problems.
Heymans and Podlubny [24] have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives or integrals on the field of the viscoelasticity, and such initial conditions are more appropriate than physically interpretable initial conditions.
The theory of fractional semilinear evolution equations is new and important branch of fractional differential equation theory, which has an extensive physical background and realistic mathematical model and hence has been emerging as an important area of investigation in recent years, see [40].

The monotone iterative method based on lower and upper solutions is an effective and flexible mechanism. It yields monotone sequences of lower and upper approximate solutions that converge to the minimal and maximal solutions between the lower and upper solutions. For fractional differential equations, the paper used the monotone iterative method of lower and upper solutions, see [5, 12, 29, 36].

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In this paper, we use a monotone iterative method to prove the existence of lower and upper solutions of semilinear equations with Riemann-Liouville derivative

$$\begin{cases} {}^{RL}D_{0^+}^{q} u(t) + Au(t) = f(t, u(t), Gu(t)), \quad t \in I' := (0, b], \\ \left(I_{0^+}^{1-q} u\right)(t) = u_0, \end{cases} \tag{1.1}$$

where ${}^{RL}D^q$ is the Riemann-Liouville fractional derivative of order q, $0 < q < 1$, with the lower limit zero, $I_{0^+}^{1-q}$ is Riemann-Liouville integral of order $1-q$ and $b > 0$, the state $u(\cdot)$ takes value in a Banach space E with norm $\|\cdot\|_E$ and $-A$ is the infinitesimal generator of semigroup in a Banach space, then $-(A + MI)$ generates a uniformly bounded semigroup for $M > 0$ large enough, $f : I \times E \times E \to E$ is a given function satisfying some assumptions. The operator G given by

$$Gu(t) = \int_0^t K(t, s)u(s)ds, \tag{1.2}$$

is a Volterra integral operator with integral kernel $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \{(t, s) : 0 \le s \le t \le b\}$. Throughout this paper, we always assume that

$$K_0 = \sup_{t \in I} \int_0^t K(t, s)ds.$$

2. Preliminaries

Let $I := [0, b]$ and E be an ordered Banach space with the norm $\|\cdot\|_E$ and partial order $\le$, whose positive cone $\mathcal{P} = \{u \in E, u \ge \theta\}$ is normal with normal constant $\mathcal{N}$. Let $C(I, E)$ and $C(I', E)$ be the spaces of E-valued continuous functions on I and I', respectively. $C(I, E)$ is endowed with the uniform norm topology

$$\|u\|_C = \sup\{|u(t)|, \ \ u(t) \in E, \ \ t \in I\}.$$

Set $L^1(I, E)$ the space of E-valued Bochner integrable functions on I with norm

$$\|f\|_{L^1} = \int_I |f(t)| dt.$$

We consider the Banach space of continuous functions

$$C_{1-q}(I, E) = \{u \in C(I', E) : \lim_{t \to 0^+} t^{1-q} u(t) \ \text{ exists } \}.$$

A norm in this space is given by

$$\|u\|_q = \sup_{t \in I} \left\{ t^{1-q} \|u(t)\|_C \right\}.$$

The following lemma is a variant of the classical Arzelà-Ascoli theorem. For Ω a subset of the space $C_q(I, E)$, define Ω_q by

$$\Omega_q = \{u_q : \ \ u \in \Omega\},$$

$$u_q(t) = \begin{cases} t^{1-q} u(t), & \text{if } \ t \in I', \\ \lim\limits_{t \to 0^+} t^{1-q} u(t), & \text{if } \ t = 0. \end{cases}$$

It is clear that $u_q \in C(I, E)$.

Lemma 2.1. *[7] A set $\Omega \subset C_{1-q}(I, E)$ is relatively compact if and only if Ω_q is relatively compact in $C(I, E)$.*

Proof. See for instance [7], Lemma 1. □

Further, we give some concepts of the fractional calculus.

Definition 2.1. *[13] If $u \in L^1([0,b],E)$, then*

$$(I^q u)(t) := \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} u(s)ds, \tag{2.1}$$

exists a.e. on $[0,b]$ and it called Riemann-Liouville fractional integral of order $q > 0$. Here Γ denotes the Gamma function. Let $0 < q < 1$. If $u \in L^1([0,b],E)$ is such that $t \to I^{1-q}u(t)$ is differentiable a.e. on $[0,b]$, then

$${}^{RL}D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} u(s)ds = \frac{d}{dt} I^{1-q} u(t), \tag{2.2}$$

and exists a.e. on $[0,b]$ and it is called the Reimann-Liouville fractional derivative of order q. The previous integral is taken in Bochner sense. Let $\phi_q(t) : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_q(t) = \begin{cases} \dfrac{t^{1-q}}{\Gamma(q)}, & \text{if} \quad t > 0, \\ 0, & \text{if} \ \ t \leq 0. \end{cases}$$

Then

$$I^q u(t) = (\phi_q * u)(t),$$

and

$${}^{RL}D^q u(t) = \frac{d}{dt}(\phi_{1-q} * u)(t).$$

$$(I^q u)(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{1-q} u(s)ds. \tag{2.3}$$

Here $a(\cdot) * b(\cdot)$ is convolution, i.e.

$$\int_0^t a(t-s)b(s)ds.$$

Lemma 2.2. *[18] Let $p, q \in \mathbb{R}_+$. Then*

$$\int_0^1 s^{q-1}(1-s)^{p-1} ds = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)},$$

and hence

$$\int_0^x s^{q-1}(x-s)^{p-1} ds = x^{q+p-1} \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}.$$

The integral in the first equation of Lemma 2.2 is known as Beta function $B(q,p)$.

We recall generalized Gronwall's inequality for fractional differential equation whose proof can be found in [37].

Lemma 2.3. *Let $v : [0,b] \to [0,+\infty)$ be a real function and $w(\cdot)$ be a nonnegative, locally integrable function on $[0,b]$. Assume that there are constants $k > 0$ and $0 < p < 1$ such that*

$$v(t) \leq w(t) + k \int_0^t \frac{v(s)}{(t-s)^p} ds. \tag{2.4}$$

Then there exists a constant $K(p)$ such that

$$v(t) \leq w(t) + K(p)k \int_0^t \frac{w(s)}{(t-s)^p} ds, \tag{2.5}$$

for every $t \in [0, b]$.

Let us recall the following definitions and properties of measure of noncompactness, for more details, we refer the reader to [3, 4, 6, 8, 26, 35].
Set $\mathcal{B}(E)$ be the space of all bounded subsets in E.

Definition 2.2. *A function $\beta : \mathcal{B}(E) \to \mathbb{R}_+$ is called a measure of noncompactness in E if*

$$\beta(\overline{co}\Omega) = \beta(\Omega),$$

for every $\Omega \in \mathcal{B}(E)$, where $\overline{co}\Omega$ denotes the closed of convex hull of Ω.

Definition 2.3. *A measure of noncompactness β is called*

(1) *monotone if $\Omega_0, \Omega_1 \in \mathcal{B}(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$,*

(2) *nonsingular, if $a \in E$ and each $\Omega \in \mathcal{B}(E)$ we have $\beta(\{a\} \cup \Omega) = \beta(\Omega)$,*

(3) *regular, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in \mathcal{B}(E)$,*

(4) *if $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset and $\lim_{n\to+\infty} \beta(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact,*

(5) *algebraically semiadditive, if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for each $\Omega_0, \Omega_1 \in \mathcal{B}(E)$.*

As of the most important examples of a measure of noncompactness possessing all these properties is the Hausdorff measure of noncompactness defined by: $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\epsilon > 0, \text{ for which } \Omega \text{ has a finite number of balls with radius} \leq \epsilon\}.$$

Notice that the Hausdorff MNC satisfies the semi-homogencity condition, i.e.:

$$\chi(\lambda\Omega) = |\lambda|\chi(\Omega),$$

for each $\lambda \in \mathbb{R}$ and each $\Omega \in \mathcal{P}(E)$.

For any $W \subset C(I, E)$, we define

$$\int_0^t W(s)ds = \left\{ \int_0^t u(s)ds : u \in W, \text{ for } t \in I = [0, b] \right\},$$

where $W(s) = \{u(s) \in E : u \in w\}$.

Lemma 2.4. *[40] If $W \subset C(I, E)$ is bounded and equicontinuous then $\overline{co}(W)$ is also bounded equicontinuous continuous on I*

Lemma 2.5. *[6] Let E be a Banach space, $\Omega \subset C(I, E)$ be bounded and equicontinuous. Then $\beta(\Omega(t))$ is continuous on I, and*

$$\beta(\Omega) = \max_{t \in I} \beta(\Omega(t)) = \beta(\Omega(I)).$$

Definition 2.4. *[7, 13] A continuous map $F : X \subset E \to E$ is said to be condensing with respect to a MNCβ (β-condensing) if for every bounded set $\Omega \subset X$ that is not relatively compact, we have*

$$\beta(F(\Omega) \ngeq \beta(\Omega).$$

Lemma 2.6. *[26] If $\{u_n\}_{n=1}^{+\infty} \subset L^1(I, E)$ satisfies $\|u_n(t)\| \leq \kappa(t)$ a.e. on I for all $n \geq 1$ with some $\kappa \in L^1(I, \mathbb{R}_+)$. Then the function $\chi(\{u_n(t)\}_{n=1}^{+\infty})$ belongs to $L^1(I, \mathbb{R}_+)$ and*

$$\chi\left(\left\{\int_I u_n(t)| \ \ n \geq 1\right\}\right) \leq 2\int_I \chi(u_n(s), \ \ n \geq 1)ds.$$

Lemma 2.7. *[40] Let E be a Banach space , $D \subset E$ be bounded. Then there exist a countable set $D_0 \subset D$, such that $\beta(D) \leq 2\beta(D_0)$.*

Based on [[7], Definition 8], we give the following the lemma.

Lemma 2.8. *Assume that $-A$ is the infinitesimal generator of C_0-semigroup $\{P(t)\}_{t\geq 0}$ of uniformly bounded linear operators in E. If $f \in C(I \times E \times E, E)$ for any $u \in C_{1-q}(I, E)$, u is a mild solution of the equation*

$$\begin{cases} {}^L D_{0^+}^q u(t) + Au(t) = f(t, u(t), Gu(t)), \quad t \in I' \\ \left(I_{0^+}^{1-q} u\right)(t) = u_0, \end{cases} \tag{2.6}$$

if and only if u satisfies the following integral equation

$$u(t) = t^{q-1} S_q(t) u_0 + \int_0^t (t-s)^{q-1} S_q(t-s) f(s, u(s), Gu(s)) ds, \tag{2.7}$$

where

$$S_q(t) = q\int_0^\infty \theta M_q(\theta) P(t^q \theta) d\theta, \tag{2.8}$$

and the function M_q is the function of Wright type

$$M_q(\theta) = \frac{1}{\pi q}\sum_{n=1}^{\infty} (-\theta)^{n-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in (0, \infty). \tag{2.9}$$

Lemma 2.9. *[7, 40] The operator $S_q(t)$ has the following properties:*

(i) For any fixed $t \geq 0$, $S_q(t)$ is linear and bounded operators, i.e., for any $u \in E$,

$$\|S_q(t)u\| \leq \frac{M}{\Gamma(q)}\|u\|. \tag{2.10}$$

(ii) $S_q(t)(t \geq 0)$ is strongly continuous for every $t \geq 0$, which means that, for all $0 < t' < t'' \leq T$, we have

$$\|S_q(t'')u - S_q(t')u\| \to 0 \quad as \quad t'' \to t'.$$

(iii) For every $t > 0$, $S_q(t)$ is also compact operator if $P(t)$ is compact.

Definition 2.5. *The continuous function $u(\cdot)$ is said to be a mild solution of* (1.1) *with $I_{0^+}^{1-q} u(0) = u_0$, if the integral equation*

$$u(t) = t^{q-1} S_q(t) u_0 + \int_0^t (t-s)^{q-1} S_q(t-s) f(s, u(s), Gu(s)) ds \tag{2.11}$$

is satisfied for $t \in I'$.

Lemma 2.10. *[5] Let $\mathcal{P}$ be a normal cone of the ordered Banach space E and $v_0, w_0 \in E$ with $v_0 \leq w_0$. Suppose that $Q : [v_0, w_0] \to E$ is a increasing has a minimal fixed point $\underline{u}$ and a maximal fixed point $\overline{u}$ in $[v_0, w_0]$, moreover, $v_n \to \underline{u}$ and $w_n \to \overline{u}$. If we let $v_n = Qv_{n-1}$ and $w_n = Qw_{n-1} (n = 1, 2, \cdots)$ which satisfy $v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq \underline{u} \leq \overline{u} \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0$.*

3. Main Results

Definition 3.1. *If a function $v_0(\cdot) \in C_{1-q}(I,E)$ satisfies*

$$\begin{cases} {}^{RL}D^{q}_{0^+}v_0(t) + Av_0(t) \leq f(t, v_0(t), Gv_0(t)), \quad t \in I', \\ \left(I^{1-q}_{0^+}v_0\right)(t) \leq u_0, \end{cases} \tag{3.1}$$

then we call it a lower solution of the problem (1.1). *If all inequalities of* (3.1) *are inverse, we call it an upper solution of the problem* (1.1).

Theorem 3.1. *Suppose that E is an ordered Banach space, its positive cone $\mathcal{P}$ is normal, and $-A$ generates a positive C_0-semigroup $\{P(t)\}(t \geq 0)$ on E, $f \in C(I \times E \times E, E)$ and $u_0 \in E$.*
If problem (1.1) *has a lower solution $v_0(\cdot) \in C_{1-q}(I,E)$ and upper solution $w_0(\cdot) \in C_{1-q}(I,E)$ with $v_0 \leq w_0$, and the following conditions are satisfied*

(H_1) *there exists a constant $\lambda > 0$ such that*

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -\lambda(u_2 - u_1),$$

for any $t \in I$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $Gv_0(t) \leq v_1 \leq v_2 \leq Gw_0(t)$.

(H_2) *there exists a constant $L \geq 0$ such that*

$$\beta(\{f(t, u_n, v_n)\}) \leq L(\beta(\{u_n\}) + \beta(\{v_n\})),$$

for any $t \in I$, and increasing or decreasing monotonic sequences $\{u_n\} \subset [v_0(t), w_0(t)]$, and $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$.

(H_3) *let $v_n = Nv_{n-1}$, $w_n = Nw_{n-1}$, $n = 1, 2, \cdots$, such that the sequences $v_n(0)$ and $w_n(0)$ are convergent.*

Then problem (1.1) *has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0.*

Proof. We consider the following system

$$\begin{cases} {}^{RL}D^{q}_{0^+}u(t) + (A + \lambda I)u(t) = f(t, u(t), Gu(t)) + \lambda u(t); \quad t \in I', \\ \left(I^{1-q}_{0^+}u\right)(t) = u_0, \end{cases} \tag{3.2}$$

for any $\lambda > 0$, $-(A + \lambda I)$ also generates a C_0-semigroup $T(t) = e^{-\lambda t}P(t)(t \geq 0)$ on E and $T(t)$ is positive and continuous in the uniform operator topology for $t > 0$. That is there exists $M^* \geq 1$ such that $\sup_{t \in [0,+\infty)} \|T(t)\| \leq M^*$. By (2.10), we have that

$$\|S^*_q(t)u\| \leq \frac{M^*}{\Gamma(q)}\|u\|, \quad t \geq 0. \tag{3.3}$$

Let $D = [v_0, w_0]$, we define a mapping $N : D \longrightarrow C_{1-q}(I,E)$ by

$$Nu(t) = t^{q-1}S^*_q(t)u_0 + \int_0^t (t-s)^{q-1}S^*_q(t-s)[f(s, u(s), Gu(s)) + \lambda u(s)]ds, \tag{3.4}$$

by Lemma 2.8, $u \in D$ is mild solution of the problem (1.1) if and only if

$$u = Nu.$$

We will divid the proof in the several steps.
Step1. We show that $N : [v_0, w_0] \to C_{1-q}(I, E)$ is increasing monotone operator. In fact, for $\forall t \in I$, $v_0 \leq u$, $v \leq w_0$, by assumptions (H_1), and (H_2), we have

$$f(t, v_0(t), Gv_0(t)) + \lambda v_0(t) \leq f(t, w_0, Gw_0(t)) + \lambda w_0(t),$$

so

$$\int_0^t (t-s)^{q-1} S_q^*(t-s)[f(s, u(s), Gu(s)) + \lambda u(s)]ds$$
$$\leq \int_0^t (t-s)^{q-1} S_q^*(t-s)[f(s, v(s), Gv(s)) + \lambda v(s)]ds.$$

Hence, from (3.4) we have

$$Nu \leq Nv.$$

We show that $v_0 \leq Nv_0$ and $Nw_0 \leq w_0$. Let $h(t) = {}^{RL}D_{0^+}^q v_0(t) + Av_0(t) + \lambda v_0(t)$, $h \in C_{1-q}(I, E)$ and $h(t) \leq f(t, v_0(t), Gv_0(t)) + \lambda v_0(t), \quad t \in I'$. By Lemma and the positivity of operators S_q^*, we have that

$$v_0(t) = t^{q-1} S_q^*(t) v_0(0) + \int_0^t (t-s)^{q-1} S_q^*(t-s) h(s) ds$$
$$\leq t^{q-1} S_q^*(t) u_0 + \int_0^t (t-s)^{q-1} S_q^*(t-s)[f(s, v_0(s), Gv_0(s)) + \lambda v_0(s)]ds$$
$$= Nv_0(t), \quad t \in I',$$

namely, $v_0 \leq Nv_0$. Similarly it can prove that $Nw_0 \leq w_0$. Thus, $N : [v_0, w_0] \to [v_0, w_0]$ is a continuous increasing monotone operator.
Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Nv_{n-1}, \quad w_n = Nw_{n-1}, \quad n = 1, 2, \cdots \tag{3.5}$$

Then, from the monotonicity of N, we have

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0. \tag{3.6}$$

Step2. We prove that $\{v_n\}$ and $\{w_n\}$ are converge in I'. Let $\Omega = \{v_n, n \in \mathbb{N}\}$ and $\Omega_0 = \{v_{n-1}, n \in \mathbb{N}\}$. Then $\Omega = N(\Omega_0)$. From $\Omega_0 = \Omega \cup \{v_0\}$ it follows that $\beta(\Omega_0(t)) = \beta(\Omega(t))$ for $t \in I'$, let $\varphi(t) := \beta(\Omega(t)), \quad t \in I'$ we will show that $\varphi(t) \equiv 0$ in I'. For $t \in I'$, by (1.2) and Lemma 2.6, we get

$$\beta(G(\Omega_0)(t)) = \beta\left(\left\{\int_0^t K(t,s) v_{n-1}(s) ds : n \in \mathbb{N}\right\}\right)$$
$$\leq 2K_0 \int_0^t \beta(\Omega_0(s)) ds$$
$$= 2K_0 \int_0^t \varphi(s) ds,$$

therefore

$$\int_0^t \beta(G(\Omega_0)) ds \leq 2bK_0 \int_0^t \varphi(s) ds.$$

For $t \in I$, from using Lemma 2.1, assumptions (H_2) and (H_3), we have

$$\begin{aligned}\widetilde{\varphi}(t) &= t^{1-q}\beta(\Omega(t)) = \beta(t^{1-q}N(\Omega_0)(t))\\ &= \beta(S_q^*(t)u_0 + t^{1-q}\int_0^t (t-s)^{q-1}S_q^*(t-s)[f(s,v_{n-1}(s),Gv_{n-1}(s)) + \lambda v_{n-1}(s)]ds)\\ &\leq \frac{2M^*b^{1-q}(L+2bLK_0+\lambda)}{\Gamma(q)}\int_0^t (t-s)^{q-1}\beta(\Omega_0(s))ds\\ &\leq \frac{2M^*b^{1-q}(L+2bLK_0+\lambda)}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{q-1}s^{1-q}\beta(\Omega(s))ds\\ &\leq \frac{2M^*b^{1-q}(L+2bLK_0+\lambda)}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{q-1}\beta(\Omega_q(s))ds\\ \widetilde{\varphi}(t) &\leq \frac{2M^*b^{1-q}(L+2bLK_0+\lambda)}{\Gamma(\alpha)}\int_0^t (t-s)^{q-1}s^{q-1}\widetilde{\varphi}(s)ds.\end{aligned} \tag{3.7}$$

Hence, by Lemma 2.3 $\widetilde{\varphi}(t) \equiv 0$ in I. So, for any $t \in I$, $\{v_n(t)\}$ is precompact and $\{v_n(t)\}$ has a convergent subsequence. By the monotonicity of (H_1), we prove that $\{v_n(t)\}$ itself is convergent, i.e., $\lim_{n\to\infty} v_n(t) = \underline{u}(t)$, $t \in I$. Similarly, $\lim_{n\to\infty} w_n(t) = \overline{u}(t)$, $t \in I$. Evidently, $\{v_n(t)\} \in C_{1-q}(I,E)$, so $\underline{u}(t)$ is bounded integrable on I. For any $t \in I$,

$$\begin{aligned}v_n(t) &= Nv_{n-1}(t)\\ &= t^{q-1}S_q^*(t)u_0 + \int_0^t (t-s)^{q-1}S_q^*(t-s)[f(s,v_{n-1}(s),Gv_{n-1}(s)) + \lambda v_{n-1}(s)]ds.\end{aligned} \tag{3.8}$$

If $n \to \infty$ in (3.8), by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\underline{u}(t) &= N(\underline{u}(t))\\ &= t^{q-1}S_q^*(t)u_0 + \int_0^t (t-s)^{q-1}S_q^*(t-s)[f(s,\underline{u}(s),G\underline{u}(s)) + \lambda \underline{u}(s)]ds.\end{aligned} \tag{3.9}$$

Thus, we have $\underline{u}(t) \in C_{1-q}(I,E)$ and $\underline{u} = N\underline{u}$. In a similar way, we can prove that there exists $\overline{u}(t) \in C_{1-q}(I,E)$ such that $\overline{u} = N\overline{u}$. Combining this with the monotonicity of (1.1), we see that $v_0 \leq \underline{u} \leq \overline{u} \leq w_0$, which implies that $\underline{u}$ and $\overline{u}$ are the minimal and maximal mild solutions of problem (1.1) in $[v_0, w_0]$. □

Corollary 3.2. *Let E be an ordered and weakly sequentially complete Banach space, its positive cone $\mathcal{P}$ is normal, and $-A$ generates a positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ on E, $f \in C(I \times E \times E, E)$, and $u_0 \in E$. If problem* (1.1) *has a lower solution $v_0(\cdot) \in C_{1-q}(I,E)$ and an upper solution $w_0(\cdot) \in C_{1-q}(I,E)$, with $v_0 \leq w_0$. Suppose also that condition (H_1)-(H_3) are satisfied. Then problem* (1.1) *has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. Since E is an ordered and weakly sequentially complete Banach space, then conditions (H_2) and (H_3) hold. In fact by Theorem 2.2 in [21] , any monotonic and order bounded sequence is precompact. By the monotonicity of (3.6), it is easy to see that $v_n(t)$ and $w_n(t)$ are convergent on I. Thus $v_n(0)$ and $w_n(0)$ are convergent, i.e., condition (H_3) holds. For $t \in I$, let $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [Gv_0(t), w_0(t)]$ be two increasing or decreasing sequences. By (H_1), $\{f(t,u_n,v_n) + \lambda u_n\}$ is an ordered monotonic and ordered bounded sequences in E. Then $\beta(\{f(t,u_n,v_n) + \lambda u_n\} = 0$, (H_2) holds, and by Theorem 3.1 our conclusion is valid. □

Theorem 3.3. *Assume that E is an ordered Banach space, its positive cone $\mathcal{P}$ is normal, and $-A$ generates a positive and equicontinuous C_0-semigroup $\{T(t)\}_{t\geq 0}$ on E, $f \in C(I\times E\times E, E)$, and $u_0 \in E$. If problem* (1.1) *has a lower solution $v_0(\cdot) \in C_{1-q}(I,E)$ and an upper solution $w_0(\cdot) \in C_{1-q}(I,E)$ with $v_0 \leq w_0$. Suppose also that conditions (H_1)-(H_2) are satisfied and*

(H_4) *there exists a nonnegative constant L_1 with*

$$\frac{2M^*b^{1-q}(L_1 + 2bL_1K_0 + \lambda)}{\Gamma(q)} < 1$$

such that

$$\beta(\{f(t,u_n,v_n)\} \leq L_1(\beta(\{u_n\}) + \beta(\{v_n\})),$$

for $\forall t \in I$, and equicontinuous countable set $\{u_n\} \subset [v_0(t), w_0(t)]$, $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$.

Then problem (1.1) *has minimal mild solution $\underline{u}$ and maximal mild solution $\overline{u}$ in $[v_0, w_0]$, and*

$$v_n(t) \to \underline{u}(t), \quad w_n(t) \to \overline{u}(t), \quad (n \to +\infty), \quad t \in I,$$

where $v_n(t) = Nv_{n-1}(t)$, $w_n(t) = Nw_{n-1}(t)$, which, satisfy

$$v_0(t) \leq v_1(t) \leq \cdots \leq v_n(t) \leq \cdots \leq \underline{u}(t) \leq \overline{u} \leq \cdots \leq w_n(t) \leq \cdots \leq w_1(t) \leq w_0(t), \quad \forall t \in I.$$

Proof. From the proof of theorem 3.1, we know that $N : [v_0, w_0] \to [v_0, w_0]$ is an equicontinuous operator. Since $T(t)(t \geq 0)$ is an equicontinuous C_0-semigroup, and $S(t)(t \geq 0)$ is also an equicontinuous C_0-semigroup, by the normality of the cone $\mathcal{P}$, there exists $\overline{M}$ such that $\|f(t,u(t),Gu(t)) + \lambda u(t)\| \leq \overline{M}, \quad u \in [v_0, w_0]$.
For any $u \in C_{1-q}(I,E)$, for $0 < t_1 < t_2 \leq b$, we get

$$\begin{aligned}
&\|t_2^{1-q}Nu(t_2) - t_1^{1-q}Nu(t_1)\| \\
&\leq \|S_q^*(t_2) - S_q^*(t_1)\|\|u_0\| + \bigg\| t_2^{1-q}\int_0^{t_2}(t_2-s)^{q-1}S_q^*(t_2-s)[f(s,u(s),Gu(s)) + \lambda u(s)]ds \\
&\quad - t_1^{1-q}\int_0^{t_1}(t_1-s)^{q-1}S_q^*(t_1-s)[f(s,u(s),Gu(s)) + \lambda u(s)]ds \bigg\|
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\leq \|S_q^*(t_2) - S_q^*(t_1)\|\|u_0\| + \bigg\| t_2^{1-q}\int_{t_1}^{t_2}(t_2-s)^{q-1}S_q^*(t_2-s)[f(s,u(s),Gu(s)) + \lambda u(s)]ds \bigg\| \\
&+ \bigg\| \int_0^{t_1}\Big[t_1^{1-q}(t_1-s)^{q-1} - t_2^{1-q}(t_2-s)^{q-1}\Big]S_q^*(t_2-s)f(s,u(s),Gu(s)) + \lambda u(s)]ds \bigg\| \\
&+ \bigg\| t_1^{1-q}\int_0^{t_1}(t_1-s)^{q-1}\Big[S_q^*(t_2-s) - S_q^*(t_1-s)\Big]f(s,u(s),Gu(s)) + \lambda u(s)]ds \bigg\| \\
&\leq \|S_q^*(t_2) - S_q^*(t_1)\|\|u_0\| + \frac{M^*b^{1-q}\overline{M}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds \\
&\quad + \frac{M^*\overline{M}}{\Gamma(q)}\int_0^{t_1}\Big[t_1^{1-q}(t_1-s)^{q-1} - t_2^{1-q}(t_2-s)^{q-1}\Big] \\
&+ \bigg\| \int_0^{t_1-\epsilon} t_1^{1-q}(t_1-s)^{q-1}\Big[S_q^*(t_2-s) - S_q^*(t_1-s)\Big](f(s,u(s),Gu(s)) + \lambda u(s))ds \bigg\| \\
&+ \| \int_{t_1-\epsilon}^{t_1} t_1^{1-q}(t_1-s)^{q-1}\Big[S_q^*(t_2-s) - S_q^*(t_1-s)\Big](f(s,u(s),Gu(s)) + \lambda u(s))ds \| \\
&\leq I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{3.11}$$

where

$$I_1 = \|S_q^*(t_2) - S_q^*(t_1)\|\|u_0\|,$$

$$I_2 = \frac{M^* b^{1-q}\overline{M}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds,$$

$$I_3 = \frac{M^*\overline{M}}{\Gamma(q+1)} \left[(t_2 - t_1) + (t_2 - t_1)^q\right],$$

$$I_4 = \sup_{s\in[0,t_1-\epsilon]} \|S_q^*(t_2 - s) - S_q^*(t_1 - s)\| \left[\frac{b\overline{M}}{\alpha}\right],$$

$$I_5 = \frac{2M^* b^{1-q}\overline{M}}{\Gamma(q)} \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{q-1} ds.$$

The continuity of $(S_q^*(t), t \geq 0)$ in t in the uniform operator topology, it is easy to see that I_1 and I_4 tends to zero independently of $u_0 \in E$ as $t_2 \to t_1$. Applying the absolute continuity of the Lebesgue integral we have I_2, I_3, I_5 tend to zero independently of $u \in \Omega$ as $t_2 \to t_1$. . Since the set $D \subset [v_0, w_0]$, $N(D) \subset [v_0, w_0]$ is bounded and equicontinuous. Therefore, by Lemma 2.7, there exists a countable set $D_0 = \{u_n\}$, such that

$$\beta(N(D)) \leq 2\beta(N(D_0)). \tag{3.12}$$

For $t \in I$, by the definition of the operator N we have

$$\begin{aligned}
& t^{1-q}\beta(N(D_0(t))) = \beta(t^{1-q}N(D_0(t))) \\
& = \beta\left(\left\{S_q^*(t)u_0 + t^{1-q}\int_0^t (t-s)^{q-1} S_q^*(t-s)[f(s, v_{n-1}(s), Gv_{n-1}(s)) + \lambda v_{n-1}(s)]ds\right\}\right) \\
& \leq \frac{2M^* b^{1-q}(L_1 + 2bL_1K_0 + \lambda)}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{q-1} s^{1-q} \beta(D_0(s)) ds \\
& \leq \frac{2M^* b^{1-q}(L_1 + 2bL_1K_0 + \lambda)}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{q-1} \beta(D_q(s)) ds \\
& \leq \frac{2M^* b^q \Gamma(q)(L_1 + 2bL_1K_0 + \lambda)}{\Gamma(2q)} \beta(D)
\end{aligned} \tag{3.13}$$

and by 3.12, we have

$$\beta(N(D)) \leq \eta\beta(D),$$

where

$$\eta = \frac{2M^* b^q \Gamma(q)(L_1 + 2bL_1K_0 + \lambda)}{\Gamma(2q)} < 1.$$

Therefore, $N : [v_0, w_0] \to [v_0, w_0]$ is a strict set contraction operator. Hence, our conclusion follows from Lemma 2.10. □

4. An example

Example 4.1. *We considere the following fractional partial differential equation:*

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha} u(t,x) = \sum_{|\alpha|\leq 2m} a_\alpha D_x^\alpha u(t,x) + f(t,x,u(t,x),Gu(t,x)), \quad (t,x) \in I \times \Omega, \\ I_{0^+}^{1-\alpha} u(0,x) = u_0, \end{cases} \tag{4.1}$$

where ${}^{RL}D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative, $0 < \alpha < 1$, $t \in I = [0, b], b > 0$, *integer* $\mathbb{N} \geq 1$, $\Omega \subset \mathbb{R}^N$ *is a bounded domain with a sufficiently smooth boundery* $\partial\Omega$, $f : I \times E \times E \to E$ *is continuous and*

$$D^{\alpha}_x = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\cdot\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ *is an n-dimensional multi-index,* $|\alpha| = \alpha_1+\alpha_1+\cdots+\alpha_n$, *coefficient function* $a_\alpha(x) \in C^{2m}(\overline{\Omega})$. *Let* $E = L^p(\Omega)$ *with* $1 < p < \infty$, $P\{u \in L^p(\Omega) : u(x) \geq 0, \quad q.e. \quad x \in \Omega\}$ *and define the operator* $A : D(A) \subset E \to E$ *is follows:*

$$D(A) = W^{2m,p} \cap W^{m,p}_0(\Omega), \quad Au = \sum_{|\alpha|\leq 2m} a_\alpha D^{\alpha}_x u.$$

Then E *is a Banach space,* $\mathcal{P}$ *is a normal cone of* E *and* $-A$ *generates a positive* C_0*-semigroup* $T(t)(t \geq 0)$ *in* E, *let* $f(t, u(t), Gu(t)) = f(t, x, u(t, x), Gu(t, x))$, $u_0 = u_0(\cdot)$, *then problem* (4.1) *can be written as abstract* (1.1).

Theorem 4.2. *If the following conditions are satisfied*

(F_1) *Let* $u_0(x) \geq 0$, $x \in \Omega$, *and there exists a function* $w = w(t, x) \in C_{1-\alpha}(I, \Omega)$ *such that*

$$\begin{cases} {}^{RL}D^{\alpha}_{0^+}u(t,x) \geq \sum_{|\alpha|\leq 2m} a_\alpha D^{\alpha}_x u(t,x) + f(t,x,u(t,x),Gu(t,x)), \\ I^{1-\alpha}_{0^+}u(0,x) = u_0. \end{cases} \tag{4.2}$$

(F_2) *There exists a constant* $\lambda > 0$ *such that*

$$f(t, x, u_2, v_2) - f(t, x, u_1, v_1) \geq -\lambda(u_2 - u_1)$$

for any $t \in I$, *and* $0 \leq u_1 \leq u_2 \leq w(t, x)$, $0 \leq v_1 \leq v_2 \leq Gw(t, x)$.

(F_3) *There exists a constant* $L > 0$ *such that*

$$\beta(\{f(t, u_n, v_n)\} \leq L(\beta(\{u_n\}) + \beta(\{v_n\})),$$

for $\forall t \in I$, *and increasing or decreasing monotonic sequences* $\{u_n\} \subset [v_0(t), w_0(t)]$ *and* $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$. *Then problem* (4.1) *has minimal and maximal mild solutions between* 0 *and* $w(x, t)$, *which can be obtained by a monotone iterative procedure starting from* 0 *and* $w(t)$ *respectively.*

Proof. *Assumption* (F_1) *implies that* $v_0 \equiv 0$ *and* $w_0 = w(x, t)$ *are lower and upper solutions of the problem* (4.1), *respectively, and from assumption* (F_2), *it is easy to verify that conditions* (H_1)-(H_2) *are satisfied under the constant* $M = 1$.
So our conclusion follows from theorem 3.1. □

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