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Existence and blow up of solutions for $m(x)$-biharmonic equation with variable exponent sources

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Abstract. In this work concerned with the existence and blow up of solutions for $m(x)-$ biharmonic equation with variable exponent sources. Under appropriate conditions, we discuss the stationary problem and potential wells. By using the Faedo-Galerkin method, we prove the global existence of the solution. We also investigate the blow up of solutions with negative initial energy.

1. Introduction

In this work, we study the subsequent $m(x)$-biharmonic heat equation, with variable exponent, of the form

$$\begin{cases} z_t - \Delta z + \Delta^2 z + \Delta^2_{m(x)} z = |z|^{p(x)-2} z, & Q = \Omega \times (0,T), \\ z(x,t) = \frac{\partial}{\partial v} z(x,t) = 0, & \partial Q = \partial\Omega \times [0,T), \\ z(x,0) = z_0(x), & \Omega, \end{cases} \tag{1}$$

where $\Delta^2_{m(x)} z$ is the $m(x)$-biharmonic operator and is defined by

$$\Delta^2_{m(x)} z = \Delta\left(|\Delta z|^{m(x)-2} \Delta z\right).$$

Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^N$, v is the outward normal on $\partial\Omega$. The exponents $m(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ such that

$$2 \le m^- \le m(x) \le m^+ < \begin{cases} \infty, \text{if } N \le 2, \\ \frac{2N}{N-2}, \text{if } N > 2, \end{cases} \tag{2}$$

and

$$\max\{2, m^+\} < p^- \le p^+ < \begin{cases} \infty, \text{if } N \le 4, \\ \frac{2N}{N-4}, \text{if } N > 4, \end{cases} \tag{3}$$

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$$\begin{cases} m^{-} = ess\inf_{x\in\Omega} m(x), & m^{+} = ess\sup_{x\in\Omega} m(x), \\ p^{-} = ess\inf_{x\in\Omega} p(x), & p^{+} = ess\sup_{x\in\Omega} p(x). \end{cases}$$

Wu et al. [33] studied the subsequent semilinear parabolic equation

$$z_t = \Delta z + z^{p(x)}.$$

The authors showed that solution blows up in finite time when the initial energy is positive. In [4], the same problem lower bounds for the time of blow up arre derived if the solutions blow up.

Qu et al. [27] studied the following fourth-order parabolic equation

$$u_t + \Delta^2 u = |u|^{p(x)}.$$

They established the asymptotic behavior of solutions. Also, Liu [17] proved the local existence and blow up of solutions the same equation.

Han [16] studied the following fourth-order parabolic equation

$$u_t + \Delta^2 u - \nabla f(\nabla u) = h(x,t,u). \tag{4}$$

The author showed that global existence and blow up in finite time of solutions are obtained when the initial data satisfy different conditions.

Pişkin and Butakın, in [23] considered the following parabolic-type Kirchhoff equation with the variable exponents

$$\left(1 + |u|^{p(x)-2}\right) u_t + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u = |u|^{q(x)-2} u.$$

They established the global existence of solutions by Faedo-Galerkin method. Later they prove the decay of solutions by Komornik's inequality.

Chuong et al. [9] investigated the problem (5)

$$u_t + \Delta^2 u - \Delta_{p(x)} u = |u|^{q(x)-2} u. \tag{5}$$

The authors proved that global existence and nonexistence of global solutions to the Cahn-Hilliard equation with variable exponent sources and arbitrary initial energy. Recently, the existence, nonexistence and decay of solutions for the equation with variable exponents was studied by many authors, see for instance [1–3, 6, 7, 10, 14, 19, 22, 24, 25, 30–32, 35, 38].

Fourth-order parabolic equations, such as those manifested in fourth-order partial differential equations, are highly versatile in their ability to mathematically represent numerous fundamental physical phenomena. Significantly, these equations prove to be particularly well-suited for elucidating the dynamic evolution of epitaxial growth in nanoscale thin films, involving intricate surface diffusion and crystal growth processes, as documented in works by [20, 29, 37] and references therein. The equation denoted by reference (4) is commonly known as the classical Cahn-Hilliard equation, originating from the modeling of phase transitions observed in binary systems, encompassing alloys, glasses, thin film epitaxy, and polymer mixtures. For a more comprehensive understanding of this subject, one can consult the scholarly contributions of [8]. For detailed information on this topics, refer to the work of [8] and papers [12, 13, 18].

Motivated by previous papers, we prove the global existence and blow up of solutions with negative initial energy by using the technique of [9].

This paper consists of four parts in addition to the introduction. In Part 2, we give the definition of the variable exponent Lebesgue and Sobolev spaces. In Part 3 we investigate the equilibrium state of equation (1) and establish the stable and unstable sets. In Part 4 of this paper focuses on the primary outcomes related to the evolution problem. The proofs for these main results are provided in the subsequent sections of the paper.

2. Preliminaries

In this part, we give some notations, lemmas and preliminary results in order to state the main results of this article. Which will be used thoughout this work. Let $\|.\|$ and $\|.\|_r$ denote usual $L^2(\Omega)$ norm and $L^r(\Omega)$ norm, respectively. Furthermore $1 \leq r \leq \infty$ and $\langle ., . \rangle$ Let us define the standard inner product of the Hilbert space $L^2(\Omega)$ as $\langle ., . \rangle$. Additionally, we represent the norm of H_0^2 as $\|.\|_{H_0^2}$.

$$\|z\|_{H_0^2} = \sqrt{\|z\|_2^2 + \|\nabla z\|_2^2 + \|\Delta z\|_2^2}.$$

In [16], H_0^2 is a Hilbert space with inner product

$$\langle z, v \rangle_{H_0^2} = \langle \Delta z, \Delta v \rangle.$$

Laterly H_0^2 is uniformly convex and the norm $\|.\|_{H_0^2}$ is equivalent to the norm $\|\Delta(.)\|_2$ because of Poincare's inequality.

Let us recall some established properties concerning the Lebesgue spaces and Sobolev spaces equipped with variable exponents (see [11], [26]).

Let $q : \Omega \to [1, \infty]$ be a measurable function, where Ω is a bounded domain of $\mathbb{R}^N$. We define the Lebesgue space with variable exponent $q(.)$ by

$$L^{q(x)}(\Omega) = \left\{ z : \Omega \to R,\ z \text{ is measurable and } \rho_{q(.)}(\lambda z) \ < \infty, \text{ for some } \lambda > 0 \right\},$$

where

$$\rho_{q(.)}(z) = \int_\Omega |z|^{q(x)}\, dx.$$

Also endowed with the Luxemburg-type norm

$$\|z\|_{q(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{z}{\lambda} \right|^{q(x)} dx \leq 1 \right\},$$

$L^{q(x)}(\Omega)$ is a Banach space.

Lemma 2.1. *[11]. Suppose that $q \in \mathcal{P}(\Omega)$ holds.*

$$\min \left\{ \|z\|_{q(.)}^{q^-}, \|z\|_{q(.)}^{q^+} \right\} \leq \rho(z) \leq \max \left\{ \|z\|_{q(.)}^{q^-}, \|z\|_{q(.)}^{q^+} \right\}, \text{ for all } z \in L^{q(.)}(\Omega).$$

For $q^+ < \infty$, the dual space of $L^{q(.)}(\Omega)$ is identified with $L^{q'(.)}(\Omega)$ with the dual variable exponent $q' \in \mathcal{T}(\Omega)$ given by

$$\frac{1}{q(x)} + \frac{1}{q'(x)} = 1 \text{ for a.e. } x \in \Omega,$$

we have $1/\infty = 0$.

The Hölder inequality in addition fulfills for variable Lebesgue spaces.

Lemma 2.2. *(Hölder inequality, [11]). Assume that $s, q, r \in \mathcal{T}(\Omega)$ hold.*

$$\|zv\|_{s(.)} \leq 2 \|z\|_{q(.)} \|v\|_{r(.)} \text{ for all } z \in L^{q(.)}(\Omega),\ v \in L^{r(.)}(\Omega),$$

$$\frac{1}{s(x)} = \frac{1}{q(x)} + \frac{1}{r(x)} \text{ for a.e. } x \in \Omega.$$

Lemma 2.3. *[11]. Suppose that $q, r \in \mathcal{T}(\Omega)$. If $q(x) \leq r(x)$ for a.e. $x \in \Omega$, then the embedding $L^{r(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous. We next define variable exponent Sobolev spaces*

$$W^{1,q(.)}(\Omega) = \left\{ z \in L^{q(.)}(\Omega) \text{ such that } \nabla z \text{ exists and } |\nabla z| \in L^{q(.)}(\Omega) \right\}.$$

This space is a Banach space with respect to the norm

$$\|z\|_{W^{1,q(.)}(\Omega)} = \left(\|z\|_{q(.)}^2 + \|\nabla z\|_{q(.)}^2 \right)^{1/2}.$$

Furthermore, let $W_0^{1,q(.)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,q(.)}(\Omega)$. The dual of $W_0^{1,q(.)}(\Omega)$ is defined as $W_0^{-1,q(.)}(\Omega)$, by the similarly usual Sobolev spaces, where $\frac{1}{q(.)} + \frac{1}{q'(.)} = 1$.

3. Stationary State

In this part, we deal with the stationary solutions of (1) which solve the problem

$$\begin{cases} \Delta^2 z - \Delta z + \Delta^2_{m(x)} z = |z|^{p(x)-2} z & \text{in } \Omega, \\ z(x) = \frac{\partial}{\partial v} z(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $m(x)$ and $p(x)$ satisfy (2)-(3). The energy functional E and the Nehari functional I are defined as follows:

$$E(z) = \frac{1}{2}\|\nabla z\|_2^2 + \frac{1}{2}\|\Delta z\|_2^2 + \int_\Omega \frac{1}{m(x)} |\Delta z|^{m(x)} dx - \int_\Omega \frac{1}{p(x)} |z|^{p(x)} dx.$$

$$I(z) = \|\nabla z\|_2^2 + \|\Delta z\|_2^2 + \int_\Omega |\Delta z|^{m(x)} dx - \int_\Omega |z|^{p(x)} dx.$$

Let E and I be functionals defined over $H_0^2(\Omega)$. They belong to the class C^1 in this space, and critical points of E correspond to weak solutions of equation (6). Moreover, the functionals E and I can be bounded as follows:

$$\begin{aligned} E(z) &\geq \frac{1}{2}\|\nabla z\|_2^2 + \frac{1}{2}\|\Delta z\|_2^2 + \frac{1}{m^+}\int_\Omega |\Delta z|^{m(x)} dx - \frac{1}{p^-}\int_\Omega |z|^{p(x)} dx \\ &= \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\nabla z\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\Delta z\|_2^2 + \left(\frac{1}{m^+} - \frac{1}{p^-}\right)\int_\Omega |\Delta z|^{m(x)} dx + \frac{1}{p^-} I(z), \end{aligned} \tag{7}$$

where

$$\begin{aligned} E(z) &\leq \frac{1}{2}\|\nabla z\|_2^2 + \frac{1}{2}\|\Delta z\|_2^2 + \frac{1}{m^-}\int_\Omega |\Delta z|^{m(x)} dx - \frac{1}{p^+}\int_\Omega |z|^{p(x)} dx \\ &= \left(\frac{1}{2} - \frac{1}{p^+}\right)\|\nabla z\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^+}\right)\|\Delta z\|_2^2 + \left(\frac{1}{m^-} - \frac{1}{p^+}\right)\int_\Omega |\Delta z|^{m(x)} dx + \frac{1}{p^+} I(z), \end{aligned} \tag{8}$$

that is

$$\begin{aligned} E(z) &= \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\nabla z\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\Delta z\|_2^2 + \int_\Omega \left(\frac{1}{m(x)} - \frac{1}{p^-}\right)|\Delta z|^{m(x)} dx \\ &\quad + \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right)|z|^{p(x)} dx + \frac{1}{p^-} I(z). \end{aligned} \tag{9}$$

Let $z \in H_0^2(\Omega) \setminus \{0\}$ and consider the fibering map $\lambda \mapsto j(\lambda) := E(\lambda z)$ for $\lambda > 0$ given by

$$j(\lambda) = \frac{\lambda^2}{2}\|\nabla z\|_2^2 + \frac{\lambda^2}{2}\|\Delta z\|_2^2 + \int_\Omega \frac{\lambda^{m(x)}}{m(x)}|\Delta z|^{m(x)}\,dx - \int_\Omega \frac{\lambda^{p(x)}}{p(x)}|z|^{p(x)}\,dx.$$

Lemma 3.1. *Suppose that (2)-(3) hold and $z \in H_0^2(\Omega)\setminus\{0\}$. $\lim_{\lambda\to 0^+} j(\lambda) = 0$ and $\lim_{\lambda\to\infty} j(\lambda) = -\infty$. There exists a $\lambda_* = \lambda_*(z) > 0$ such that $j(\lambda)$ reaches its maximum at $\lambda = \lambda_*$ and consequently, we have $I(\lambda_* z) = 0$. Additionally, the following conditions hold based on the value of $I(z)$:*

i. $0 < \lambda_* < 1$, *if* $I(z) < 0$,

ii. $\lambda_* = 1$, *if* $I(z) = 0$, *and*

iii. $\lambda_* > 1$, *if* $I(z) > 0$.

Proof. We obtain

$$\begin{aligned} j(\lambda) \quad &\geq \quad \frac{\lambda^2}{2}\|\nabla z\|_2^2 + \frac{\lambda^2}{2}\|\Delta z\|_2^2 + \min\left\{\lambda^{m^-}, \lambda^{m^+}\right\}\int_\Omega \frac{\lambda^{m(x)}}{m(x)}|\Delta z|^{m(x)}\,dx \\ &\quad - \max\left\{\lambda^{m^-}, \lambda^{m^+}\right\}\int_\Omega \frac{\lambda^{p(x)}}{p(x)}|z|^{p(x)}\,dx, \end{aligned}$$

and

$$\begin{aligned} j(\lambda) \quad &\leq \quad \frac{\lambda^2}{2}\|\nabla z\|_2^2 + \frac{\lambda^2}{2}\|\Delta z\|_2^2 + \max\left\{\lambda^{m^-}, \lambda^{m^+}\right\}\int_\Omega \frac{\lambda^{m(x)}}{m(x)}|\Delta z|^{m(x)}\,dx \\ &\quad - \min\left\{\lambda^{m^-}, \lambda^{m^+}\right\}\int_\Omega \frac{\lambda^{p(x)}}{p(x)}|z|^{p(x)}\,dx. \end{aligned}$$

This, along with $p^- > \max\{2, m^+\}$ and $\int_\Omega \frac{1}{p(x)}|z|^{p(x)}\,dx > 0$, implies statement (i). Furthermore, for sufficiently small $\lambda > 0$, we likewise have $j(\lambda) > 0$. In other words, there exists a $\lambda_* > 0$ such that $j(\lambda_*) = \sup_{\lambda>0} j(\lambda)$. By Fermat's Theorem, we have $j'(\lambda_*) = 0$. This, leads to $I(\lambda_* z) = 0$ based on the relationship $I(\lambda z) = \lambda j'(\lambda)$.

Consequently, we establish the final statement of (ii). By employing the definition of I, we obtain:

$$\begin{aligned} 0 \quad &= \quad I(\lambda_* z) \\ &= \quad \lambda_*^2\|\nabla z\|_2^2 + \lambda_*^2\|\Delta z\|_2^2 + \int_\Omega \lambda_*^{m(x)}|\Delta z|^{m(x)}\,dx - \int_\Omega \lambda_*^{p(x)}|z|^{p(x)}\,dx \\ &= \quad \left(\lambda_*^2 - \lambda_*^{p^-}\right)\|\nabla z\|_2^2 + \left(\lambda_*^2 - \lambda_*^{p^-}\right)\|\Delta z\|_2^2 + \int_\Omega \left(\lambda_*^2 - \lambda_*^{p^-}\right)|\Delta z|^{m(x)}\,dx \\ &\quad + \int_\Omega \left(\lambda_*^2 - \lambda_*^{p^-}\right)|z|^{p(x)}\,dx + \lambda_*^{p^-} I(z), \end{aligned}$$

which can be represented as

$$\lambda_*^{p^-} I(z) \quad = \quad \left(\lambda_*^{p^-} - \lambda_*^2\right)\|\nabla z\|_2^2 + \left(\lambda_*^{p^-} - \lambda_*^2\right)\|\Delta z\|_2^2 + \int_\Omega \left(\lambda_*^{p^-} - \lambda_*^2\right)|\Delta z|^{m(x)}\,dx + \int_\Omega \left(\lambda_*^{p^-} - \lambda_*^2\right)|z|^{p(x)}\,dx.$$

Since $p^- > \max\{2, m^+\}$,the above equality shows that $0 < \lambda_* < 1$, $\lambda_* = 1$ and $\lambda_* > 1$ provided that $I(z) < 0$, $I(z) = 0$ and $I(z) > 0$, respectively. This concludes the proof. □

Let us introduce the Nehari manifold, which is associated with the energy functional E

$$\mathcal{N} = \left\{ z \in H_0^2(\Omega) \setminus \{0\} : I(z) = 0 \right\}.$$

Lemma 3.1 ensures that $\mathcal{N}$ is non-empty. Consequently, we can establish a definition for $\mathcal{N}$ as follows:

$$d = \inf_{z \in \mathcal{N}} E(z). \tag{10}$$

The subsequent lemma holds significant importance in the proofs of our main results, particularly for the case of low initial energy.

Lemma 3.2. *Let assumptions of (2)-(3) holds and $z \in H_0^2(\Omega) \setminus \{0\}$. Then*

$$E(z) - \frac{1}{p^-} I(z) \geq \frac{d}{\max\left\{\lambda_*^2, \lambda_*^{m^-}, \lambda_*^{p^+}\right\}},$$

where λ_* is as in Lemma 3.1.

Proof. For any $z \in H_0^2(\Omega) \setminus \{0\}$, by Lemma 3.1, there exists $\lambda_* \in (0, \infty)$ such that $I(\lambda_* z) = 0$. By the definition of d and replacing z by $\lambda_* z$ in (9), one has

$$\begin{aligned}
d &\leq E(\lambda_* z) \\
&= \left(\frac{1}{2} - \frac{1}{p^-}\right) \lambda_*^2 \|\nabla z\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right) \lambda_*^2 \|\Delta z\|_2^2 \\
&\quad + \int_\Omega \lambda_*^{m(x)} \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\Delta z|^{m(x)} dx - \int_\Omega \lambda_*^{p(x)} \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |z|^{p(x)} dx \\
&\leq \max\left\{\lambda_*^2, \lambda_*^{m^-}, \lambda_*^{p^+}\right\} . \left[\left(\frac{1}{2} - \frac{1}{p^-}\right) \|\nabla z\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right) \|\Delta z\|_2^2 \right. \\
&\quad \left. + \int_\Omega \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\Delta z|^{m(x)} dx - \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |z|^{p(x)} dx \right] \\
&= \max\left\{\lambda_*^2, \lambda_*^{m^-}, \lambda_*^{p^+}\right\} \left[E(z) - \frac{1}{p^-} I(z)\right].
\end{aligned}$$

Consequently, this establishes the desired outcome. The proof is completed. □

Lemma 3.3. *Assume that (2)-(3) hold. Then we get*
i) $d = \inf_{z \in H_0^2(\Omega) \setminus \{0\}} \sup_{\lambda > 0} E(\lambda z)$.
ii) d is a positive quantity.
iii) There exists $z^* \in \mathcal{N}$, $z^*(x) \geq 0$ a.e. in Ω so that $E(z^*) = d$.

Proof. For any $z \in H_0^2(\Omega) \setminus \{0\}$, by Lemma 3.1 we get

$$\sup_{\lambda > 0} E(\lambda z) = E(\lambda_* z). \tag{11}$$

By definition of $\mathcal{N}$, it follows from Lemma 3.1 that $\lambda_* z \in \mathcal{N}$. Thus

$$E(\lambda_* z) \geq \inf_{z \in N} E(z) = d. \tag{12}$$

Combining (11) and (12),

$$\inf_{z \in H_0^2(\Omega) \setminus \{0\}} \sup_{\lambda > 0} E(\lambda z) \geq d. \tag{13}$$

Furthermore, for any $z \in \mathcal{N}$, by Lemma 3.1, one has $\lambda_* = 1$,

$$\sup_{\lambda>0} E(\lambda z) = E(z). \tag{14}$$

We remove from (13) and (14) that i) holds.

Next, we will prove statement ii). As shown in (3), the function $p(x)$ satisfies the conditions, allowing us to establish a continuous embedding of $H_0^2(\Omega)$ into $L^{p(.)}(\Omega)$. Let $S_{p(.)}$ denote the optimal embedding constant, i.e.,

$$S_{p(.)} = \sup_{z \in H_0^2(\Omega)\setminus\{0\}} \frac{\|z\|_{p(.)}}{\|\Delta z\|_2}.$$

Let any $z \in H_0^2(\Omega) \setminus \{0\}$ such that $I(z) \le 0$. We observe that

$$\begin{aligned} \|\Delta z\|_2^2 &\le \int_\Omega |z|^{p(x)} dx \\ &\le \max\left\{\|z\|_{p(.)}^{p^-}, \|z\|_{p(.)}^{p^+}\right\} \\ &\le \max\left\{S_{p(.)}^{p^-} \|\Delta z\|_2^{p^-}, S_{p(.)}^{p^+} \|\Delta z\|_2^{p^+}\right\}. \end{aligned}$$

Next we have to show that $\|\Delta z\|_2 > 0$ and $p^- > 2$. So we get

$$\|\Delta z\|_2 \ge \delta_1, \tag{15}$$

where

$$\delta_1 = \min\left\{S_{p(.)}^{\frac{p^-}{2-p^-}}, S_{p(.)}^{\frac{p^+}{2-p^+}}\right\}.$$

Constant $z \in \mathcal{N}$, we have $z \in H_0^2(\Omega) \setminus \{0\}$ and $I(z) = 0$. By from (7) and (15), we get

$$\begin{aligned} E(z) &\ge \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\nabla z\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\Delta z\|_2^2 + \left(\frac{1}{m^+} - \frac{1}{p^-}\right)\int_\Omega |\Delta z|^{m(x)} dx + \frac{1}{p^-} I(z) \\ &\ge \left(\frac{1}{2} - \frac{1}{p^-}\right)\left(\|\nabla z\|_2^2 + \|\Delta z\|_2^2\right) \\ &\ge \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\Delta z\|_2^2 \\ &\ge \left(\frac{1}{2} - \frac{1}{p^-}\right)\delta_1^2. \end{aligned} \tag{16}$$

Then by the definition d, we get

$$d \ge \left(\frac{1}{2} - \frac{1}{p^-}\right)\delta_1^2 > 0.$$

As a consequently, we show iii). By (10) there exists. Let $\{z_n\}_{n=1}^\infty \subset \mathcal{N}$ be a minimizing sequence of E such that $\lim_{n\to\infty} E(z_n) = d$. It is evident that $|z_n| \in \mathcal{N}$ and $E(|z_n|) = E(z_n)$. Therefore, we may assume that $z_n(x) \ge 0$ almost everywhere in Ω for all $n \in \mathbb{N}^*$.

Since $\lim_{n\to\infty} E(z_n) = d$ and using (16), we deduce that $\{z_n\}$ is bounded in $H_0^2(\Omega)$. As $H_0^2(\Omega)$ is reflexive, the embeddings $H_0^2(\Omega) \hookrightarrow W_0^{2,m(.)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ are compact (by (2) and (3)). Consequently, there exists a subsequence of $\{z_n\}$, which we will still denote by $\{z_n\}$ and an element $z^* \in H_0^2(\Omega)$ such that:

$$\begin{aligned} z_n &\rightharpoonup z^* \text{ weakly in } H_0^1(\Omega), \\ z_n &\rightharpoonup z^* \text{ weakly in } H_0^2(\Omega), \\ z_n &\rightharpoonup z^* \text{ strongly in } W_0^{2,m(.)}(\Omega), \\ z_n &\rightharpoonup z^* \text{ strongly in } L^{p(.)}(\Omega), \\ z_n(x) &\to z^*(x) \text{ a.e. in } \Omega. \end{aligned}$$

Then we get $z^*(x) \geq 0$ a.e. in Ω and

$$\|\nabla z^*\|_2 \leq \lim_{n\to\infty} \inf \|\nabla z_n\|_2,$$

$$\|\Delta z^*\|_2 \leq \lim_{n\to\infty} \inf \|\Delta z_n\|_2,$$

$$\int_\Omega |\Delta z^*|^{m(x)}\, dx = \lim_{n\to\infty} \int_\Omega |\Delta z_n|^{m(x)}\, dx,$$

$$\int_\Omega \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\Delta z^*|^{m(x)}\, dx = \lim_{n\to\infty} \int_\Omega \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\Delta z_n|^{m(x)}\, dx,$$

$$\int_\Omega |z^*|^{p(x)}\, dx = \lim_{n\to\infty} \int_\Omega |z_n|^{p(x)}\, dx,$$

$$\int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |z^*|^{p(x)}\, dx = \lim_{n\to\infty} \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |z_n|^{p(x)}\, dx.$$

We use z_n instead of z (9) and notice that $z_n \in \mathcal{N}$,

$$\begin{aligned} d &= \lim_{n\to\infty} \inf E(z_n) \\ &\geq \left(\frac{1}{2} - \frac{1}{p^-}\right) \|\nabla z^*\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right) \|\Delta z^*\|_2^2 \\ &\quad + \int_\Omega \left(\frac{1}{m(x)} - \frac{1}{p^-}\right) |\Delta z^*|^{m(x)}\, dx + \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right) |z^*|^{p(x)}\, dx \\ &= E(z^*) - \frac{1}{p^-} I(z^*). \end{aligned} \tag{17}$$

Assume that $I(z^*) < 0$. Then, according to Lemmas 3.1 and 3.2, there exists $\lambda_* \in (0,1)$ so that

$$E(z^*) - \frac{1}{p^-} I(z^*) \geq \frac{d}{\max\left\{\lambda_*^2, \lambda_*^{m^-}, \lambda_*^{p^+}\right\}} > d.$$

This contradicts (17), and so

$$I(z^*) \geq 0. \tag{18}$$

As $z_n \in \mathcal{N}$ we get $I(z_n) = 0$, which implies that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \inf I(z_n) \\ &= \lim_{n\to\infty} \inf \left(\|\nabla z_n\|_2^2 + \|\Delta z_n\|_2^2 + \int_\Omega |\Delta z_n|^{m(x)} dx - \int_\Omega |z_n|^{p(x)} dx \right) \\ &\geq \|\nabla z^*\|_2^2 + \|\Delta z^*\|_2^2 + \int_\Omega |\Delta z^*|^{m(x)} dx - \int_\Omega |z^*|^{p(x)} dx \\ &= I(z^*). \end{aligned}$$

Combining with (18), we can deduce that $I(z^*) = 0$. Now, let's demonstrate that $z^* \in \mathcal{N}$. To establish this, we must demonstrate that $z^* \neq 0$. Since $z_n \in \mathcal{N}$ and (15), we can infer from:

$$\int_\Omega |z_n|^{p(x)} dx = \|\nabla z_n\|_2^2 + \|\Delta z_n\|_2^2 + \int_\Omega |\Delta z_n|^{m(x)} dx \geq \delta_1^2.$$

Passing to the limit, we get

$$\int_\Omega |z^*|^{p(x)} dx \geq \delta_1^2 > 0.$$

This leads to $z^* \neq 0$. Consequently, $z^* \in \mathcal{N}$ and therefore $E(z^*) \geq d$. By (17) and $I(z^*) = 0$, we have $E(z^*) \leq d$. Hence $E(z^*) = d$. The proof is now complete. □

Next, we introduce the stable set $\mathcal{W}$ and unstable set $\mathcal{U}$ which are analogous to the works of Sattinger [28], Payne and Sattinger [21].

$$\mathcal{W} = \left\{ z \in H_0^2(\Omega) : E(z) < d,\ I(z) > 0 \right\} \cup \{0\},$$

$$\mathcal{U} = \left\{ z \in H_0^2(\Omega) : E(z) < d,\ I(z) < 0 \right\}.$$

We also introduce

$$\mathcal{N}_- = \left\{ z \in H_0^2(\Omega) : I(z) < 0 \right\}, \quad \mathcal{N}_+ = \left\{ z \in H_0^2(\Omega) : I(z) > 0 \right\},$$

and the open sub levels of E,

$$E^k = \left\{ z \in H_0^2(\Omega) : E(z) < k \right\}.$$

Furthermore, the variational characterization of d also implies that

$$\mathcal{N}_k := \mathcal{N} \cap E^k \neq \emptyset \text{ for all } k > d.$$

For $k > d$, we now define

$$\lambda_k = \inf \{\|z\|_2 : z \in \mathcal{N}_k\} \text{ and } \Lambda_k = \sup \{\|z\|_2 : z \in \mathcal{N}_k\}. \tag{19}$$

The non-increasing nature of $k \mapsto \lambda_k$ and non-decreasing nature of $k \mapsto \Lambda_k$ are evident. The following lemma demonstrates that λ_k and Λ_k are finite positive values, thus confirming the nontriviality of the result in Theorem 4.7.

Lemma 3.4. *Suppose that (2)-(3) hold. Then for any $k > d$, λ_k and Λ_k defined in (19) satisfy $0 < \lambda_k \leq \Lambda_k < \infty$.*

Proof. Let's begin by proving $\Lambda_k < \infty$. For any $k > d$ and $z \in \mathcal{N}_k$, we have $E(z) < k$ and $I(z) = 0$. By (7) and utilizing the embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$, we get

$$\begin{aligned} k &> E(z) \geq \left(\frac{1}{2} - \frac{1}{p^-}\right) \|\nabla z\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right) \|\Delta z\|_2^2 \\ &\quad + \left(\frac{1}{m^+} - \frac{1}{p^-}\right) \int_\Omega |\Delta z|^{m(x)} dx + \frac{1}{p^-} I(z) \\ &\geq \left(\frac{1}{2} - \frac{1}{p^-}\right) \|\Delta z\|_2^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{p^-}\right) S_2^{-2} \|z\|_2^2, \end{aligned} \tag{20}$$

then

$$\|z\|_2 \leq S_2 \sqrt{\frac{2kp^-}{p^- - 2}},$$

where $S_2 > 0$ is the optimal embedding constant, i.e.,

$$S_2 = \sup_{z \in H_0^2(\Omega) \setminus \{0\}} \frac{\|z\|_2}{\|\Delta z\|_2}. \tag{21}$$

To indicate that

$$\Lambda_k \leq S_2 \sqrt{\frac{2kp^-}{p^- - 2}} < \infty.$$

To establish that $\lambda_k > 0$. We use the Gagliardo-Nirenberg inequality. There exists a positive constant A_0 depending only on Ω, N, p^- and p^+ such that

$$\begin{aligned} \|z\|_{p^-}^{p^-} &\leq A_0 \|\Delta z\|_2^{\theta^- p^-} \|z\|_2^{(1-\theta^-)p^-}, \\ \|z\|_{p^+}^{p^+} &\leq A_0 \|\Delta z\|_2^{\theta^+ p^+} \|z\|_2^{(1-\theta^+)p^+}, \end{aligned}$$

where $\theta^\pm = \frac{N(p^\pm - 2)}{4p^\pm} \in (0,1)$ by (3). Then, since $z \in \mathcal{N}_k$ and $\mathcal{N}_k \subset \mathcal{N}$, it follows that

$$\begin{aligned} \|\Delta z\|_2^2 &\leq \int_\Omega |z|^{p(x)} dx \\ &\leq \int_\Omega \left(|z|^{p^-} + |z|^{p^+}\right) dx \\ &\leq 2 \max\left\{\|z\|_{p^-}^{p^-}, \|z\|_{p^+}^{p^+}\right\} \\ &\leq 2A_0 \max\left\{\|\Delta z\|_2^{\theta^- p^-} \|z\|_2^{(1-\theta^-)p^-}, \|\Delta z\|_2^{\theta^+ p^+} \|z\|_2^{(1-\theta^+)p^+}\right\}. \end{aligned}$$

Taking this into account and noticing that $\|\Delta z\|_2 > 0$ and $\theta^\pm < 1$, we obtain

$$\|z\|_2 \geq \min\left\{(2A_0)^{\frac{1}{(\theta^- - 1)p^-}} \|\Delta z\|_2^{\frac{2-\theta^- p^-}{(1-\theta^-)p^-}}, (2A_0)^{\frac{1}{(\theta^+ - 1)p^+}} \|\Delta z\|_2^{\frac{2-\theta^+ p^+}{(1-\theta^+)p^+}}\right\}. \tag{22}$$

Moreover, it follows from (15) and (20) that

$$\delta_1 \leq \|\Delta z\|_2 \leq \sqrt{\frac{2kp^-}{p^- - 2}} := \delta_2, \text{ for all } z \in \mathcal{N}_k.$$

Thus, when combined with (22), it implies

$$\|z\|_2 \geq \min \left\{ \begin{array}{c} (2A_0)^{\frac{1}{(\theta^- - 1)p^-}} \min \left\{ \delta_1^{\frac{2-\theta^- p^-}{(1-\theta^-)p^-}}, \delta_2^{\frac{2-\theta^- p^-}{(1-\theta^-)p^-}} \right\}, \\ (2A_0)^{\frac{1}{(\theta^+ - 1)p^+}} \min \left\{ \delta_1^{\frac{2-\theta^+ p^+}{(1-\theta^+)p^+}}, \delta_2^{\frac{2-\theta^+ p^+}{(1-\theta^+)p^+}} \right\} \end{array} \right\} > 0.$$

Therefore, $\lambda_k > 0$ by the definition of λ_k. This completes the proof. □

As a result, here we present the following lemma, which plays a crucial role in the proofs of our main results for the case of high initial energy.

Lemma 3.5. *Suppose that (2)-(3) hold, we can derive the following results*
i) The point 0 is away from both $\mathcal{N}$ and $\mathcal{N}_-$, i.e., dist $(0, \mathcal{N}) > 0$ and dist $(0, \mathcal{N}_-) > 0$.
ii)The set $\mathcal{N}_+ \cap E^k$ is bounded in $H_0^2(\Omega)$ for any $k > 0$.

Therefore from (15), obtain

$$\begin{aligned} dist\,(0, \mathcal{N}) &= \inf_{z \in N} \|\Delta z\|_2 \geq \delta_1 > 0, \\ dist\,(0, \mathcal{N}_-) &= \inf_{z \in N_-} \|\Delta z\|_2 \geq \delta_1 > 0. \end{aligned}$$

Let's proceed with the proof of ii). For any $z \in \mathcal{N}_+ \cap E^k$, we have $E(z) < k$ and $I(z) > 0$. Utilizing (7), we obtain

$$\begin{aligned} k &> E(z) \geq \left(\frac{1}{2} - \frac{1}{p^-}\right) \|\nabla z\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right) \|\Delta z\|_2^2 \\ &\quad + \left(\frac{1}{m^+} - \frac{1}{p^-}\right) \int_\Omega |\Delta z|^{m(x)} dx + \frac{1}{p^-} I(z) \\ &\geq \left(\frac{1}{2} - \frac{1}{p^-}\right) \|\Delta z\|_2^2, \end{aligned}$$

we get

$$\|\Delta z\|_2 < \sqrt{\frac{2kp^-}{p^- - 2}},$$

and completes the proof.

4. Evolution Problem

Definition 4.1. *Consider a function $z(t)$ defined on the domain $\Omega \times [0, T)$, where $T > 0$. We say $z(t)$ is a weak solution to equation (1) in $\Omega \times [0, T)$ if the following conditions hold:*

1. The function $z(t)$ is essentially bounded in time (up to T) and has square-integrable second-order spatial derivatives on Ω with homogeneous Dirichlet boundary conditions. Formally $z(t) \in L^\infty\left(0, T; H_0^2(\Omega)\right)$.

2. The time derivative of $z(t)$ denoted by z_t is square-integrable over time (up to T) and the spatial distribution is also square-integrable on Ω. Formally $z_t \in L^2\left(0,T;L^2(\Omega)\right)$.

3. The initial condition of the function $z(t)$ is given by $z(0)=z_0$ where $z(0)=z_0\in H_0^2(\Omega)$. This initial condition represents the state of the system at time $t=0$, satisfying homogeneous Dirichlet boundary conditions on Ω.

$$\langle z_t,v\rangle+\langle\nabla z,\nabla v\rangle+\langle\Delta z,\Delta v\rangle+\left\langle|\Delta z|^{m(x)-2}\Delta z,\Delta v\right\rangle=\left\langle|z|^{p(x)-2}z,v\right\rangle,\ \text{a.e. } t\in(0,T) \tag{23}$$

for any $v\in H_0^2(\Omega)$. Furthermore ,

$$\int_0^t \left\|z'(s)\right\|_2^2 ds+E(z(t))=E(z_0),\ 0\le t<T. \tag{24}$$

Definition 4.2. *Suppose that $z(t)$ be a weak solution to the problem (1). We define the maximal existence time $T_{\max}$ of $z(t)$ as follows*

i)If $z(t)$ exists for $0\le t<\infty$, then $T_{\max}=\infty$.

ii)If there exists $t_0>0$ such that $z(t)$ exists for $0\le t<t_0$, but does not exist at t_0, then $T_{\max}=t_0$.

Lemma 4.3. *Assuming that (2)-(3) hold and $J(z_0)<d$, we have the following statements:*

i) If $I(z_0)<0$, then $I(z(t))<0$ for all $t\in[0,T_{\max})$.

ii) If $I(z_0)\ge 0$, then $I(z(t))\ge 0$ for all $t\in[0,T_{\max})$.

Proof. It is important to note that $z\notin\mathcal{N}$, for all $t\in[0,T_{\max})$ since $E(z(t))\le E(z_0)<d$.

For i). Assume the opposite that there exists $t_0\in(0,T_{\max})$ such that $I(z(t))<0$ for all $t\in[0,t_0)$ and $I(z(t_0))=0$. Then using (15), we get $\|\Delta z(t)\|_2\ge\delta_1$, for all $t\in[0,t_0)$. As $t\to t_0$, we have $\|\Delta z(t_0)\|_2\ge\delta_1$, which implies $z(t_0)\ne 0$. Consequently, $z(t_0)\in\mathcal{N}$, leading to a contradiction.

For ii). Assuming the contrary, we consider the existence of $t_1\in(0,T_{\max})$ such that $I(z(t_1))<0$. Given that $I(z_0)\ge 0$ this implies the existence of $t_2\in[0,t_1)$ for which $I(z(t_2))=0$, as $z(t_2)\notin\mathcal{N}$. Since $z(t_2)=0$ it follows that $z(t)=0$ for $t\in[t_2,T_{\max})$. Consequently, we have $z(t_1)=0$ which contradicts the initial assumption of $I(z(t_1))<0$. Thus, the proof is complete. □

Let us define the set

$$S=\left\{\phi\in H_0^2(\Omega):\phi\text{ is a stationary solution of (1)}\right\},$$

and define the $\omega-$limit set $\omega(z_0)$ of the initial data $z_0\in W_0^{2,m(.)}(\Omega)$ by

$$\omega(z_0)=\left\{\omega\in H_0^2(\Omega):\exists\{t_n\}\text{ with } t_n\to\infty\text{ such that } z(t_n)\to\omega\right\}.$$

Let $z(t)$ be a solution to (1) related with $z_0\in H_0^2(\Omega)$ over the maximal existence time interval $[0,T_{\max})$. Subsequently, we define the sets.

$$\begin{aligned}
G &= \left\{z_0\in H_0^2(\Omega):z(t)\text{ exists globally, i.e. } T_{\max}=\infty\right\},\\
G_0 &= \left\{z_0\in G:z(t)\to 0\text{ in } H_0^2(\Omega)\text{ as } t\to\infty\right\},\\
B &= \left\{z_0\in H_0^2(\Omega):z(t)\text{ blows up in finite time, i.e. } T_{\max}<\infty\right\}.
\end{aligned}$$

The main results are stated as follows.

Theorem 4.4. *Assume that (2)-(3) hold. If $E(z_0)<d$ and $I(z_0)\ge 0$, then the maximal existence time $T_{\max}=\infty$. Furthermore, $z(t)$ satisfies the following decay estimates:*

$$\begin{aligned}
\|z(t)\|_2 &\le \|z_0\|_2 e^{-\alpha t},\\
\|\Delta z(t)\|_2 &\le \sqrt{\frac{2q^-}{q^--2}\left(E(z_0)+\|z_0\|_2^2\right)e^{-\beta t}},\\
\sqrt{E(z(t))+\|z(t)\|_2^2} &\le \sqrt{\left(E(z_0)+\|z_0\|_2^2\right)e^{-\beta t}},
\end{aligned}$$

where α and β are positive constants.

Theorem 4.5. *Suppose that (2)-(3) hold.*

i)If $z_0 \in H_0^2(\Omega) \setminus \{0\}$ holds $E(z_0) \leq 0$, then $T_{\max} < \infty$. Additionally, it is possible to obtain an upper bound for the maximal existence time.

$$T_{\max} \leq C_{\max}\left\{\|z_0\|_2^{2-p^-}, \|z_0\|_2^{2-p^+}\right\},$$

where

$$C = \frac{p^- \max\left\{S_{p(.),2}^{p^-}, S_{p(.),2}^{p^+}\right\}}{(p^- - 2)(p^- - \max\{2, m^+\})} > 0, \tag{25}$$

and $S_{p(.),2}$ represents the optimal embedding constant of $L^{p(.)}(\Omega) \hookrightarrow L^2(\Omega)$ when $p > 2$, i.e.,

$$S_{p(.),2} = \sup_{z \in L^{p(.)}(\Omega)\setminus\{0\}} \frac{\|z\|_2}{\|z\|_{p(.)}}. \tag{26}$$

ii) If $0 < E(z_0) < d$ and $I(z_0) < 0$, then $T_{\max} < \infty$.

Theorem 4.6. *Let's denote that (2)-(3) are satisfied, and $z(t)$ is a global solution to (1). Then there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ and a function $\phi \in S$ such that*

$$\lim_{n\to\infty} \left\|\Delta z(t_n) - \Delta\phi\right\|_2 = 0.$$

Our next result provides an abstract criterion for the vanishing and global nonexistence of solutions to (1) based on the variational values λ_k and Λ_k.

Theorem 4.7. *Assume that (2)-(3) hold and $E(z_0) > d$. If $z_0 \in \mathcal{N}_+$ and $\|z_0\|_2 \leq \lambda_{E(z_0)}$, then $z_0 \in G_0$. If $z_0 \in \mathcal{N}_-$ and $\|z_0\|_2 \leq \Lambda_{E(z_0)}$, then $z_0 \in B$.*

Lastly, there is a characterization of the initial data z_0 with arbitrary high energy $E(z_0)$ which results in the phenomenon of blow-up in finite time.

Theorem 4.8. *Under the conditions (2)-(3) suppose $z_0 \in H_0^2(\Omega)$ satisfies $E(z_0) > d$ and the following inequality holds:*

$$\|z_0\|_2^2 \geq \frac{2p^- S_2^2}{p^- - 2} E(z_0). \tag{27}$$

Then, it follows that $z_0 \in \mathcal{N}_- \cap B$. Here S_2 is the constant given in (21).

5. The proof of Theorem 4.4

Assume that $z := z(x,t)$ be a solution of equation (1) on the interval $[0, T_{\max})$ related with to the initial data z_0. Our goal is to demonstrate the uniform boundedness in time of z in the function space $H_0^2(\Omega)$, which will imply $T_{\max} = \infty$ by the continuation principle. To begin, we note that since $E(z_0) < d$ and $I(z_0) \geq 0$, according to Lemma 4.3, we have the inequality $I(z) \geq 0$ for all t in $[0, T_{\max})$. We proceed to utilize the non-increasing property of $E(z)$ and and equation (7).

$$\begin{aligned} E(z_0) &\geq E(z) \\ &\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\|\nabla z\|_2^2+\left(\frac{1}{2}-\frac{1}{p^-}\right)\|\Delta z\|_2^2 \\ &\quad +\left(\frac{1}{m^+}-\frac{1}{p^-}\right)\int_\Omega |\Delta z|^{m(x)}\,dx-\left(\frac{1}{p^+}-\frac{1}{p^-}\right)\int_\Omega |z|^{p(x)}\,dx+\frac{1}{p^-}I(z) \\ &\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\|\nabla z\|_2^2+\left(\frac{1}{2}-\frac{1}{p^-}\right)\|\Delta z\|_2^2 \\ &\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\|\Delta z\|_2^2, \end{aligned} \tag{28}$$

which implies

$$\|\Delta z\|_2^2 \leq \sqrt{\frac{2p^-E(z_0)}{p^- - 2}}.$$

Next, we will establish the decay estimates of z. Let's consider two cases:

Case 1: If there exists a $t_0 \geq 0$ such that $z(t_0)=0$, then we can deduce that $z=0$ for all $t \geq t_0$ and thus, the proof is complete.

Case 2: Let's assume that $z \neq 0$ for all $t \geq 0$. Since $I(z) \geq 0$ due to Lemma 3.1, we can find a parameter $\lambda_* \geq 1$ such that $I(\lambda_* z)=0$.

$$\begin{aligned} \lambda_*^{p^-}I(z) &= \lambda_*^{p^-}I(z)-I(\lambda_* z) \\ &= \left(\lambda_*^{p^-}-\lambda_*^2\right)\|\nabla z\|_2^2+\left(\lambda_*^{p^-}-\lambda_*^2\right)\|\Delta z\|_2^2 \\ &\quad +\int_\Omega \left(\lambda_*^{p^-}-\lambda_*^{m(x)}\right)|\Delta z|^{m(x)}\,dx+\int_\Omega\left(\lambda_*^{p(x)}-\lambda_*^{p}\right)|z|^{p(x)}\,dx \\ &\geq \left(\lambda_*^{p^-}-\lambda_*^2\right)\|\nabla z\|_2^2+\left(\lambda_*^{p^-}-\lambda_*^2\right)\|\Delta z\|_2^2+\int_\Omega\left(\lambda_*^{p^-}-\lambda_*^{m(x)}\right)|\Delta z|^{m(x)}\,dx. \end{aligned}$$

Dividing the above inequality by $\lambda_*^{p^-}$, we have

$$\begin{aligned} I(z) &\geq \left(1-\lambda_*^{2-p^-}\right)\|\nabla z\|_2^2+\left(1-\lambda_*^{2-p^-}\right)\|\Delta z\|_2^2 \\ &\quad +\left(1-\lambda_*^{m^+-p^-}\right)\int_\Omega |\Delta z|^{m(x)}\,dx. \end{aligned} \tag{29}$$

Next, we proceed to estimate the value of λ_*. To do this, we apply Lemma 3.2 and take into account that $\lambda_* \geq 1$,

$$E(z(t))-\frac{1}{q^-}I(z(t)) \geq \frac{d}{\max\left\{\lambda_*^2,\lambda_*^{m^-},\lambda_*^{p^+}\right\}}=\frac{d}{\lambda_*^{p^+}}. \tag{30}$$

Moreover, we can utilize the non-increasing property of $E(z)$ and take into account that $I(z) \geq 0$, to deduce the following:

$$E(z)-\frac{1}{p^-}I(z) \leq E(z_0).$$

This together with (30),

$$\lambda_* \geq \left(\frac{d}{E(z_0)}\right)^{1\backslash p^+} > 1. \tag{31}$$

It follows from (29) and (31) that

$$\begin{aligned} I(z) &\geq \left(1-\left(\frac{d}{E(z_0)}\right)^{\frac{2-p^-}{p^+}}\right)\|\nabla z\|_2^2 + \left(1-\left(\frac{d}{E(z_0)}\right)^{\frac{2-p^-}{p^+}}\right)\|\nabla z\|_2^2 \\ &\quad + \left(1-\left(\frac{d}{E(z_0)}\right)^{\frac{m^+-p^-}{p^+}}\right)\int_\Omega |\Delta z|^{m(x)}\,dx, \end{aligned}$$

where

$$I(z) \geq C_1\|\nabla z\|_2^2\ ,\ I(z) \geq C_2\|\Delta z\|_2^2 \text{ and } I(z) \geq C_3\int_\Omega |\Delta z|^{m(x)}\,dx, \tag{32}$$

then

$$C_1 = 1-\left(\frac{d}{E(z_0)}\right)^{\frac{2-p^-}{p^+}}\ ,\ C_2 = 1-\left(\frac{d}{E(z_0)}\right)^{\frac{2-p^-}{p^+}} \text{ and } C_3 = 1-\left(\frac{d}{E(z_0)}\right)^{\frac{m^+-p^-}{p^+}}.$$

We now deal with the exponential decay of $\|z\|_2$. Taking $v = z$ in (23), we get

$$\begin{aligned} \frac{d}{dt}\|z\|_2^2 &= -2\left(\|\nabla z\|_2^2 + \|\Delta z\|_2^2 + \int_\Omega |\Delta z|^{p(x)}\,dx - \int_\Omega |z|^{q(x)}\,dx\right) \\ &= -2I(z). \end{aligned}$$

From this and (32), it follows that

$$\begin{aligned} \frac{d}{dt}\|z\|_2^2 &\leq -2\left(C_1\|\nabla z\|_2^2 + C_2\|\Delta z\|_2^2\right) \leq -2C_2\|\Delta z\|_2^2 \\ &\leq -2C_2\|\Delta z\|_2^2 \leq -2C_2S_2^{-2}\|z\|_2^2, \end{aligned}$$

where S_2 is the constant given in (21). This implies that

$$\|z\|_2 \leq \|z_0\|_2\, e^{-\alpha t},$$

where $\alpha = C_2S_2^{-2} > 0$.

Next, we will focus on the exponential decay of $E(z)$ and $\|\Delta z\|_2$. By utilizing equation (8) we can derive the following:

$$\begin{aligned} E(z) &\leq \left(\frac{1}{2}-\frac{1}{p^+}\right)\|\nabla z\|_2^2 + \left(\frac{1}{2}-\frac{1}{p^+}\right)\|\Delta z\|_2^2 \\ &\quad + \left(\frac{1}{m^-}-\frac{1}{p^+}\right)\int_\Omega |\Delta z|^{m(x)}\,dx + \frac{1}{p^+}I(z). \end{aligned}$$

This together with (32) immediately yields

$$E(z) \leq C_4 I(z), \tag{33}$$

where

$$C_4 = \frac{1}{C_1}\left(\frac{1}{2} - \frac{1}{p^+}\right) + \frac{1}{C_2}\left(\frac{1}{2} - \frac{1}{p^+}\right) + \frac{1}{C_3}\left(\frac{1}{m^-} - \frac{1}{p^+}\right) + \frac{1}{p^+} > 0.$$

Let us define an auxiliary function

$$L(t) = E(z) + \|z\|_2^2, \text{ for } t \geq 0. \tag{34}$$

Then by (28) and (34), we get

$$L(t) \leq E(z) + S_2^{-2}\|z\|_2^2 \leq C_5 E(z). \tag{35}$$

Here $C_5 = 1 + \frac{2p^-}{p^- - 2} S_2^2 > 0$ and S_2 is the constant given in (21). It follows from (33), (34) and (35) that

$$\frac{d}{dt} L(t) = -\left\|z'\right\|_2^2 - 2I(z) \leq -\frac{2}{C_4} E(z) \leq -\frac{2}{C_4 C_5} L(t).$$

As a result, we can conclude that

$$L(t) \leq L(0) e^{-2\beta t},$$

where $\beta = \frac{1}{C_4 C_5} > 0$. The inequality shown above can be reformulated as

$$E(z) + \|z\|_2^2 \leq \left(E(z_0) + \|z_0\|_2^2\right) e^{-2\beta t}. \tag{36}$$

The proof is complete.

6. The proof of Theorem 4.5

We deal with following two cases by utilizing different methods:
Case 1: $z_0 \in H_0^2(\Omega) \setminus \{0\}$ with $E(z_0) \leq 0$. We define the function

$$f(t) = \|z\|_2^2, \text{ for all } t \in [0, T_{\max}).$$

By the definition of E and I, we get

$$\begin{aligned} E(z) &\geq \frac{1}{2}\|\nabla z\|_2^2 + \frac{1}{2}\|\Delta z\|_2^2 \\ &\quad + \frac{1}{m^+}\int_\Omega |\Delta z|^{m(x)} dx - \frac{1}{p^-}\int_\Omega |z|^{p(x)} dx \\ &\geq \frac{1}{\max\{2, m^+\}}\left(\|\nabla z\|_2^2 + \|\Delta z\|_2^2 + \int_\Omega |\Delta z|^{m(x)} dx\right) - \frac{1}{p^-}\int_\Omega |z|^{p(x)} dx \\ &= \left(\frac{1}{\max\{2, m^+\}} - \frac{1}{p^-}\right)\int_\Omega |z|^{p(x)} dx + \frac{1}{\max\{2, m^+\}} I(z). \end{aligned}$$

Considering this fact and observing that $E(z) \leq E(z_0) \leq 0$,

$$\begin{aligned} f'(t) &= -2I(z) \\ &\geq -2\max\{2, m^+\} E(z) + 2\left(1 - \frac{\max\{2, m^+\}}{p^-}\right)\int_\Omega |z|^{p(x)} dx \qquad (37) \\ &\geq 2\left(1 - \frac{\max\{2, m^+\}}{p^-}\right)\int_\Omega |z|^{p(x)} dx. \end{aligned}$$

From the obtained value $p^- > \max\{2, m^+\}$, it follows that $f'(t) \geq 0$ for all $t \in [0, T_{\max})$. This implies that

$$f(t) \geq f(0) = \|z_0\|_2^2 > 0, \text{ for all } t \in [0, T_{\max}). \tag{38}$$

Then by (38) we can estimate $\int_\Omega |z|^{p(x)} dx$ as follows:

$$\begin{aligned} \int_\Omega |z|^{p(x)} dx &\geq \min\left\{\|z\|_{p(.)}^{p^-}, \|z\|_{p(.)}^{p^+}\right\} \\ &\geq \min\left\{S_{p(.),2}^{-p^-}\|z\|_2^{p^-}, S_{p(.),2}^{-p^+}\|z\|_2^{p^+}\right\} \\ &\geq \min\left\{S_{p(.),2}^{-p^-}, S_{p(.),2}^{-p^+}\right\}\min\left\{\|z\|_2^{p^-}, \|z\|_2^{p^+}\right\} \\ &= \min\left\{S_{p(.),2}^{-p^-}, S_{p(.),2}^{-p^+}\right\}\min\left\{1, f^{\frac{p^+-p^-}{2}}(t)\right\} f^{\frac{p^-}{2}}(t) \\ &\geq \min\left\{S_{p(.),2}^{-p^-}, S_{p(.),2}^{-p^+}\right\}\min\left\{1, \|z_0\|_2^{p^+-p^-}\right\} f^{\frac{p^-}{2}}(t), \end{aligned} \tag{39}$$

where $S_{p(.),2}$ is defined in (26). It follows from (37) and (39) that

$$f'(t) \geq C_0 f^{\frac{p^-}{2}}(t), \tag{40}$$

where

$$C_0 = 2\left(1 - \frac{\max\{2, m^+\}}{p^-}\right)\min\left\{S_{p(.),2}^{-p^-}, S_{p(.),2}^{-p^+}\right\}\min\left\{1, \|z_0\|_2^{p^+-p^-}\right\} > 0.$$

Given that $f(t) > 0$, we can divide the inequality (40) by $f^{\frac{p^-}{2}}(t)$, yielding

$$f'(t) f^{-p^-/2}(t) \geq C_0.$$

By integrating the above inequality over the interval $[0, t]$, we obtain

$$f^{1-\frac{p^-}{2}}(t) \leq f^{1-\frac{p^-}{2}}(0) - \left(\frac{p^-}{2} - 1\right)C_0 t, \text{ for all } t \in [0, T_{\max}).$$

This and $f^{1-\frac{p^-}{2}}(t) > 0$ imply

$$t < \frac{2}{(p^- - 2)C_0}\|z_0\|_2^{2-p^-}, \text{ for all } t \in [0, T_{\max}).$$

Therefore, we have

$$T_{\max} \leq \frac{2}{(p^- - 2)C_0}\|z_0\|_2^{2-p^-} = C\max\left\{\|z_0\|_2^{2-p^-}, \|z_0\|_2^{2-p^+}\right\},$$

here, C represents the constant given in equation (25).

Case 2: Assuming a contradiction, let us suppose that $T_{\max} = \infty$ given that $0 < E(z_0) < d$ and $I(z_0) < 0$. Due to $I(z_0) < 0$, by Lemma 4.3, we deduce that $I(z(t)) < 0$ for all $t \geq 0$. Then, with the aid of Lemmas 3.1 and 3.2, we can conclude that there exists $\lambda_* \in (0, 1)$ such that

$$E(z) - \frac{1}{p^-}I(z) \geq \frac{d}{\max\left\{\lambda_*^2, \lambda_*^{m^-}, \lambda_*^{p^+}\right\}} > d,$$

which implies that

$$\frac{d}{dt}\|z\|_2^2 = -2I(z) > 2p^-(d - E(z)) \geq 2p^-(d - E(z_0)).$$

Then we obtain

$$\|z\|_2^2 = \|z_0\|_2^2 + \int_0^t \frac{d}{ds}\|z(s)\|_2^2 ds \geq \|z_0\|_2^2 + 2p^-(d - E(z_0))\,t.$$

From the previous results and the fact that $E(z_0) < d$, we obtain $\lim_{t\to\infty}\|z\|_2^2 = \infty$. Thus, we can choose sufficiently large $t_0 > 0$ such that

$$\|z(t_0)\|_2^2 > \frac{p^-}{(p^- - 2)}\|z_0\|_2^2.$$

Let

$$T = \frac{\int_0^{t_0}\|z(s)\|_2^2 ds}{\left(\frac{p^-}{2} - 1\right)\left(\|z(t_0)\|_2^2 - \frac{p^-}{(p^- - 2)}\|z_0\|_2^2\right)} + t_0 \geq t_0 > 0. \tag{41}$$

We now define the auxiliary function $F : [0, T] \longrightarrow (0, \infty)$ by

$$F(t) = \int_0^t \|z(s)\|_2^2 ds + (T - t)\|z_0\|_2^2. \tag{42}$$

Then

$$F'(t) = \|z\|_2^2 - \|z_0\|_2^2 = 2\int_0^t \langle z'(s), z(s)\rangle ds,$$

and

$$\begin{aligned} F''(t) &= 2\langle z'(t), z(t)\rangle = -2I(z(t)) \\ &> 2p^-(d - E(z(t))) \\ &= 2p^-(d - E(z_0)) + 2p^-\int_0^t \|z'(s)\|_2^2 ds \\ &\geq 2p^-\int_0^t \|z'(s)\|_2^2 ds. \end{aligned} \tag{43}$$

From equations (42) and (43) we can deduce

$$F(t)F''(t) \geq 2p^-\int_0^t \|z'(s)\|_2^2 ds \int_0^t \|z(s)\|_2^2 ds. \tag{44}$$

Moreover, by Cauchy-Schwarz inequality, we get

$$\int_0^t \|z'(s)\|_2^2 \, ds \int_0^t \|z(s)\|_2^2 \, ds \geq \left(\int_0^t \langle z'(s), z(s) \rangle \, ds \right)^2 = \frac{1}{4} (F(t))^2 . \tag{45}$$

Combining (44)-(45), we have

$$F(t) F''(t) \geq \frac{p^-}{2} \left(F'(t) \right)^2, \text{ for all } t \in [0, T] . \tag{46}$$

Setting $G(t) = F^{1-\frac{p^-}{2}}(t)$, we get

$$G'(t) = \left(1 - \frac{p^-}{2} \right) \frac{F'(t)}{F^{\frac{p^-}{2}}(t)}, \quad G''(t) = \left(1 - \frac{p^-}{2} \right) \frac{F(t) F''(t) - \frac{p^-}{2} (F'(t))^2}{F^{1+\frac{p^-}{2}}(t)} .$$

Consequently, utilizing equation (46), we can establish $G''(t) \leq 0$, for all $t \in [0, T]$ thanks to . Hence, $G(t)$ is concave on the interval $[0, T]$. This implies that

$$G(t) \leq G(t_0) + G'(t_0)(t - t_0), \text{ for all } t \in [0, T] .$$

Substituting t by T in the previous inequality and taking into account equation (41), we obtain

$$\begin{aligned} G(T) &\leq G(t_0) + G'(t_0)(T - t_0) \\ &= F^{-\frac{p^-}{2}}(t_0) \left[F(t_0) - \left(\frac{p^-}{2} - 1 \right) (T - t_0) F'(t_0) \right] = 0. \end{aligned}$$

This contradicts $G(T) > 0$, and the proof is complete.

7. The proof of Theorem 4.6

Assuming z is a global weak solution to equation (1). We can apply property i). In Theorem 4.6 to deduce that $E(z) \geq 0$ for all $t \geq 0$. Thus,

$$\int_0^t \|z'\|_2^2 \, ds = E(z_0) - E(z) \leq E(z_0) .$$

Letting $t \to \infty$, one has

$$\int_0^\infty \|z'\|_2^2 \, ds \leq E(z_0) < \infty .$$

Consequently, we can find a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \|z'(t_n)\|_2 = 0, \tag{47}$$

which implies that $\|z'(t_n)\|_2 \leq A$ for all $n \in \mathbb{N}$, for some a constant A. Then

$$\begin{aligned} |I(z(t_n))| &= |\langle z'(t_n), z(t_n) \rangle| && (48) \\ &\leq \|z'(t_n)\|_2 \|z(t_n)\|_2 && (49) \\ &\leq \|z'(t_n)\|_2 S_2 \|\Delta z(t_n)\|_2 && (50) \\ &\leq A S_2 \|\Delta z(t_n)\|_2 . && (51) \end{aligned}$$

By utilizing the constant S_2 given in equation (21) and taking into account the non-increasing property of $E(z)$, equation (51) and replacing z with $z(t_n)$ in equation (7), we can establish

$$\begin{aligned} E(z_0) &\geq E(z(t_n)) \\ &\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\|\nabla z(t_n)\|_2^2+\left(\frac{1}{2}-\frac{1}{p^-}\right)\|\Delta z(t_n)\|_2^2+\left(\frac{1}{m^+}-\frac{1}{p^-}\right)\int_\Omega |\Delta z(t_n)|^{m(x)}\,dx+\frac{1}{p^-}I(z(t_n)) \\ &\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\|\Delta z(t_n)\|_2^2-\frac{AS_2}{p^-}\|\Delta z(t_n)\|_2, \end{aligned}$$

which implies that

$$\|\Delta z(t_n)\|_2 \leq \frac{AS_2+\sqrt{A_2^2S_2^2+2p^-(p^--2)E(z_0)}}{p^--2}. \tag{52}$$

The above inequality guarantees that the sequence $\{z(t_n)\}$ is bounded in the function space $H_0^2(\Omega)$. Given that $H_0^2(\Omega)$ is a reflexive space, the embeddings $H_0^2(\Omega) \hookrightarrow W_0^{2,m(.)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ are compact (by (2) and (3)). Consequently, there exists a subsequence of $\{z(t_n)\}$ and a function $\phi \in H_0^2(\Omega)$ such that

$$z(t_n) \rightharpoonup \phi \text{ weakly in } H_0^1(\Omega), \tag{53}$$

$$z(t_n) \rightharpoonup \phi \text{ weakly in } H_0^2(\Omega), \tag{54}$$

$$z(t_n) \longrightarrow \phi \text{ strongly in } W_0^{2,m(.)}(\Omega), \tag{55}$$

$$z(t_n) \longrightarrow \phi \text{ strongly in } L^{p(.)}(\Omega). \tag{56}$$

For any $v \in H_0^2(\Omega)$. We replace z with $z(t_n)$ in the equation (1). By multiplying equation(1) by v and integrating by parts, we obtain

$$\begin{aligned} &\left|\langle \nabla z(t_n),\nabla v\rangle+\langle \Delta z(t_n),\Delta v\rangle+\left\langle |\Delta z(t_n)|^{m(x)-2}\Delta z(t_n),\Delta v\right\rangle-\left\langle |z(t_n)|^{p(x)-2}z(t_n),v\right\rangle\right| \\ = \;& |\langle z'(t_n),v\rangle| \leq \|z'(t_n)\|_2\|v\|_2. \end{aligned}$$

From this and (47), it follows that

$$\lim_{n\to\infty}\left(\langle \nabla z(t_n),\nabla v\rangle+\langle \Delta z(t_n),\Delta v\rangle+\left\langle |\Delta z(t_n)|^{m(x)-2}\Delta z(t_n),\Delta v\right\rangle-\left\langle |z(t_n)|^{p(x)-2}z(t_n),v\right\rangle\right)=0,$$

which, together with (53),(54), (55) and (56) yields

$$\phi \in S. \tag{57}$$

By (47), (50) and (52), we have

$$\lim_{n\to\infty} I(z(t_n)) = 0,$$

which, together with (55), (56) and (57), implies

$$\begin{aligned} \lim_{n\to\infty}(\|\nabla z(t_n)\|_2+\|\Delta z(t_n)\|_2) &= -\lim_{n\to\infty}\int_\Omega |\Delta z(t_n)|^{m(x)}\,dx+\lim_{n\to\infty}\int_\Omega |z(t_n)|^{p(x)}\,dx \\ &= -\int_\Omega |\Delta\phi|^{m(x)}\,dx+\lim_{n\to\infty}\int_\Omega |\phi|^{p(x)}\,dx=\|\Delta\phi\|_2. \end{aligned} \tag{58}$$

Note that $H_0^2(\Omega)$ is uniformly convex. Then by (54) and (58), we imply (see ([5], Proposition 3.32)).

$$z(t_n) \longrightarrow \phi \text{ strongly in } H_0^2(\Omega).$$

Thus, the proof is now complete.

8. The proof of Theorems 4.7 and 4.8

We can establish the Theorems 4.7 and 4.8 by adopting the ideas presented in [15, 34]. Here's how the proof proceeds:

The proof of Theorem 4.7. Suppose that $z_0 \in \mathcal{N}_+$ and $\|z_0\|_2 \leq \lambda_{E(z_0)}$. We first prove that $z \in \mathcal{N}_+$ for all $t \in [0, T_{\max})$. Indeed, assume on the contrary that there is $t_0 > 0$ such that for all $t \in [0, t_0)$ and $z(t_0) \in \mathcal{N}$. Then for all $t \in [0, t_0)$, we get

$$0 < |I(z)| = |\langle z', z\rangle| \leq \|z'\|_2 \|z\|_2,$$

which gives $\|z'\|_2 > 0$. From this and (24), we have $E(z(t_0)) < E(z_0)$, i.e., $z(t_0) \in E^{E(z_0)}$. So $\|z(t_0)\|_2 \geq \lambda_{E(z_0)}$. Moreover, for all $t \in [0, t_0)$, we obtain

$$\frac{d}{dt}\|z\|_2^2 = -2I(z) < 0.$$

The obtained inequality implies that $\|z(t_0)\|_2 < \|z_0\|_2 \leq \lambda_{E(z_0)}$. This leads to a contradiction, which establishes the claim that $z \in \mathcal{N}_+$ for all $t \in [0, T_{\max})$. Considering the strictly decreasing property of $E(z)$ it follows that $z \in \mathcal{N}_+ \cap E^{E(z_0)}$ for all $t \in [0, T_{\max})$. Additionally, based on property ii). In Lemma 3.5, z remains bounded in $H_0^2(\Omega)$ for all $t \in [0, T_{\max})$. Consequently, $T_{\max} = \infty$ indicating that $z_0 \in G$. Next, we aim to prove that $z_0 \in G_0$. Let any $w \in \omega(z_0)$,

$$\|w\|_2 < \lambda_{E(z_0)} \text{ and } E(w) < E(z_0),$$

which implies that $\omega(z_0) \cap \mathcal{N} = \emptyset$ by definition of $\lambda_{E(z_0)}$. And thus, $\omega(z_0) = \{0\}$, i.e., $z_0 \in G_0$.

Now we suppose that $z_0 \in \mathcal{N}_-$ and $\|z_0\|_2 \geq \Lambda_{E(z_0)}$. By analogous arguments as above, in addition to $z \in \mathcal{N}_-$ for all $t \in [0, T_{\max})$. Assume on the contrary that $T_{\max} = \infty$, then for every $w \in \omega(z_0)$, one has

$$\|w\|_2 > \Lambda_{E(z_0)} \text{ and } E(w) < E(z_0),$$

which gives $\omega(z_0) \cap \mathcal{N} = \emptyset$ by the definition of $\Lambda_{E(z_0)}$. However, since $dist(0, \mathcal{N}_-) > 0$, we also have $0 \notin \omega(z_0)$. And thus $\omega(z_0) = \emptyset$, which contradicts $T_{\max} = \infty$. Thus $z_0 \in B$. The proof is complete.

The proof of Theorem 4.8. Let any $z \in H_0^2(\Omega) \setminus \{0\}$. By using (7), we get

$$\begin{aligned} E(z) &\geq \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\nabla z\|_2^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\Delta z\|_2^2 + \left(\frac{1}{m^+} - \frac{1}{p^-}\right)\int_\Omega |\Delta z|^{m(x)}\,dx + \frac{1}{p^-}I(z) \\ &> \left(\frac{1}{2} - \frac{1}{p^-}\right)\|\Delta z\|_2^2 + \frac{1}{p^-}I(z) \\ &\geq \left(\frac{1}{2} - \frac{1}{p^-}\right)S_2^{-2}\|z\|_2^2 + \frac{1}{p^-}I(z), \end{aligned} \tag{59}$$

where S_2 is defined in (21). Replacing z by z_0 in (59) and using (27), we have

$$E(z_0) > \left(\frac{1}{2} - \frac{1}{p^-}\right)S_2^{-2}\|z_0\|_2^2 + \frac{1}{p^-}I(z_0) \geq E(z_0) + \frac{1}{p^-}I(z_0),$$

which gives $I(z_0) < 0$, i.e.,

$$z_0 \in \mathcal{N}_-. \tag{60}$$

For any $z \in \mathcal{N}_{E(z_0)}$, we know that $I(z) = 0$ and $E(z) < E(z_0)$. Utilizing equation (59), we can deduce

$$\|z\|_2^2 \leq \frac{2p^- S_2^2}{p^- - 2}E(z_0),$$

which together with (27), implies $\|z\|_2 \leq \|z_0\|_2$. By taking the supremum over $z \in \mathcal{N}_{E(z_0)}$, we have

$$\Lambda_{E(z_0)} \leq \|z_0\|_2. \tag{61}$$

Then by Theorem 4.7, it follows from (60) and (61) that $z_0 \in \mathcal{N}_- \cap B$. The proof is complete.

References

[1] R. Abita, Bounds for Blow-up Solutions of a Semilinear Pseudo-Parabolic Equation with A Memory Term and Logarithmic Nonlinearity in Variable Space, Math. Scand., 128 (3) (2022).

[2] S. N. Antontsev, J. Ferreira, E. Pişkin, Existence and blow up of solutions for a strongly damped Petrovsky equation with variable-exponent nonlinearities, Electron. J. Differ. Equ., 2021(2021) 1-18.

[3] S. N. Antontsev, J. Ferreira, E. Pişkin, S. M. S. Cordeiro, Existence and non-existence of solutions for Timoshenko-type equations with variable exponents, Nonlinear Anal. Real World Appl., 61 (2021) 103341: 1-13.

[4] K. Baghaei, M. B. Ghaemi, M. Hesaaraki, Lower Bounds for the blow-up time in a semilinear parabolic problem involving a variable source, Appl. Math. Lett., 27 (2014), 49-52.

[5] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2010.

[6] A. Braik, Y. Miloudi, K. Zennir, A finite-time blow-up result for a class of solutions with positive initial energy for coupled system of heat equations with memories, Math. Meth. Appl. Sci., 41 (2018) 1674-1682.

[7] G. Butakın, E. Pişkin, Existence and Blow up of Solutions of a Viscoelastic m(x)-Biharmonic Equation with Logarithmic Source Term, Miskolc Math. Notes, (in press).

[8] J. W. Cahn, J. E. Hilliard, Free energy of a nonuniform system.I. Interfacial free energy, J. Chem. Phys., 28 (1958), 258-267.

[9] Q. V. Chuong, L. C. Nhan, L. X. Truong, Existence and Nonexistence of Global solution to the Cahn-Hilliard Equation with variable exponent sources, Electron. J. Differ. Equ., 46 (2022), 1-22.

[10] Q. V. Chuong, L. C. Nhan, L. X. Truong, Existence and non-existence of global solutions of pseudo-parabolic equations involving $p(x)$-Laplacian and logarithmic nonlinearity, J. Elliptic Parabol. Equ., 8 (2022), 483-512.

[11] L. Diening, P. Harjulehto, P.Hasto, M. Ruziçka, Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics 2017, Springer-Verlag, Heidelberg, 2011.

[12] N.C. Eddine, P.D. Nguyen, M.A. Ragusa, Existence and multiplicity of solutions for a class of critical anisotropic elliptic equations of Schrodinger-Kirchhoff-type, Math. Methods App. Sci., 46(16) (2023), 16782-16801.

[13] N.C. Eddine, M.A. Ragusa, Generalized critical Kirchhoff-type potential systems with Neumann boundary conditions, Appl. Anal., 101(11) (2022), 3958-3988.

[14] J. Ferreira, W. S. Panni, S. A. Messaoudi, E. Pişkin, M. Shahrouzi, Existence and Asymptotic Behavior of Beam-Equation Solutions with Strong Damping and p(x)-Biharmonic Operator, J. Math. Phys., Analysis, Geometry, 18(4) (2022) 488-513.

[15] F. Gazzola, T. Weth, Finite time blow up and global solutions for semilinear parabolic equations with initial data at high energy level, Diff. Int. Eq., 18 (2005), 961-990.

[16] Y. Han, A class of fourth-order parabolic equation with arbitrary initial energy, Nonlinear Anal. RWA, 43 (2018), 451-466.

[17] Y. Liu, Existence and Blow-up of solutions to a parabolic equation with nonstandard growth conditions, Bull. Aust. Math. Soc., 99 (2) (2019), 242-249.

[18] N. Moujane, M. El Ouaarabi, C. Allalou, Elliptic Kirchhoff-type system with two convections terms and under Dirichlet boundary conditions, Filomat, 37(28) (2023), 9693-9707.

[19] L. Narayanan, G. Soundararajan, Nonexistence of Global Solutions of a Viscoelastic $p(x)$-Laplacian Equation with Logarithmic Nonlinearity, IAIP Conf. Proc. 2451, 020024-1–020024-10; https://doi.org/10.1063/5.0095323.

[20] M. Ortiz, E. A. Repetto, H. Si, A continuum model of kinetic roughening and coarsening in thin films, J. Mech. Phys. Solids, 47(4) (1999), 697-730.

[21] L. E. Payne, D. H. Sattinger, Saddle points and instability of nonlinear hyperbolic equations, Israel J. Math., 22 (1975), 273-303.

[22] E. Pişkin, Finite time blow up of solutions for a strongly damped nonlinear Klein-Gordon equation with variable exponents, Honam Math. J., 40(4) (2018) 771-783.

[23] E. Pişkin, G. Butakın, Existence and decay of solutions for a parabolic-type Kirchhoff equation with variable exponents, J. Math. Sci. Model., 6 (1) (2023) 32-41.

[24] E. Pişkin, G. Butakın, Blow-up phenomena for a p(x)-biharmonic heat equation with variable exponent, Mathematica Moravica, 27(2) (2023) 25-32.

[25] E. Pişkin, Y. Dinç, C. Tunç, Lower and Upper Bounds for the Blow up Time for Generalized Heat Equations with Variable Exponents, Palest. J. Math., 10(2) (2021) 601-608.

[26] E. Pişkin, B. Okutmuştur, An Introduction to Sobolev Spaces, Bentham Science, 2021.

[27] C. Qu, W. Zhou, B. Liang, Asymptotic behavior for a fourth-order parabolic equation modeling thin film growth, Appl. Math. Lett., 78 (2018), 141-146.

[28] D. H. Sattinger, Stability of nonlinear hyperbolic equations, Arch. Rational Mech. Anal., 28 (1968), 226-244.

[29] T. P. Schulze, R. V. Kohn, A geometric model for coarsening during spiral-mode growth of thin films, Physica D., 132(4) (1999), 520-542.

[30] M. Shahrouzi, J. Ferreira, E. Pişkin, K. Zennir, On the behavior of solutions for a class of nonlinear viscoelastic fourth-order p(x)-Laplacian equation, Mediterr. J. Math., 20 (2023) 1-28.

[31] M. Shahrouzi, J. Ferreira, E. E. Pişkin, General decay and blow up of solutions for a plate viscoelastic p(x)-Kirchhoff type equation with variable exponent nonlinearities and boundary feedback, Quaest. Math., (in press).

[32] M. Shahrouzi, J. Ferreira, E. Pişkin, Stability result for a variable-exponent viscoelastic double-Kirchhoff type inverse source problem with nonlocal degenerate damping term, Ric. di Mat., https://doi.org/10.1007/s11587-022-00713-5 (in press).

[33] X. Wu, B. Guo, W. Gao, Blow-up of solutions for a semilinear parabolic equation involving variable source and positive initial energy, Appl. Math. Lett., 26 (2013) 539-543.

[34] R. Z. Xu, J. Su, Global existence and nite time blow-up for a class of semilinear pseudoparabolic equations, J. Funct. Anal., 264 (2013) 2732-2763.

[35] N. Yılmaz, S. M. Gözen, E. Pişkin, B. Okutmuştur, A blow up of solutions for a system of Klein-Gordon equations with variable exponent: Theoretical and Numerical Results, Math. Appl., 51(1) (2023) 143-164.
[36] N. Yılmaz, E. Pişkin, E. Çelik, Well-Posedness and Blow-Up of Solutions for a Variable Exponent Nonlinear Petrovsky Equation, Adv. Math. Phys., Volume 2023, Article ID 8866861, 1-13.
[37] A. Zangwill, Some causes and a consequence of epitaxial roughening, J. Cryst. Growth, 163(1-2) (1996), 8-21.
[38] K. Zennir,T. Miyasita, Lifespan of solutions for a class of pseudo-parabolic equation with weak-memory, Alex. Eng. J., 59 (2020) 957-964.