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On the initial value problem for parabolic equations with memory terms of fractional type

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Abstract. In this paper, we are interested in considering Cauchy problem for parabolic equation with memory term. The memory component contains a fractional Laplacian operator term. First, we represent the mild solution as a Fourier series. Next, we consider the well-posedness of the Cauchy problem when the initial data and source function are in Gevrey spaces. Under appropriate given data, we also investigate the continuity of the solution according to the parameter k. The another results of this paper is to show the ill-posedness in the sense of Hadamard and give some regularized methods. In the homogeneous case, we use the quasi-boundary value method to regularize the problem and obtain the error estimate when the observation data in L^2. In the case of in-homogeneous source term, we use the truncation method to approximate the problem with observed data in L^p.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with sufficiently smooth boundary $\partial\Omega$. Let T be a positive number. In this paper, we are interested in the parabolic equation with fractional Laplacian term in the memory kernel as follows

$$\begin{cases} y_t - \Delta y + k\int_0^t (-\Delta)^\theta y(x,s)ds = G(x,t), & \text{in} \quad \Omega\times(0,T], \\ y|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \end{cases} \tag{1}$$

with the initial condition

$$y(x,0) = f(x), \quad x \in \Omega. \tag{2}$$

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The function G is called source function, the function f is the initial condition. They will be defined later in spaces in later theorems. Equation (1) as one important class of diffusion equations, which can describe physical properties problems in many areas such as population dynamics and heat conduction in memory materials (see [1–10, 15–17]) and some research directions related to this topic [19–21, 23, 25–28].

In [13], S. Guerrero and O.Y. Imanuvilov investigated the control system for the heat equation with memory

$$\begin{cases} y_t - \Delta y = k\displaystyle\int_0^t \Delta u(x,r)dr, \ (x,t) \in \Omega \times (0,T), \\ y = 0, \ (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = f(x), \ x \in \Omega. \end{cases} \tag{3}$$

Problem (3) appears as a natural model in a number of tools developed to approximate Navier-Stokes system see [22]. The authors in [13] showed that the null controllability of system (3) is not true for all initial conditions f. A. Halanay and L. Pandolfi [14] considered the heat equation with memory in a bounded region as follows

$$y_t = ay + \Delta y + \int_0^t k(t-r)\Delta y(r)dr + G(x,t), \quad y(x,0) = \xi(x), \tag{4}$$

where a a is a real constant and k is of class C^1. They obtained approximate controllability of the system. In [29], H. Zhao, J. Zhang studied the following parabolic integro-differential equation:

$$\begin{cases} y_t - \Delta y = \displaystyle\int_0^t k(x,r)u(x,r)dr + \chi_\omega u, \ (x,t) \in \Omega \times (0,T), \\ y = 0, \ (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = f(x), \ x \in \Omega \end{cases} \tag{5}$$

where χ_ω is called characteristic function on the nonempty set $\omega \subset \Omega$. The function y is the state, the function u is the control and $k(x,t)$ is a memory kernel. The approximate controllability of a parabolic integro-differential equation system has already been shown for the case where the memory kernel is a constant. In the interesting paper of Zhou [30], the author studied the following integro-differential equation of parabolic type

$$\begin{cases} y_t - \Delta y + ay = \displaystyle\int_0^t M(t-s)\Delta y(x,r)dr + \chi_\omega u, \ (x,t) \in \Omega \times (0,T), \\ y = 0, \ (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = f(x), \ x \in \Omega \end{cases} \tag{6}$$

where $a \in \mathbb{R}$ is a real constant, $y = y(x,t)$ is the state variable, $u = u(x,t)$ is the control variable, and M is the integral kernel. The main goal of this paper is to show the existence of a control function which control state variable and the integral term of the neighborhood of two given final configurations.

In [18], the authors studied observability/controllability properties for a viscoelastic string, described by the equation

$$y_{tt} - \Delta y = \int_0^t k(t-r)\Delta u(x,r)dr, \ (x,t) \in \Omega \times (0,T), \tag{7}$$

with $\Omega = (0,\pi)$ and the following initial condition

$$y(x,0) = \xi_0(x), \quad y_t(x,0) = \xi_1(x) \tag{8}$$

and

$$y(0,t) = g(t), \quad y(\pi,t) = 0. \tag{9}$$

In [11], Grimmer considered the following integrodifferential equation

$$y'(t) = Ay(t) + \int_0^t B(t-s)y(s)ds + h(t), \quad y(0) = y_0 \in X, t \geq 0 \tag{10}$$

where A generates a C_0-semigroup on a Banach space X, the function $h : \mathbb{R}^+ \to X$ is a continuous function. Using the resolvent operators theory, the author obtained a variation of parameters formula which enabled him to obtain some results concerning the existence, the regularity and the asymptotic behavior of solutions to (10).

The main contributions of the paper are as follows

- The first result concerns the well-posedness of the problem when the input data belongs to Gevrey space. Furthermore, we prove the continuity of the solution with respect to the parameter k. When k is an irrational number, we cannot know its exact value but can only know its approximate value. More specifically, an irrational number can only be approximated by a sequence of rational numbers. Therefore, the question here is: When k is close to k′, is u_k close to $u_{k'}$ or not?
- The second results show the ill-posedness of the problem in the sense of Hadamard. We regularize problem by two method: quasi-boundary method and truncation method. The new point of the truncation method is the approxiamation of the solution when the observed data belongs to the L^p space. To demonstrate the main results, we had to go through a number of rather complex and sophisticated assessments. The appearance of the memory component $k\int_0^t(-\Delta)^\theta y(x,s)ds$ has also made the problem much more complicated.

This paper is organized as follows. In section 2, we introduce some basic knowledge about the function spaces needed to use in the paper. In section 3, we give the mild solution formula of Problem (1)-(2) in the form of a Fourier series. Section 4 introduce the well-posed of the problem when the Cauchy data in the Gevrey space. Finally, in section 5, we provide two regularize method to approximate our problem.

2. Preliminary results

This section provide some notation and the functional spaces which will be used throughout this article. Recall that the spectral problem

$$\begin{cases} \Delta e_n(x) = -\lambda_n e_n(x), & x \in \Omega, \\ e_n(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \dots$ with $\lambda_n \to \infty$ as $n \to \infty$ and corresponding eigenfunctions $e_n \in H_0^1(\Omega)$.

Let us define fractional powers of $(-\Delta)^\theta$ for $0 < \theta < 2$ as follows (see [24])

$$(-\Delta)^\theta f = \sum_{n=1}^{\infty} \lambda_n^\theta \Big(\int_\Omega f(x)e_n(x)\mathrm{d}x \Big) e_n(x) \tag{11}$$

and its domain is as follows

$$\mathcal{D}\Big((-\Delta)^\theta\Big) = \Big\{ f \in L^2(\Omega), \quad \sum_{n=1}^{\infty} \lambda_n^{2\theta} \Big(\int_\Omega f(x)e_n(x)\mathrm{d}x \Big)^2 < \infty \Big\}. \tag{12}$$

Definition 2.1. *(Hilbert scale space). We recall the Hilbert scale space* $\mathbb{H}^s(\Omega)$ *given as follows*

$$\mathbb{H}^s(\Omega) = \left\{ f \in L^2(\Omega) \,\Big|\, \sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_\Omega f(x) e_n(x) \mathrm{d}x \Big)^2 < \infty \right\},$$

for any $s \geq 0$. *It is well-known that* $\mathbb{H}^s(\Omega)$ *is a Hilbert space corresponding to the norm*

$$\|f\|_{\mathbb{H}^s(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_\Omega f(x) e_n(x) \mathrm{d}x \Big)^2 \right)^{1/2}, \ f \in \mathbb{H}^s(\Omega).$$

Definition 2.2. *The Gevrey space is defined by*

$$\mathbb{G}^b_{\mu,M}(\Omega) := \left\{ f \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2b} \exp\left(2M\lambda_n^{\mu}\right) f_n^2 < \infty \right\}, \tag{13}$$

where the norm of it is given as follows

$$\|f\|_{\mathbb{G}^b_{\mu,M}(\Omega)} = \sqrt{\sum_{n=1}^{\infty} \lambda_n^{2b} \exp\left(2M\lambda_n^{\mu}\right) f_n^2}. \tag{14}$$

Definition 2.3. *Let* $C([0,T];\mathcal{B})$ *be the set of all continuous functions which map* $[0,T]$ *into* $\mathcal{B}$ *where* $\mathcal{B}$ *is a Banach space. It is a Banach space endowed with the usual supremum norm. For any* $\theta > 0$, *we introduce the following Hölder continuous space of exponent* θ

$$C^\theta([0,T];\mathcal{B}) = \left\{ v \in C([0,T];\mathcal{B}) : \sup_{0 \leq s < t \leq T} \frac{\|v(.,t) - v(.,s)\|_{\mathcal{B}}}{|t-s|^\theta} < \infty \right\}.$$

which is equipped with the norm

$$\|v\|_{C^\theta([0,T];\mathcal{B})} = \sup_{0 \leq s < t \leq T} \frac{\|v(.,t) - v(.,s)\|_{\mathcal{B}}}{|t-s|^\theta}.$$

3. The mild solution

3.1. The mild solution for parabolic problem with memory

Our aim in this section is to give the form of the solution $u(x,t)$ to problem (1)-(2). Let us assume that $y(x,t) = \sum_{n=1}^{\infty} y_n(t) e_n(x)$, where

$$y_n(t) = \int_\Omega y(x,t) e_n(x) dx.$$

Since the main equation of (1) and noting that

$$-\int_\Omega \Delta y(x,t) e_n(x) dx = \lambda_n y_n(t), \quad \int_\Omega (-\Delta)^\theta y(x,t) e_n(x) dx = -\lambda_n^\theta y_n(t),$$

we get that

$$\frac{d}{dt} y_n(t) + \lambda_n y_n(t) - k\lambda_n^\theta \int_0^t y_n(\tau) d\tau = G_n(t), \tag{15}$$

where $G_n(t)$ is Fourier series of G. Let $v_n(t) = \int_0^t y_n(s)ds$. Hence, we find that

$$\frac{d}{dt}v_n(t) = y_n(t). \tag{16}$$

and

$$\frac{d}{dt}v_n(0) = y_n(0) = f_n, \quad v_n(0) = 0.$$

This implies that

$$v_n''(t) + \lambda_n v_n'(t) - k\lambda_n^{\theta} v_n(t) = G_n(t). \tag{17}$$

Set

$$\mathbb{C}_n = \frac{-\lambda_n + \sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}}{2}, \quad \mathbb{D}_n = \frac{-\lambda_n - \sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}}{2}.$$

The solution v_n is given by a general form as follows

$$v_n(t) = a_n e^{\mathbb{C}_n t} + b_n e^{\mathbb{D}_n t} + \alpha_n(t)e^{\mathbb{C}_n t} + \beta_n(t)e^{\mathbb{D}_n t}. \tag{18}$$

Here a_n, b_n are two constants. The function α_n and θ_n are as follows

$$\alpha_n'(t) = -\frac{G_n(t)e^{\mathbb{D}_n t}}{\left(\mathbb{D}_n - \mathbb{C}_n\right)e^{(\mathbb{C}_n+\mathbb{D}_n)t}} = \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}}e^{-\mathbb{C}_n t}G_n(t). \tag{19}$$

and

$$\beta_n'(t) = \frac{F_n(t)e^{\mathbb{C}_n t}}{\left(\mathbb{D}_n - \mathbb{C}_n\right)e^{(\mathbb{C}_n+\mathbb{D}_n)t}} = \frac{-1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}}e^{-\mathbb{D}_n t}G_n(t). \tag{20}$$

Then we get

$$\alpha_n(t) = \int_0^t \alpha_n'(\tau)d\tau + \alpha_n(0) = \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}}\int_0^t e^{-\mathbb{C}_n \tau}G_n(\tau)d\tau + \alpha_n(0), \tag{21}$$

and

$$\beta_n(t) = \int_0^t \beta_n'(\tau)d\tau + \beta_n(0) = \frac{-1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}}\int_0^t e^{-\mathbb{D}_n \tau}G_n(\tau)d\tau + \beta_n(0). \tag{22}$$

Combining (18), (21) and (22), we derive that

$$\begin{aligned} v_n(t) &= a_n e^{\mathbb{C}_n t} + b_n e^{\mathbb{D}_n t} \\ &+ \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}}\int_0^t \left(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\right)G_n(\tau)d\tau + \alpha_n(0)e^{\mathbb{C}_n t} + \beta_n(0)e^{\mathbb{D}_n t}. \end{aligned} \tag{23}$$

Since (18), we find that

$$v_n(0) = a_n + b_n + \alpha_n(0) + \beta_n(0) = 0. \tag{24}$$

Taking the first derivative of (18), one has

$$\begin{aligned} v_n'(t) &= \mathbb{C}_n a_n e^{\mathbb{C}_n t} + \mathbb{D}_n b_n e^{\mathbb{D}_n t} + \alpha_n(t)\mathbb{C}_n e^{\mathbb{C}_n t} \\ &+ \alpha_n'(t)e^{\mathbb{C}_n t} + \beta_n(t)\mathbb{D}_n e^{\mathbb{D}_n t} + \beta_n'(t)e^{\mathbb{D}_n t}. \end{aligned} \tag{25}$$

By taking $t = 0$ into the above equality, we derive that

$$\Big(a_n + \alpha_n(0)\Big)C_n + \Big(b_n + \beta_n(0)\Big)D_n = f_n. \tag{26}$$

Solving systems (24) and (26), we find that

$$a_n + \alpha_n(0) = \frac{f_n}{\mathbb{C}_n - \mathbb{D}_n} = \frac{f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \tag{27}$$

and

$$b_n + \theta_n(0) = \frac{-f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}. \tag{28}$$

Combining (23), (27) and (28), we derive that

$$\begin{aligned} v_n(t) &= \Big(a_n + \alpha_n(0)\Big)e^{\mathbb{C}_n t} + \Big(b_n + \beta_n(0)\Big)e^{\mathbb{D}_n t} \\ &\quad + \frac{1}{\lambda_n^2 + 4k\lambda_n^\theta}\int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\Big)G_n(\tau)d\tau. \end{aligned} \tag{29}$$

Hence, we have immediately that

$$v_n(t) = \frac{f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\Big(e^{\mathbb{C}_n t} - e^{\mathbb{D}_n t}\Big) + \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\Big)G_n(\tau)d\tau. \tag{30}$$

By taking the derivative of v_n, we get

$$\begin{aligned} y_n(t) = v_n'(t) &= \frac{f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big) \\ &\quad + \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\int_0^t \Big(\mathbb{C}_n e^{\mathbb{C}_n(t-\tau)} - \mathbb{D}_n e^{\mathbb{D}_n(t-\tau)}\Big)G_n(\tau)d\tau \end{aligned} \tag{31}$$

where we have used the fact that

$$\frac{d}{dt}\int_0^t G(t,s)ds = G(t,t) + \int_0^t G_t(t,s)ds.$$

Thus, using the definition of Fourier series, we find that

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} \frac{f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)e_n(x) \\ &\quad + \sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\Big)G_n(\tau)d\tau\right]e_n(x). \end{aligned} \tag{32}$$

Lemma 3.1. *We have the following inequality*

$$\frac{2k}{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}\lambda_n^{\theta-1} \le \mathbb{C}_n \le 2k\lambda_n^{\theta-1} \tag{33}$$

and

$$\lambda_n \le -\mathbb{D}_n \le \Big(1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}\Big)\frac{\lambda_n}{2}. \tag{34}$$

Proof. It is obvious to see that

$$\mathbb{C}_n = \frac{-\lambda_n + \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}{2} = \frac{2k\lambda_n^\theta}{\lambda_n + \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \le 2k\lambda_n^{\theta-1}. \tag{35}$$

In addition, we give another observation

$$\mathbb{C}_n = \frac{2k\lambda_n^{\theta-1}}{1 + \sqrt{1 + 4k\lambda_n^{\theta-2}}} \ge \frac{2k\lambda_n^{\theta-1}}{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}} \tag{36}$$

where we note that $\lambda_n^{\theta-2} \le \lambda_1^{\theta-2}$ since $\theta < 2$. In addition, we know that

$$-\mathbb{D}_n = \frac{\lambda_n + \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}{2} = \frac{\lambda_n}{2}\Big(1 + \sqrt{1 + 4k\lambda_n^{\theta-2}}\Big) \le \Big(1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}\Big)\frac{\lambda_n}{2}. \tag{37}$$

and using the inequality $\sqrt{1 + 4k\lambda_n^{\theta-2}} \ge 1$, we get that

$$-\mathbb{D}_n = \frac{\lambda_n + \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}{2} = \frac{\lambda_n}{2}\Big(1 + \sqrt{1 + 4k\lambda_n^{\theta-2}}\Big) \ge \lambda_n. \tag{38}$$

□

4. Well-posedness of the problem

The main goal of this section is to investigate the well-posedness of the problem (1)-(2).

Theorem 4.1. *Let $f \in \mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)$ and $G \in L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))$. Then for any $0 \le t \le T$, we get*

$$\begin{aligned}\Big\|y(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le\ & \sqrt{8}k\Big\|f\Big\|_{\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)} + \frac{\Big(1 + 4k\lambda_1^{\theta-2}\Big)}{2}\Big\|f\Big\|_{\mathbb{H}^s(\Omega)}\\ &+ \sqrt{8}k\Big\|G\Big\|_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))} + \frac{\Big(1 + 4k\lambda_1^{\theta-2}\Big)}{2}\Big\|G\Big\|_{L^2(0,T;\mathbb{H}^s(\Omega))}.\end{aligned} \tag{39}$$

Proof. From (32), we get that

$$y(x,t) = y^{(I)}(x,t) + y^{(II)}(x,t) \tag{40}$$

where

$$y^{(I)}(x,t) = \sum_{n=1}^{\infty} \frac{f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)e_n(x). \tag{41}$$

and

$$y^{(II)}(x,t) = \sum_{n=1}^{\infty}\Bigg[\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\Big)G_n(\tau)d\tau\Bigg]e_n(x). \tag{42}$$

Using Parseval's equality $\sum_n |\langle x, e_n\rangle|^2 = \|x\|^2$, we infer that

$$\Big\|y^{(I)}(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{\infty} \lambda_n^{2s}\frac{|f_n|^2}{\lambda_n^2 + 4k\lambda_n^\theta}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)^2. \tag{43}$$

In view of the inequality $(a+b)^2 \le 2a^2 + 2b^2$ and using Lemma 3.1, one has

$$\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)^2 \le 2\mathbb{C}_n^2 e^{2\mathbb{C}_n t} + 2\mathbb{D}_n^2 e^{2\mathbb{D}_n t} \le 8k^2 \lambda_n^{2\theta-2} e^{4k\lambda_n^{\theta-1} t} + \frac{\Big(1+4k\lambda_1^{\theta-2}\Big)^2}{4} \lambda_n^2, \tag{44}$$

where we note that $\mathbb{D}_n \le 0$. This implies that

$$\frac{\lambda_n^{2s}}{\lambda_n^2 + 4k\lambda_n^\theta}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)^2 \le 8k^2 \lambda_n^{2s+2\theta-4} e^{4kT\lambda_n^{\theta-1}} + \frac{\Big(1+4k\lambda_1^{\theta-2}\Big)^2}{4} \lambda_n^{2s}. \tag{45}$$

Combining (43) and (45), we obtain

$$\Big\| y^{(I)}(.,t) \Big\|_{\mathbb{H}^s(\Omega)}^2 \le 8k^2 \Big\| f \Big\|_{\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)}^2 + \frac{\Big(1+4k\lambda_1^{\theta-2}\Big)^2}{4} \Big\| f \Big\|_{\mathbb{H}^s(\Omega)}^2. \tag{46}$$

Thus, one gets

$$\Big\| y^{(I)}(.,t) \Big\|_{\mathbb{H}^s(\Omega)} \le \sqrt{8}k \Big\| f \Big\|_{\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)} + \frac{\Big(1+4k\lambda_1^{\theta-2}\Big)}{2} \Big\| f \Big\|_{\mathbb{H}^s(\Omega)}. \tag{47}$$

Using the inequality $e^a \ge C_\mu a^\mu$, $\mu > 0$, $a > 0$, we have immediately

$$\Big\| f \Big\|_{\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)}^2 = \sum_{n=1}^\infty \lambda_n^{2s+2\theta-4} e^{4kT\lambda_n^{\theta-1}} f_n^2 \ge \sum_{n=1}^\infty C(\theta)(4kT)^{\frac{4-2\theta}{\theta-1}} \lambda_n^{4-2\theta} \lambda_n^{2s+2\theta-4} = C(\theta)(4kT)^{\frac{4-2\theta}{\theta-1}} \sum_{n=1}^\infty \lambda_n^{2s} f_n^2. \tag{48}$$

This implies that

$$\Big\| f \Big\|_{\mathbb{H}^s(\Omega)} \le C(\theta)(4kT)^{\frac{\theta-2}{\theta-1}} \Big\| f \Big\|_{\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)}. \tag{49}$$

Hence, we find that $f \in \mathbb{H}^s(\Omega)$ if $f \in \mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)$. Let us now to treat the function $y^{(II)}$. Indeed, using (45) and Hölder inequality, we derive that

$$\begin{aligned} \Big\| y^{(II)}(.,t) \Big\|_{\mathbb{H}^s(\Omega)}^2 &= \sum_{n=1}^\infty \lambda_n^{2s} \Bigg[\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)} \Big) G_n(\tau) d\tau \Bigg]^2 \\ &\le \sum_{n=1}^\infty \frac{\lambda_n^{2s}}{\lambda_n^2 + 4k\lambda_n^\theta} \Big(\int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)} \Big)^2 d\tau \Big) \Big(\int_0^t G_n^2(\tau) d\tau \Big) \\ &\le 8k^2 \Bigg(\int_0^t \sum_{n=1}^\infty \lambda_n^{2s+2\theta-4} e^{4kT\lambda_n^{\theta-1}} G_n^2(\tau) d\tau \Bigg) + \frac{\Big(1+4k\lambda_1^{\theta-2}\Big)^2}{4} \Bigg(\int_0^t \sum_{n=1}^\infty \lambda_n^{2s} G_n^2(\tau) d\tau \Bigg). \end{aligned} \tag{50}$$

In view of Parseval's equality, we deduce that

$$\Big\| y^{(II)}(.,t) \Big\|_{\mathbb{H}^s(\Omega)}^2 \le 8k^2 \Big\| G \Big\|_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))}^2 + \frac{\Big(1+4k\lambda_1^{\theta-2}\Big)^2}{4} \Big\| G \Big\|_{L^2(0,T;\mathbb{H}^s(\Omega))}^2.$$

Hence, one gets

$$\left\|y^{(II)}(.,t)\right\|_{\mathbb{H}^s(\Omega)} \le \sqrt{8}k\left\|G\right\|_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))} + \frac{\left(1+4k\lambda_1^{\theta-2}\right)}{2}\left\|G\right\|_{L^2(0,T;\mathbb{H}^s(\Omega))}. \tag{51}$$

By collecting (40), (47) and (51), we completes the proof of Theorem. □

Theorem 4.2. *Let $f \in \mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega)$ and $G \in L^2(0,T;\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega))$ for $\varepsilon > 0$. Then we deduce that $y \in C^{\min(\varepsilon,\frac{1}{2})}([0,T];\mathbb{H}^s(\Omega))$ and*

$$\left\|y\right\|_{C^{\min(\varepsilon,\frac{1}{2})}([0,T];\mathbb{H}^s(\Omega))} \le \overline{M}_2\left(\left\|f\right\|_{\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega)} + \left\|G\right\|_{L^2(0,T;\mathbb{G}^{s-1}_{\theta-1,4kT}(\Omega))} + \left\|G\right\|_{L^2(0,T;\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega))}\right). \tag{52}$$

Here $\overline{M}_2$ is a postive constant which depends on $k, \varepsilon, \lambda_1, \theta, T$.

Proof. From (32), we rewrite the solution y as follows

$$y(x,t) = y^{(I)}(x,t) + y^{(II)}(x,t) \tag{53}$$

where $y^{(I)}$ and $y^{(II)}$ are defined in (41) and (42) respectively. Let $0 \le t \le t+h \le T$. It is obvious to see that

$$\begin{aligned} &y^{(I)}(x,t+h) - y^{(I)}(x,t) \\ &\quad = \sum_{n=1}^{\infty} \frac{f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\left(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{C}_n e^{\mathbb{C}_n(t+h)} - \mathbb{D}_n e^{\mathbb{D}_n t} + \mathbb{D}_n e^{\mathbb{D}_n(t+h)}\right)e_n(x). \end{aligned} \tag{54}$$

Thus, using Parseval's equality, we have immediately that

$$\begin{aligned} &\left\|y^{(I)}(.,t+h) - y^{(I)}(.,t)\right\|^2_{\mathbb{H}^s(\Omega)} \\ &\quad = \sum_{n=1}^{\infty} \frac{\lambda_n^{2s}}{\lambda_n^2 + 4k\lambda_n^\theta} f_n^2\left(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{C}_n e^{\mathbb{C}_n(t+h)} - \mathbb{D}_n e^{\mathbb{D}_n t} + \mathbb{D}_n e^{\mathbb{D}_n(t+h)}\right)^2. \end{aligned} \tag{55}$$

Looking at the inequality $1 - e^{-\rho} \le C_\varepsilon \rho^\varepsilon$ for $\rho \ge 0, \varepsilon > 0$, we know that

$$\left|\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{C}_n e^{\mathbb{C}_n(t+h)}\right| = \mathbb{C}_n e^{\mathbb{C}_n(t+h)}\left(1 - e^{-\mathbb{C}_n h}\right) \le C_\varepsilon e^{\mathbb{C}_n T}|\mathbb{C}_n|^{1+\varepsilon}h^\varepsilon, \tag{56}$$

and

$$\left|-\mathbb{D}_n e^{\mathbb{D}_n t} + \mathbb{D}_n e^{\mathbb{D}_n(t+h)}\right| = -\mathbb{D}_n e^{\mathbb{D}_n t}\left(1 - e^{\mathbb{D}_n h}\right) \le C_\varepsilon|-\mathbb{D}_n|^{1+\varepsilon}h^\varepsilon. \tag{57}$$

Using Lemma 3.1, we obtain

$$\begin{aligned} &\left|\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{C}_n e^{\mathbb{C}_n(t+h)} - \mathbb{D}_n e^{\mathbb{D}_n t} + \mathbb{D}_n e^{\mathbb{D}_n(t+h)}\right| \le C_\varepsilon e^{\mathbb{C}_n T}|\mathbb{C}_n|^{1+\varepsilon}h^\varepsilon + C_\varepsilon|-\mathbb{D}_n|^{1+\varepsilon}h^\varepsilon \\ &\qquad \le C_\varepsilon e^{2k\lambda_n^{\theta-1}T}\left[\left(2k\lambda_n^{\theta-1}\right)^{1+\varepsilon} + \left(\frac{1+\sqrt{1+4k\lambda_1^{\theta-2}}}{2}\right)^{1+\varepsilon}\lambda_n^{1+\varepsilon}\right]h^\varepsilon. \end{aligned} \tag{58}$$

Since $1 \le \theta < 2$, we note that

$$\lambda_n^{(\theta-1)(1+\varepsilon)} = \lambda_n^{1+\varepsilon}\left(1 + \lambda_n^{-(2-\theta)(1+\varepsilon)}\right) \le \left(1 + \lambda_1^{-(2-\theta)(1+\varepsilon)}\right)\lambda_n^{1+\varepsilon}.$$

Thus, there exists a positive constant $\overline{M}_1$ which depends on $k, \varepsilon, \lambda_1, \theta$ such that

$$\left|\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{C}_n e^{\mathbb{C}_n (t+h)} - \mathbb{D}_n e^{\mathbb{D}_n t} + \mathbb{D}_n e^{\mathbb{D}_n (t+h)}\right| \leq \overline{M}_1 e^{2k\lambda_n^{\theta-1}T} \lambda_n^{1+\varepsilon} h^{\varepsilon}. \tag{59}$$

Combining (55) and (59), we infer that

$$\left\| y^{(I)}(.,t+h) - y^{(I)}(.,t)\right\|^2_{\mathbb{H}^s(\Omega)} \leq |\overline{M}_1|^2 h^{2\varepsilon} \sum_{n=1}^{\infty} e^{4kT\lambda_n^{\theta-1}} \lambda_n^{2s+2\varepsilon} f_n^2. \tag{60}$$

Hence, we have immediately that

$$\left\| y^{(I)}(.,t+h) - y^{(I)}(.,t)\right\|_{\mathbb{H}^s(\Omega)} \leq \overline{M}_1 h^{\varepsilon} \left\| f \right\|_{\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega)}. \tag{61}$$

From (42), we see that

$$\begin{aligned}
y^{(II)}(x,t+h) - y^{(II)}(x,t) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}} \Big[\int_0^{t+h} \Big(e^{\mathbb{C}_n(t+h-\tau)} - e^{\mathbb{D}_n(t+h-\tau)} \Big) G_n(\tau) d\tau \Big] e_n(x) \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}} \Big[\int_0^{t} \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)} \Big) G_n(\tau) d\tau \Big] e_n(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}} \Big[\int_t^{t+h} \Big(e^{\mathbb{C}_n(t+h-\tau)} - e^{\mathbb{D}_n(t+h-\tau)} \Big) G_n(\tau) d\tau \Big] e_n(x) \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}} \Big[\int_0^{t} \Big(e^{\mathbb{C}_n(t+h-\tau)} - e^{\mathbb{C}_n(t-\tau)} + e^{\mathbb{D}_n(t-\tau)} - e^{\mathbb{D}_n(t+h-\tau)} \Big) G_n(\tau) d\tau \Big] e_n(x) \\
&= \mathscr{L}_1(x,t) + \mathscr{L}_2(x,t).
\end{aligned} \tag{62}$$

Applying Hölder inequality, the norm of $\mathscr{L}_1$ is bounded by

$$\begin{aligned}
\left\| \mathscr{L}_1(.,t) \right\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n=1}^{\infty} \frac{\lambda_n^{2s}}{\lambda_n^2 + 4k\lambda_n^{\theta}} \Big[\int_t^{t+h} \Big(e^{\mathbb{C}_n(t+h-\tau)} - e^{\mathbb{D}_n(t+h-\tau)} \Big) G_n(\tau) d\tau \Big]^2 \\
&\leq \sum_{n=1}^{\infty} \frac{\lambda_n^{2s}}{\lambda_n^2 + 4k\lambda_n^{\theta}} \Big[\int_t^{t+h} d\tau \Big] \Big[\int_t^{t+h} \Big(e^{\mathbb{C}_n(t+h-\tau)} - e^{\mathbb{D}_n(t+h-\tau)} \Big)^2 G_n^2(\tau) d\tau \Big].
\end{aligned} \tag{63}$$

It is obvious to see that

$$\Big(e^{\mathbb{C}_n(t+h-\tau)} - e^{\mathbb{D}_n(t+h-\tau)} \Big)^2 \leq e^{2\mathbb{C}_n T} \leq e^{4k\lambda_n^{\theta-1}T}.$$

This implies that

$$\left\| \mathscr{L}_1(.,t) \right\|^2_{\mathbb{H}^s(\Omega)} \leq h \int_0^T \Big(\sum_{n=1}^{\infty} \lambda_n^{2s-2} e^{4k\lambda_n^{\theta-1}T} G_n^2(\tau) d\tau \Big). \tag{64}$$

In view of Parseval's equality, we have the following bound for the first term $\mathscr{L}_1$

$$\left\| \mathscr{L}_1(.,t) \right\|_{\mathbb{H}^s(\Omega)} \leq \sqrt{h} \left\| G \right\|_{L^2(0,T;\mathbb{G}^{s-1}_{\theta-1,4kT}(\Omega))}. \tag{65}$$

Let us now estimate the second term $\mathscr{L}_2$. Using (59) and Hölder inequality, one has

$$\begin{aligned}\left\|\mathscr{L}_2(.,t)\right\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n=1}^{\infty} \frac{\lambda_n^{2s}}{\lambda_n^2+4k\lambda_n^{\theta}}\left[\int_0^t \left(e^{\mathbb{C}_n(t+h-\tau)}-e^{\mathbb{C}_n(t-\tau)}+e^{\mathbb{D}_n(t-\tau)}-e^{\mathbb{D}_n(t+h-\tau)}\right)G_n(\tau)d\tau\right]^2\\ &\le \sum_{n=1}^{\infty} T\frac{\lambda_n^{2s}}{\lambda_n^2+4k\lambda_n^{\theta}}\left[\int_0^t \left(e^{\mathbb{C}_n(t+h-\tau)}-e^{\mathbb{C}_n(t-\tau)}+e^{\mathbb{D}_n(t-\tau)}-e^{\mathbb{D}_n(t+h-\tau)}\right)^2|G_n(\tau)|^2d\tau\right]\\ &\le \sum_{n=1}^{\infty} T\frac{\lambda_n^{2s}}{\lambda_n^2+4k\lambda_n^{\theta}}\left[\int_0^t \left(\overline{M}_1 e^{2k\lambda_n^{\theta-1}T}\lambda_n^{1+\varepsilon}h^{\varepsilon}\right)^2|G_n(\tau)|^2d\tau\right]\\ &\le T|\overline{M}_1|^2h^{2\varepsilon}\int_0^T\Big(\sum_{n=1}^{\infty} e^{4k\lambda_n^{\theta-1}T}\lambda_n^{2s+2\varepsilon}G_n^2(\tau)\Big)d\tau. \end{aligned} \tag{66}$$

This implies that

$$\left\|\mathscr{L}_2(.,t)\right\|_{\mathbb{H}^s(\Omega)} \le \overline{M}_1\sqrt{T}h^{\varepsilon}\left\|G\right\|_{L^2(0,T;\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega))}. \tag{67}$$

Combining (42), (65), and (67), we obtain

$$\begin{aligned}\left\|y^{(II)}(.,t+h)-y^{(II)}(.,t)\right\|_{\mathbb{H}^s(\Omega)} \le\ & \sqrt{h}\left\|G\right\|_{L^2(0,T;\mathbb{G}^{s-1}_{\theta-1,4kT}(\Omega))}\\ &+\overline{M}_1\sqrt{T}h^{\varepsilon}\left\|G\right\|_{L^2(0,T;\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega))}.\end{aligned} \tag{68}$$

This estimate together with (53) and (61) yields to

$$\begin{aligned}\left\|y(.,t+h)-y(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\le \left\|y^{(I)}(.,t+h)-y^{(I)}(.,t)\right\|_{\mathbb{H}^s(\Omega)}+\left\|y^{(II)}(.,t+h)-y^{(II)}(.,t)\right\|_{\mathbb{H}^s(\Omega)}\\ &\le \overline{M}_1h^{\varepsilon}\left\|f\right\|_{\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega)}+\sqrt{h}\left\|G\right\|_{L^2(0,T;\mathbb{G}^{s-1}_{\theta-1,4kT}(\Omega))}\\ &\quad+\overline{M}_1\sqrt{T}h^{\varepsilon}\left\|G\right\|_{L^2(0,T;\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega))}.\end{aligned} \tag{69}$$

It is easy to observe that

$$h^{\varepsilon-\min(\varepsilon,\frac12)} \le \max\left(1, T^{\varepsilon-\min(\varepsilon,\frac12)}\right) \tag{70}$$

and

$$h^{\frac12-\min(\varepsilon,\frac12)} \le \max\left(1, T^{\frac12-\min(\varepsilon,\frac12)}\right). \tag{71}$$

Thus, combining all previous observations, we get that

$$\begin{aligned}\frac{\left\|y(.,t+h)-y(.,t)\right\|_{\mathbb{H}^s(\Omega)}}{h^{\min(\varepsilon,\frac12)}} \le\ & \max\left(1, T^{\varepsilon-\min(\varepsilon,\frac12)}, T^{\frac12-\min(\varepsilon,\frac12)}\right)\Bigg[\overline{M}_1\left\|f\right\|_{\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega)}\\ &+\left\|G\right\|_{L^2(0,T;\mathbb{G}^{s-1}_{\theta-1,4kT}(\Omega))}+\overline{M}_1\sqrt{T}\left\|G\right\|_{L^2(0,T;\mathbb{G}^{s+\varepsilon}_{\theta-1,4kT}(\Omega))}\Bigg].\end{aligned} \tag{72}$$

From the definition (2.3), we know that $y \in C^{\min(\varepsilon,\frac12)}([0,T];\mathbb{H}^s(\Omega))$, and we also obtain the desired result (52). The proof is completed. □

The following theorem gives us the continuous result of the solution according to the parameter k.

Theorem 4.3. *Let* $f \in \mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega) \cap \mathbb{G}^{s+2\theta-3}_{\theta-1,4kT}(\Omega)$ *and* $G \in L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)) \cap L^2(0,T;\mathbb{H}^s(\Omega))$. *Let* $0 < k, k'$. *Let* u^k *and* $u^{k'}$ *be the solution to Problem* (1)-(2). *Then we obtain*

$$\begin{aligned}\Big\|u^k(.,t) - u^{k'}(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\le \sqrt{2}\Big(\sqrt{\frac{k}{k'}} + 2\Big)|k-k'|\Big\|f\Big\|_{\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)}\\ &+ \sqrt{32}k|k-k'|T\Big\|f\Big\|_{\mathbb{G}^{s+2\theta-3}_{\theta-1,4kT}(\Omega)} + \overline{M}_1|k-k'|\Big\|f\Big\|_{\mathbb{H}^s(\Omega)}\\ &+ \Big(\frac{|k-k'|}{2\sqrt{kk'}}\Big)\sqrt{8}k\Big\|G\Big\|_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))} + \Big(\frac{|k-k'|}{2\sqrt{kk'}}\Big)\frac{\big(1+4k\lambda_1^{\theta-2}\big)}{2}\Big\|G\Big\|_{L^2(0,T;\mathbb{H}^s(\Omega))}\\ &+ \sqrt{8t^3}|k-k'|\Bigg(\Big\|G\Big\|_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))} + \Big\|G\Big\|_{L^2(0,T;\mathbb{H}^{s+\theta-2}(\Omega))}\Bigg). \end{aligned} \tag{73}$$

Proof. Let us re-notify the expressions as follows

$$\mathbb{C}_{n,k} = \frac{-\lambda_n + \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}{2}, \quad \mathbb{D}_{n,k} = \frac{-\lambda_n - \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}{2}. \tag{74}$$

It is obvious to see that

$$\begin{aligned}\mathbb{C}_{n,k} - \mathbb{C}_{n,k'} &= \frac{-\lambda_n + \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}{2} - \frac{-\lambda_n + \sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}}{2}\\ &= \frac{4(k-k')\lambda_n^\theta}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}}.\end{aligned} \tag{75}$$

Hence, we find that

$$\Big|\mathbb{C}_{n,k} - \mathbb{C}_{n,k'}\Big| \le 2|k-k'|\lambda_n^{\theta-1}. \tag{76}$$

By a similar techniques, we obtain that

$$\Big|\mathbb{D}_{n,k} - \mathbb{D}_{n,k'}\Big| \le 2|k-k'|\lambda_n^{\theta-1}. \tag{77}$$

Let us continue to have

$$\begin{aligned}\Bigg|\frac{\mathbb{C}_{n,k}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} - \frac{\mathbb{C}_{n,k'}}{\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}}\Bigg| &\le \mathbb{C}_{n,k}\frac{\Big|\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta} - \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}\Big|}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}}\\ &+ \frac{\Big|\mathbb{C}_{n,k} - \mathbb{C}_{n,k'}\Big|}{\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}} = \mathbb{E}_1 + \mathbb{E}_2.\end{aligned} \tag{78}$$

In view of Lemma 3.1, one has

$$\begin{aligned}\mathbb{E}_1 &\le 2k\lambda_n^{\theta-1}\frac{4|k-k'|\lambda_n^\theta}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}\Big(\Big|\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta} + \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}\Big|\Big)}\\ &\le 8k|k-k'|\frac{\lambda_n^{2\theta-1}}{8\sqrt{kk'}\lambda_n^\theta\lambda_n} = \sqrt{\frac{k}{k'}}|k-k'|\lambda_n^{\theta-2}.\end{aligned} \tag{79}$$

In addition, using (76), we find that

$$\mathbb{E}_2 \le 2|k-k'|\lambda_n^{\theta-2}. \tag{80}$$

Combining (78), (79) and (80), we derive that

$$\left|\frac{\mathbb{C}_{n,k}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}} - \frac{\mathbb{C}_{n,k'}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\right| \le \Big(\sqrt{\frac{k}{k'}}+2\Big)|k-k'|\lambda_n^{\theta-2}. \tag{81}$$

In a similar way, we also have

$$\begin{aligned}\left|\frac{-\mathbb{D}_{n,k}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}} - \frac{-\mathbb{D}_{n,k'}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\right| &\le |-\mathbb{D}_{n,k}|\frac{\left|\sqrt{\lambda_n^2+4k'\lambda_n^\theta}-\sqrt{\lambda_n^2+4k\lambda_n^\theta}\right|}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}\sqrt{\lambda_n^2+4k'\lambda_n^\theta}} \\ &\quad + \frac{\left|\mathbb{D}_{n,k}-\mathbb{D}_{n,k'}\right|}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}} = \mathbb{E}_3+\mathbb{E}_4. \end{aligned} \tag{82}$$

In view of Lemma 3.1, one has

$$\begin{aligned}\mathbb{E}_3 &\le \Big(1+\sqrt{1+4k\lambda_1^{\theta-2}}\Big)\frac{\lambda_n}{2}\frac{4|k-k'|\lambda_n^\theta}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}\sqrt{\lambda_n^2+4k'\lambda_n^\theta}\Big(\Big|\sqrt{\lambda_n^2+4k'\lambda_n^\theta}+\sqrt{\lambda_n^2+4k\lambda_n^\theta}\Big|\Big)} \\ &\le 2\Big(1+\sqrt{1+4k\lambda_1^{\theta-2}}\Big)|k-k'|\frac{\lambda_n^{\theta+1}}{8\sqrt{kk'}\lambda_n^\theta\lambda_n} = \frac{1+\sqrt{1+4k\lambda_1^{\theta-2}}}{4\sqrt{kk'}}|k-k'|. \end{aligned} \tag{83}$$

By a similar to $\mathbb{E}_2$ as in (80), we know that

$$\mathbb{E}_4 \le 2|k-k'|\lambda_n^{\theta-2} \le 2\lambda_1^{\theta-2}|k-k'|. \tag{84}$$

By collecting (82), (83) and (84), one gets

$$\left|\frac{-\mathbb{D}_{n,k}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}} - \frac{-\mathbb{D}_{n,k'}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\right| \le \Big(\frac{1+\sqrt{1+4k\lambda_1^{\theta-2}}}{4\sqrt{kk'}}+2\lambda_1^{\theta-2}\Big)|k-k'|. \tag{85}$$

Let us denote

$$u^k(x,t) = \sum_{n=1}^\infty \frac{f_n}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}}\Big(\mathbb{C}_{n,k}e^{\mathbb{C}_{n,k}t} - \mathbb{D}_{n,k}e^{\mathbb{D}_{n,k}t}\Big)e_n(x), \tag{86}$$

and

$$w^k(x,t) = \sum_{n=1}^\infty \left[\frac{1}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}}\int_0^t \Big(e^{\mathbb{C}_{n,k}(t-\tau)} - e^{\mathbb{D}_{n,k}(t-\tau)}\Big)G_n(\tau)d\tau\right]e_n(x). \tag{87}$$

Let us divide into two steps.

Step 1. Estimate of the term $\left\|u^k(.,t)-u^{k'}(.,t)\right\|_{\mathbb{H}^s(\Omega)}$.

From (86), we have

$$\begin{aligned}\left\|u^k(.,t)-u^{k'}(.,t)\right\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{n=1}^\infty \lambda_n^{2s}\Big(\frac{\mathbb{C}_{n,k}e^{\mathbb{C}_{n,k}t}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}} - \frac{\mathbb{C}_{n,k'}e^{\mathbb{C}_{n,k'}t}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\Big)^2|f_n|^2 \\ &\quad + \sum_{n=1}^\infty \lambda_n^{2s}\Big(\frac{-\mathbb{D}_{n,k}e^{\mathbb{D}_{n,k}t}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}} - \frac{-\mathbb{D}_{n,k'}e^{\mathbb{D}_{n,k'}t}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\Big)^2|f_n|^2 \\ &= \mathbb{P}_1+\mathbb{P}_2. \end{aligned} \tag{88}$$

Let us now treat the term $\mathbb{P}_1$ first. It is easy to check that

$$\left(\frac{\mathbb{C}_{n,k}e^{\mathbb{C}_{n,k}t}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}}-\frac{\mathbb{C}_{n,k'}e^{\mathbb{C}_{n,k'}t}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\right)^2 \le 2e^{2\mathbb{C}_{n,k}t}\left|\frac{\mathbb{C}_{n,k}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}}-\frac{\mathbb{C}_{n,k'}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\right|^2 + 2|\mathbb{C}_{n,k'}|^2\frac{\left(e^{\mathbb{C}_{n,k}t}-e^{\mathbb{C}_{n,k'}t}\right)^2}{\lambda_n^2+4k'\lambda_n^\theta} = \mathbb{E}_5+\mathbb{E}_6. \tag{89}$$

For considering $\mathbb{E}_5$, we have the following estimation by using Lemma 3.1 and the inequality (81)

$$\mathbb{E}_5 \le 2e^{4k\lambda_n^{\theta-1}T}\Big(\sqrt{\frac{k}{k'}}+2\Big)^2|k-k'|^2\lambda_n^{2\theta-4}. \tag{90}$$

Let us continue to consider the term $\mathbb{E}_6$. Indeed, using the inequality $|e^a-e^b|\le|a-b|\max(e^a,e^b)$, we find that

$$\left|e^{\mathbb{C}_{n,k}t}-e^{\mathbb{C}_{n,k'}t}\right| \le |\mathbb{C}_{n,k}-\mathbb{C}_{n,k'}|t\max\left(e^{\mathbb{C}_{n,k}t},e^{\mathbb{C}_{n,k'}t}\right). \tag{91}$$

Using Lemma 3.1, we obtain

$$\mathbb{E}_6 \le \frac{8k^2\lambda_n^{2\theta-2}(2|k-k'|\lambda_n^{\theta-1})^2t^2e^{4k\lambda_n^{\theta-1}t}}{\lambda_n^2+4k'\lambda_n^\theta} \le 32k^2|k-k'|^2\lambda_n^{4\theta-6}T^2e^{4k\lambda_n^{\theta-1}t}. \tag{92}$$

Combining (89), (90) and (92), we derive that

$$\left(\frac{\mathbb{C}_{n,k}e^{\mathbb{C}_{n,k}t}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}}-\frac{\mathbb{C}_{n,k'}e^{\mathbb{C}_{n,k'}t}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\right)^2 \le 2e^{4k\lambda_n^{\theta-1}T}\Big(\sqrt{\frac{k}{k'}}+2\Big)^2|k-k'|^2\lambda_n^{2\theta-4} + 32k^2|k-k'|^2\lambda_n^{4\theta-6}T^2e^{4k\lambda_n^{\theta-1}t}. \tag{93}$$

Based on the above bound, we infer that

$$\mathbb{P}_1 \le 2\Big(\sqrt{\frac{k}{k'}}+2\Big)^2|k-k'|^2\sum_{n=1}^\infty\lambda_n^{2s+2\theta-4}e^{4k\lambda_n^{\theta-1}T}f_n^2 + 32k^2|k-k'|^2T^2\sum_{n=1}^\infty\lambda_n^{2s+4\theta-6}e^{4k\lambda_n^{\theta-1}T}f_n^2. \tag{94}$$

In view of Parseval's equality, we find that

$$\mathbb{P}_1 \le 2\Big(\sqrt{\frac{k}{k'}}+2\Big)^2|k-k'|^2\left\|f\right\|^2_{\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)} + 32k^2|k-k'|^2T^2\left\|f\right\|^2_{\mathbb{G}^{s+2\theta-3}_{\theta-1,4kT}(\Omega)}. \tag{95}$$

Let us return to the study the term $\mathbb{P}_2$. It is obvious to see that

$$\left(\frac{-\mathbb{D}_{n,k}e^{\mathbb{D}_{n,k}t}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}}-\frac{-\mathbb{D}_{n,k'}e^{\mathbb{D}_{n,k'}t}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\right)^2 \le 2e^{2\mathbb{D}_{n,k}t}\left|\frac{-\mathbb{D}_{n,k}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}}-\frac{-\mathbb{D}_{n,k'}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\right|^2 + 2|\mathbb{D}_{n,k'}|^2\frac{\left(e^{\mathbb{D}_{n,k}t}-e^{\mathbb{D}_{n,k'}t}\right)^2}{\lambda_n^2+4k'\lambda_n^\theta} = \mathbb{E}_7+\mathbb{E}_8. \tag{96}$$

Since the fact that $\mathbb{D}_{n,k} \le 0$, we follows from (85) that

$$\mathbb{E}_7 \le 2\Big(\frac{1+\sqrt{1+4k\lambda_1^{\theta-2}}}{4\sqrt{kk'}} + 2\lambda_1^{\theta-2}\Big)^2 |k-k'|^2. \tag{97}$$

We continue to bound $\mathbb{E}_8$. By looking at the inequality $|e^{-a}-e^{-b}| \le |a-b|$ for any $,b \ge 0$, we follows from Lemma 3.1 that

$$\begin{aligned}\mathbb{E}_8 &\le \Big(1+\sqrt{1+4k\lambda_1^{\theta-2}}\Big)^2 \frac{\lambda_n^2}{2} \frac{(2|k-k'|\lambda_n^{\theta-1})^2}{\lambda_n^2} \\ &= 2\Big(1+\sqrt{1+4k\lambda_1^{\theta-2}}\Big)^2 |k-k'|^2 \lambda_n^{2\theta-2} \le 2\Big(1+\sqrt{1+4k\lambda_1^{\theta-2}}\Big)^2 \lambda_1^{2\theta-2}|k-k'|^2. \end{aligned} \tag{98}$$

Combining (96), (97), (98), we derive that

$$\Bigg(\frac{-\mathbb{D}_{n,k}e^{\mathbb{D}_{n,k}t}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}} - \frac{-\mathbb{D}_{n,k'}e^{\mathbb{D}_{n,k'}t}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\Bigg)^2 \le \overline{M}_1^2|k-k'|^2, \quad 0 \le t \le T. \tag{99}$$

Here

$$\overline{M}_1 = 2\Big(\frac{1+\sqrt{1+4k\lambda_1^{\theta-2}}}{4\sqrt{kk'}} + 2\lambda_1^{\theta-2}\Big)^2 + 2\Big(1+\sqrt{1+4k\lambda_1^{\theta-2}}\Big)^2\lambda_1^{2\theta-2}.$$

This follows from (99) that

$$\mathbb{P}_2 = \sum_{n=1}^\infty \lambda_n^{2s}\Bigg(\frac{-\mathbb{D}_{n,k}e^{\mathbb{D}_{n,k}t}}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}} - \frac{-\mathbb{D}_{n,k'}e^{\mathbb{D}_{n,k'}t}}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\Bigg)^2 |f_n|^2 \le \overline{M}_1^2|k-k'|^2\sum_{n=1}^\infty \lambda_n^{2s}f_n^2. \tag{100}$$

Using Parseval's equality, one gets

$$\mathbb{P}_2 \le \overline{M}_1^2|k-k'|^2\Big\|f\Big\|^2_{\mathbb{H}^s(\Omega)}. \tag{101}$$

By collecting some known results (88), (95) and (101), we derive that

$$\begin{aligned}\Big\|u^k(.,t)-u^{k'}(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} &\le 2\Big(\sqrt{\frac{k}{k'}}+2\Big)^2|k-k'|^2\Big\|f\Big\|^2_{\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)} \\ &\quad + 32k^2|k-k'|^2T^2\Big\|f\Big\|^2_{\mathbb{G}^{s+2\theta-3}_{\theta-1,4kT}(\Omega)} + \overline{M}_1^2|k-k'|^2\Big\|f\Big\|^2_{\mathbb{H}^s(\Omega)}.\end{aligned} \tag{102}$$

Hence, using the inequality $\sqrt{a^2+b^2+c^2} \le a+b+c$, for any $a,b,c \ge 0$, we deduce that

$$\begin{aligned}\Big\|u^k(.,t)-u^{k'}(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\le \sqrt{2}\Big(\sqrt{\frac{k}{k'}}+2\Big)|k-k'|\Big\|f\Big\|_{\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega)} \\ &\quad + \sqrt{32}k|k-k'|T\Big\|f\Big\|_{\mathbb{G}^{s+2\theta-3}_{\theta-1,4kT}(\Omega)} + \overline{M}_1|k-k'|\Big\|f\Big\|_{\mathbb{H}^s(\Omega)}.\end{aligned} \tag{103}$$

Step 2. Estimation of the term $\Big\|w^k(.,t)-w^{k'}(.,t)\Big\|_{\mathbb{H}^s(\Omega)}$.

From (87), we find that

$$\begin{aligned} w^k(x,t)-w^{k'}(x,t) &= \sum_{n=1}^\infty \Bigg[\frac{1}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}}\int_0^t \Big(e^{\mathbb{C}_{n,k}(t-\tau)}-e^{\mathbb{D}_{n,k}(t-\tau)}\Big)G_n(\tau)d\tau\Bigg]e_n(x) \\ &\quad -\sum_{n=1}^\infty \Bigg[\frac{1}{\sqrt{\lambda_n^2+4k'\lambda_n^\theta}}\int_0^t \Big(e^{\mathbb{C}_{n,k'}(t-\tau)}-e^{\mathbb{D}_{n,k'}(t-\tau)}\Big)G_n(\tau)d\tau\Bigg]e_n(x) \\ &= \mathbb{E}_9 + \mathbb{E}_{10}. \end{aligned} \tag{104}$$

Here

$$\mathbb{E}_9 = \sum_{n=1}^{\infty} \Big[\Big(\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} - \frac{1}{\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}} \Big) \int_0^t \Big(e^{\mathbb{C}_{n,k}(t-\tau)} - e^{\mathbb{D}_{n,k}(t-\tau)} \Big) G_n(\tau) d\tau \Big] e_n(x) \tag{105}$$

and

$$\mathbb{E}_{10} = \sum_{n=1}^{\infty} \Big[\frac{1}{\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}} \int_0^t \Big(e^{\mathbb{C}_{n,k}(t-\tau)} - e^{\mathbb{D}_{n,k}(t-\tau)} - e^{\mathbb{C}_{n,k'}(t-\tau)} + e^{\mathbb{D}_{n,k'}(t-\tau)} \Big) G_n(\tau) d\tau \Big] e_n(x) \tag{106}$$

Let us consider the term $\mathbb{E}_9$. It is obvious to see that

$$\begin{aligned}
\Big| \frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} - \frac{1}{\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}} \Big| &= \frac{\Big| \sqrt{\lambda_n^2 + 4k'\lambda_n^\theta} - \sqrt{\lambda_n^2 + 4k\lambda_n^\theta} \Big|}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} \sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}} \\
&= \frac{4|k-k'|\lambda_n^\theta}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} \sqrt{\lambda_n^2 + 4k'\lambda_n^\theta} \Big(\Big| \sqrt{\lambda_n^2 + 4k'\lambda_n^\theta} + \sqrt{\lambda_n^2 + 4k\lambda_n^\theta} \Big| \Big)} \\
&\le \frac{4|k-k'|\lambda_n^\theta}{8\sqrt{kk'}\lambda_n^\theta \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} = \frac{|k-k'|}{2\sqrt{kk'} \sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}.
\end{aligned} \tag{107}$$

This implies that

$$\begin{aligned}
\Big\| \mathbb{E}_9 \Big\|_{\mathbb{H}^s(\Omega)}^2 &= \sum_{n=1}^{\infty} \lambda_n^{2s} \Big[\Big(\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} - \frac{1}{\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}} \Big) \int_0^t \Big(e^{\mathbb{C}_{n,k}(t-\tau)} - e^{\mathbb{D}_{n,k}(t-\tau)} \Big) G_n(\tau) d\tau \Big]^2 \\
&\le \Big(\frac{|k-k'|}{2\sqrt{kk'}} \Big)^2 \sum_{n=1}^{\infty} \lambda_n^{2s} \Big[\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)} \Big) G_n(\tau) d\tau \Big]^2 \\
&= \Big(\frac{|k-k'|}{2\sqrt{kk'}} \Big)^2 \Big\| y^{(II)}(.,t) \Big\|_{\mathbb{H}^s(\Omega)}^2
\end{aligned} \tag{108}$$

where in the last line, we have used (50). In view of (51), we find that

$$\Big\| \mathbb{E}_9 \Big\|_{\mathbb{H}^s(\Omega)} \le \Big(\frac{|k-k'|}{2\sqrt{kk'}} \Big) \sqrt{8}k \Big\| G \Big\|_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))} + \Big(\frac{|k-k'|}{2\sqrt{kk'}} \Big) \frac{\Big(1 + 4k\lambda_1^{\theta-2} \Big)}{2} \Big\| G \Big\|_{L^2(0,T;\mathbb{H}^s(\Omega))}. \tag{109}$$

Let us now treat the term $\mathbb{E}_{10}$. Indeed, from (76) and (91), one gets

$$\Big| e^{\mathbb{C}_{n,k}t} - e^{\mathbb{C}_{n,k'}t} \Big| \le |\mathbb{C}_{n,k} - \mathbb{C}_{n,k'}| t \max \Big(e^{\mathbb{C}_{n,k}t}, e^{\mathbb{C}_{n,k'}t} \Big) \le 2t \exp \Big(2k\lambda_n^{\theta-1} \Big) |k-k'| \lambda_n^{\theta-1}. \tag{110}$$

In addition, using (77), we also have

$$\Big| e^{\mathbb{D}_{n,k}t} - e^{\mathbb{D}_{n,k'}t} \Big| \le |\mathbb{D}_{n,k} - \mathbb{D}_{n,k'}| t \max \Big(e^{\mathbb{D}_{n,k}t}, e^{\mathbb{D}_{n,k'}t} \Big) \le 2t|k-k'|\lambda_n^{\theta-1}. \tag{111}$$

From two latter observations, we arrive at

$$\begin{aligned}
&\Big\| \mathbb{E}_{10} \Big\|_{\mathbb{H}^s(\Omega)}^2 \\
&= \sum_{n=1}^{\infty} \lambda_n^{2s} \Big[\frac{1}{\sqrt{\lambda_n^2 + 4k'\lambda_n^\theta}} \int_0^t \Big(e^{\mathbb{C}_{n,k}(t-\tau)} - e^{\mathbb{D}_{n,k}(t-\tau)} - e^{\mathbb{C}_{n,k'}(t-\tau)} + e^{\mathbb{D}_{n,k'}(t-\tau)} \Big) G_n(\tau) d\tau \Big]^2 \\
&\le \sum_{n=1}^{\infty} \frac{\lambda_n^{2s}}{\lambda_n^2 + 4k'\lambda_n^\theta} t \int_0^t \Big(e^{\mathbb{C}_{n,k}(t-\tau)} - e^{\mathbb{D}_{n,k}(t-\tau)} - e^{\mathbb{C}_{n,k'}(t-\tau)} + e^{\mathbb{D}_{n,k'}(t-\tau)} \Big)^2 |G_n(\tau)|^2 d\tau \\
&\le 4t^3 |k-k'|^2 \int_0^t \sum_{n=1}^{\infty} \frac{\lambda_n^{2s+2\theta-2}}{\lambda_n^2 + 4k'\lambda_n^\theta} \Big(\exp \Big(2k\lambda_n^{\theta-1} \Big) + 1 \Big)^2 |G_n(\tau)|^2 d\tau.
\end{aligned} \tag{112}$$

In view of Parseval's equality, one has

$$\Big\|\mathbb{E}_{10}\Big\|^2_{\mathbb{H}^s(\Omega)} \le 8t^3|k-k'|^2\Big\|G\Big\|^2_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))} + 8t^3|k-k'|^2\Big\|G\Big\|^2_{L^2(0,T;\mathbb{H}^{s+\theta-2}(\Omega))}. \tag{113}$$

This implies that

$$\Big\|\mathbb{E}_{10}\Big\|_{\mathbb{H}^s(\Omega)} \le \sqrt{8t^3}|k-k'|\left(\Big\|G\Big\|^2_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))} + \Big\|G\Big\|_{L^2(0,T;\mathbb{H}^{s+\theta-2}(\Omega))}\right). \tag{114}$$

Combining (104), (109) and (114), we obtain

$$\begin{aligned}\Big\|w^k(.,t)-w^{k'}(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\le \Big\|\mathbb{E}_9\Big\|_{\mathbb{H}^s(\Omega)} + \Big\|\mathbb{E}_{10}\Big\|_{\mathbb{H}^s(\Omega)}\\ &\le \Big(\frac{|k-k'|}{2\sqrt{kk'}}\Big)\sqrt{8}k\Big\|G\Big\|_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))} + \Big(\frac{|k-k'|}{2\sqrt{kk'}}\Big)\frac{\left(1+4k\lambda_1^{\theta-2}\right)}{2}\Big\|G\Big\|_{L^2(0,T;\mathbb{H}^s(\Omega))}\\ &+ \sqrt{8t^3}|k-k'|\left(\Big\|G\Big\|^2_{L^2(0,T;\mathbb{G}^{s+\theta-2}_{\theta-1,4kT}(\Omega))} + \Big\|G\Big\|_{L^2(0,T;\mathbb{H}^{s+\theta-2}(\Omega))}\right).\end{aligned} \tag{115}$$

Combining (103) and (115), we obtain the desired result (73). □

5. Regularization and error estimate

The main purpose of this section is to introduce two regularize method for Problem (1)-(2) under two cases: homogeneous case and inhomogeneous case.

5.1. Quasi-boundary value method

In this section, we study the following problem

$$\begin{cases} y_t^\varepsilon - \Delta y^\varepsilon + k\int_0^t (-\Delta)^\theta y^\varepsilon(x,s)ds = 0, & \text{in } \Omega\times(0,T],\\ y^\varepsilon|_{\partial\Omega} = 0, & \text{in } \Omega,\end{cases} \tag{116}$$

with the nonlocal condition

$$y^\varepsilon(x,0) + \beta y^\varepsilon(x,T) = f^\varepsilon(x), \quad x\in\Omega. \tag{117}$$

Here $\beta = \beta(\varepsilon)$ is the parameter regularization which satisfies that

$$\lim_{\varepsilon\to 0}\beta = 0.$$

Theorem 5.1. *Let $f^\varepsilon \in L^2(\Omega)$ such that*

$$\Big\|f^\varepsilon - f\Big\|_{L^2(\Omega)} \le \varepsilon, \quad \varepsilon > 0. \tag{118}$$

Then Problem (116)-(117) has a unique solution $y^\varepsilon \in L^\infty(0,T;L^2(\Omega))$. Let us assume that Problem (1)-(2) has a unique solution $y(.,T)\in \mathbb{H}^{2-\theta}(\Omega)$. Then if we choose $\beta = \varepsilon^p$, $0<p<1$, then we obtain

$$\Big\|y(.,t)-y^\varepsilon(.,t)\Big\|_{L^2(\Omega)} \le \Big(\frac{1+\sqrt{1+4k\lambda_1^{\theta-2}}}{2}\varepsilon + \varepsilon^{1-p}\Big) + C_3\varepsilon^{p-\frac{pt}{T}}\Big\|y(.,T)\Big\|_{\mathbb{H}^{2-\theta}(\Omega)}. \tag{119}$$

Here C_3 is a positive constant which depends only on $\lambda_1, k, \theta, t, T$.

Proof. Let us give the explicit fomula of the solution of Problem (116)-(117). From (32), we get that

$$y_n^\varepsilon(t) = \sum_{n=1}^\infty \frac{y_n^\varepsilon(0)}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big) e_n(x). \tag{120}$$

Thus, using (120) and (117), we know that

$$\left(\frac{\beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} + 1\right) y_n^\varepsilon(0) = f_n^\varepsilon. \tag{121}$$

Hence, we have immediately that

$$y_n^\varepsilon(0) = \frac{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)} f_n^\varepsilon. \tag{122}$$

From (120) and (122), we infer that

$$y^\varepsilon(t) = \sum_{n=1}^\infty \frac{\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)} f_n^\varepsilon e_n(x). \tag{123}$$

It is obvious to see that

$$\frac{\mathbb{C}_n e^{\mathbb{C}_n t}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)} \le \frac{\mathbb{C}_n e^{\mathbb{C}_n t}}{\beta \mathbb{C}_n e^{\mathbb{C}_n T}} \le \frac{1}{\beta}. \tag{124}$$

and using Lemma 3.1, we get that

$$\frac{-\mathbb{D}_n e^{\mathbb{D}_n t}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)} \le \frac{\Big(1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}\Big)\frac{\lambda_n}{2}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \le \frac{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}{2}. \tag{125}$$

Two latter observations implies that

$$\frac{\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)} \le \frac{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}{2} + \frac{1}{\beta}. \tag{126}$$

Thus, we obtain that the following bound for y^ε

$$\begin{aligned}\Big\|y^\varepsilon(t)\Big\|_{L^2(\Omega)}^2 &= \sum_{n=1}^\infty \Big(\frac{\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)}\Big)^2 \Big(f_n^\varepsilon\Big)^2 \\ &\le \Big(\frac{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}{2} + \frac{1}{\beta}\Big)^2 \sum_{n=1}^\infty \Big(f_n^\varepsilon\Big)^2.\end{aligned} \tag{127}$$

Hence, we arrive at

$$\Big\|y^\varepsilon(.,t)\Big\|_{L^2(\Omega)} \le \left(\frac{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}{2} + \frac{1}{\beta}\right) \Big\|f^\varepsilon\Big\|_{L^2(\Omega)}. \tag{128}$$

Since the right above is independent of t, we deduce that $y^\varepsilon \in L^\infty(0,T;L^2(\Omega))$. In order to give the error (119), we need to introduce the following function

$$z^\varepsilon(t) = \sum_{n=1}^{\infty} \frac{\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)} f_n e_n(x). \tag{129}$$

We divide the proof into two steps.

Step 1. Error $\Big\|y^\varepsilon(t) - z^\varepsilon(t)\Big\|_{L^2(\Omega)}$.

Combining (123) and (129) and using Parseval's equality, we infer that

$$\Big\|y^\varepsilon(t) - z^\varepsilon(t)\Big\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} \Big(\frac{\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)}\Big)^2 \Big(f_n^\varepsilon - f_n\Big)^2. \tag{130}$$

This inequality together with (130) yields to

$$\Big\|y^\varepsilon(.,t) - z^\varepsilon(.,t)\Big\|_{L^2(\Omega)}^2 \le \Big(\frac{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}{2} + \frac{1}{\beta}\Big)^2 \sum_{n=1}^{\infty} \Big(f_n^\varepsilon - f_n\Big)^2. \tag{131}$$

Hence, we find that

$$\begin{aligned}\Big\|y^\varepsilon(.,t) - z^\varepsilon(.,t)\Big\|_{L^2(\Omega)} &\le \Big(\frac{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}{2} + \frac{1}{\beta}\Big)\Big\|f^\varepsilon - f\Big\|_{L^2(\Omega)} \\ &\le \Big(\frac{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}{2} + \frac{1}{\beta}\Big)\varepsilon. \end{aligned} \tag{132}$$

Step 2. Error $\Big\|y(.,t) - z^\varepsilon(.,t)\Big\|_{L^2(\Omega)}$.

From (41) and (129), we find that

$$\begin{aligned} y(x,t) - z^\varepsilon(x,t) &= \sum_{n=1}^{\infty} \frac{f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big) e_n(x) \\ &\quad - \sum_{n=1}^{\infty} \frac{\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)} f_n e_n(x) \\ &= \sum_{n=1}^{\infty} \frac{\beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}\Big(\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)\Big)} f_n e_n(x). \end{aligned} \tag{133}$$

Since the fact that

$$y(x,T) = \sum_{n=1}^{\infty} \frac{f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big) e_n(x), \tag{134}$$

we know that

$$y(x,t) - z^\varepsilon(x,t) = \sum_{n=1}^{\infty} \frac{\beta\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\Big(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\Big)} y_n(T) e_n(x). \tag{135}$$

It is obvious to see that

$$\left|\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\right| \le \left(2k\lambda_n^{\theta-1} + \left(1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}\right)\frac{\lambda_n}{2}\right)e^{\mathbb{C}_n t} \le C_1(\lambda_1, k, \theta)\lambda_n e^{\mathbb{C}_n t}. \tag{136}$$

In addition, we get

$$\begin{aligned}
&\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\left(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\right) \ge \lambda_n + \beta\mathbb{C}_n e^{\mathbb{C}_n T} \\
&\ge \lambda_n + \beta\frac{2k}{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}\lambda_n^{\theta-1}e^{\mathbb{C}_n T} = \lambda_n^{\theta-1}\left[\lambda_n^{2-\theta} + \beta\frac{2k}{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}e^{\mathbb{C}_n T}\right] \\
&\ge \lambda_n^{\theta-1}\left(\lambda_1^{2-\theta} + \beta C_2(\lambda_1, k, \theta)e^{\mathbb{C}_n T}\right),
\end{aligned} \tag{137}$$

where

$$C_2(\lambda_1, k, \theta) = \frac{2k}{1 + \sqrt{1 + 4k\lambda_1^{\theta-2}}}.$$

Combining (136) and (137), we arrive at

$$\left|\frac{\left(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\right)}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\left(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\right)}\right| \le C_1(\lambda_1, k, \theta)\lambda_n^{2-\theta}\frac{e^{\mathbb{C}_n t}}{\lambda_1^{2-\theta} + \beta C_2(\lambda_1, k, \theta)e^{\mathbb{C}_n T}}. \tag{138}$$

It is easy to verify that

$$\frac{e^{\mathbb{C}_n t}}{\lambda_1^{2-\theta} + \beta C_2(\lambda_1, k, \theta)e^{\mathbb{C}_n T}} = \frac{e^{\mathbb{C}_n t}}{\left(\lambda_1^{2-\theta} + \beta C_2(\lambda_1, k, \theta)e^{\mathbb{C}_n T}\right)^{\frac{t}{T}}\left(\lambda_1^{2-\theta} + \beta C_2(\lambda_1, k, \theta)e^{\mathbb{C}_n T}\right)^{\frac{T-t}{T}}}. \tag{139}$$

Moreover, we get that

$$\begin{aligned}
&\left(\lambda_1^{2-\theta} + \beta C_2(\lambda_1, k, \theta)e^{\mathbb{C}_n T}\right)^{\frac{t}{T}}\left(\lambda_1^{2-\theta} + \beta C_2(\lambda_1, k, \theta)e^{\mathbb{C}_n T}\right)^{\frac{T-t}{T}} \\
&\ge \beta^{\frac{t}{T}}\left(C_2(\lambda_1, k, \theta)\right)^{\frac{t}{T}}\left(\lambda_1^{2-\theta}\right)^{\frac{T-t}{T}}e^{\mathbb{C}_n t}.
\end{aligned} \tag{140}$$

This follows from (138) that

$$\left|\frac{\left(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\right)}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\left(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\right)}\right| \le C_3\lambda_n^{2-\theta}\beta^{-\frac{t}{T}} \tag{141}$$

where C_3 depends on $\lambda_1, k, \theta, t, T$. Thus, we remind (135) in order to obtain that

$$\begin{aligned}
\left\|y(.,t) - z^\varepsilon(.,t)\right\|_{L^2(\Omega)}^2 &= \beta^2\sum_{n=1}^\infty\left|\frac{\left(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\right)}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta} + \beta\left(\mathbb{C}_n e^{\mathbb{C}_n T} - \mathbb{D}_n e^{\mathbb{D}_n T}\right)}\right|^2|y_n(T)|^2 \\
&\le C_3^2\beta^{2-\frac{2t}{T}}\sum_{n=1}^\infty\lambda_n^{4-2\theta}|y_n(T)|^2.
\end{aligned} \tag{142}$$

In view of Paseval's equality, one has

$$\Big\|y(.,t) - z^{\varepsilon}(.,t)\Big\|_{L^2(\Omega)} \le C_3\beta^{1-\frac{t}{T}}\Big\|y(.,T)\Big\|_{\mathbb{H}^{2-\theta}(\Omega)}. \tag{143}$$

Combining (132) and (143), we infer that

$$\begin{aligned}\Big\|y(.,t) - y^{\varepsilon}(.,t)\Big\|_{L^2(\Omega)} &\le \Big\|y(.,t) - z^{\varepsilon}(.,t)\Big\|_{L^2(\Omega)} + \Big\|y^{\varepsilon}(.,t) - z^{\varepsilon}(.,t)\Big\|_{L^2(\Omega)}\\ &\le \Big(\frac{1+\sqrt{1+4k\lambda_1^{\theta-2}}}{2} + \frac{1}{\beta}\Big)\varepsilon + C_3\beta^{1-\frac{t}{T}}\Big\|y(.,T)\Big\|_{\mathbb{H}^{2-\theta}(\Omega)}.\end{aligned} \tag{144}$$

Under the choice $\beta = \varepsilon^p,\ 0 < p < 1$, we get the desired result (119). □

5.2. The truncation method

In this subsection, we provide a truncation method to approximate the solution to Problem (1). Compared to Theorem (5.1), the truncation method has the advantage that we can estimate the error in the space $\mathbb{H}^s$.

By applying Fourier truncation method, we introduce the following regularized solution

$$\begin{aligned}Y_{\varepsilon}(x,t) &= \sum_{\lambda_n \le M_\varepsilon} \frac{f_n^{\varepsilon}}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)e_n(x)\\ &\quad + \sum_{\lambda_n \le M_\varepsilon}\Bigg[\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^{\theta}}}\int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\Big)G_n^{\varepsilon}(\tau)d\tau\Bigg]e_n(x).\end{aligned} \tag{145}$$

Theorem 5.2. *Let (f,G) be the function which is noised by $(f^{\varepsilon}, G^{\varepsilon})$ such that*

$$\Big\|f^{\varepsilon} - f\Big\|_{L^2(\Omega)} + \Big\|G^{\varepsilon} - G\Big\|_{L^2(0,T;L^2(\Omega))} \le \varepsilon. \tag{146}$$

Let us choose M_ε such that

$$\lim_{\varepsilon\to 0}\Big(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^s\Big)\varepsilon = 0, \quad \lim_{\varepsilon\to 0} M_\varepsilon = +\infty. \tag{147}$$

Let us assume that $y \in L^{\infty}(0,T;\mathbb{H}^{s+\vartheta}(\Omega))$ for any $s \ge 0$ and $\vartheta > 0$. Then we get

$$\Big\|Y_{\varepsilon}(.,t) - y(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le \Big(\overline{C}_2 + \overline{C}_3\Big)\Big(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^s\Big)\varepsilon + (M_\varepsilon)^{-\vartheta}\Big\|y\Big\|_{L^{\infty}(0,T;\mathbb{H}^{s+\vartheta}(\Omega))}. \tag{148}$$

Here $1 < \theta \le 2$ and $0 \le s \le 2 - \theta$.

Remark 5.3. *Let us choose $M_\varepsilon > 0$ such that*

$$M_\varepsilon > 0 = \Big(\frac{1}{2hkT}\Big)^{\frac{1}{\theta-1}}\Big(\log(\frac{1}{\varepsilon})\Big)^{\frac{1}{\theta-1}}, \quad 0 < h < 1. \tag{149}$$

Then the error $\Big\|Y_{\varepsilon}(.,t) - y(.,t)\Big\|_{\mathbb{H}^s(\Omega)}$ is of order

$$\max\Bigg(\Big(\log(\frac{1}{\varepsilon})\Big)^{\frac{-\vartheta}{\theta-1}}, \varepsilon\Big(\log(\frac{1}{\varepsilon})\Big)^{\frac{s}{\theta-1}}, \varepsilon^{1-h}\Bigg). \tag{150}$$

Proof. Let us set the following function W_ε as follows

$$W_\varepsilon(x,t) = \sum_{\lambda_n \le M_\varepsilon} \frac{f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big) e_n(x) + \sum_{\lambda_n \le M_\varepsilon} \left[\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\Big) G_n(\tau) d\tau\right] e_n(x). \tag{151}$$

Using triangle inequality, we get that

$$\Big\|Y_\varepsilon(.,t) - y(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le \Big\|Y_\varepsilon(.,t) - W_\varepsilon(.,t)\Big\|_{\mathbb{H}^s(\Omega)} + \Big\|y(.,t) - W_\varepsilon(.,t)\Big\|_{\mathbb{H}^s(\Omega)}. \tag{152}$$

Let us divide two steps.

Step 1. Estimation of $\Big\|Y_\varepsilon(.,t) - W_\varepsilon(.,t)\Big\|_{\mathbb{H}^s(\Omega)}$.

From (145) and (151), we obtain the following bound

$$\begin{aligned} Y_\varepsilon(x,t) - W_\varepsilon(x,t) &= \sum_{\lambda_n \le M_\varepsilon} \frac{f_n^\varepsilon - f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big) e_n(x) \\ &+ \sum_{\lambda_n \le M_\varepsilon} \left[\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}} \int_0^t \Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\Big)\Big(G_n^\varepsilon(\tau) - G_n(\tau)\Big) d\tau\right] e_n(x) \\ &= \mathcal{F}_1(x,t) + \mathcal{F}_2(x,t). \end{aligned} \tag{153}$$

Using Parseval's equality and (45), we infer that

$$\begin{aligned} \Big\|\mathcal{F}_1(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{\lambda_n \le M_\varepsilon} \lambda_n^{2s} \left(\frac{f_n^\varepsilon - f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)\right)^2 \\ &\le \sum_{\lambda_n \le M_\varepsilon} \left[8k^2 \lambda_n^{2s+2\theta-4} e^{4kT\lambda_n^{\theta-1}} + \frac{\Big(1 + 4k\lambda_1^{\theta-2}\Big)^2}{4} \lambda_n^{2s}\right] \Big(f_n^\varepsilon - f_n\Big)^2. \end{aligned} \tag{154}$$

If $s + \theta \le 2$ then we get

$$\lambda_n^{2s+2\theta-4} e^{4kT\lambda_n^{\theta-1}} \le \lambda_1^{2s+2\theta-4} e^{4kT(M_\varepsilon)^{\theta-1}} \tag{155}$$

and

$$\lambda_n^{2s} \le (M_\varepsilon)^{2s} \tag{156}$$

for $\lambda_n \le M_\varepsilon$. Thus, we find that

$$\begin{aligned} \Big\|\mathcal{F}_1(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} &\le |\overline{C}_2|^2 \Big(e^{4kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{2s}\Big) \sum_{\lambda_n \le M_\varepsilon} \Big(f_n^\varepsilon - f_n\Big)^2 \\ &\le |\overline{C}_2|^2 \Big(e^{4kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{2s}\Big) \Big\|f^\varepsilon - f\Big\|^2_{L^2(\Omega)} \end{aligned} \tag{157}$$

where $\overline{C}_2 = \overline{C}_2(k, \lambda_1, s, \theta)$. Hence, we obtain

$$\Big\|\mathcal{F}_1(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le \overline{C}_2 \Big(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^s\Big) \varepsilon. \tag{158}$$

Using Parseval's equality and (45) and Hölder inequality, we infer that

$$\Big\|\mathcal{F}_2(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} = \sum_{\lambda_n \le M_\varepsilon} \lambda_n^{2s}\Bigg[\frac{1}{\sqrt{\lambda_n^2+4k\lambda_n^\theta}}\int_0^t \Big(e^{\mathbb{C}_n(t-\tau)}-e^{\mathbb{D}_n(t-\tau)}\Big)\Big(G_n^\varepsilon(\tau)-G_n(\tau)\Big)d\tau\Bigg]^2$$

$$\le \sum_{\lambda_n \le M_\varepsilon}\Bigg(\int_0^t \frac{\lambda_n^{2s}}{\lambda_n^2+4k\lambda_n^\theta}\Big(e^{\mathbb{C}_n(t-\tau)}-e^{\mathbb{D}_n(t-\tau)}\Big)^2 d\tau\Bigg)\Bigg(\int_0^t \Big(G_n^\varepsilon(\tau)-G_n(\tau)\Big)^2 d\tau\Bigg). \tag{159}$$

Using (45) and (155), (156), we obtain that

$$\int_0^t \frac{\lambda_n^{2s}}{\lambda_n^2+4k\lambda_n^\theta}\Big(e^{\mathbb{C}_n(t-\tau)}-e^{\mathbb{D}_n(t-\tau)}\Big)^2 d\tau$$

$$\le 8Tk^2\lambda_n^{2s+2\theta-4}e^{4kT\lambda_n^{\theta-1}} + \frac{T\Big(1+4k\lambda_1^{\theta-2}\Big)^2}{4}\lambda_n^{2s}$$

$$\le 8Tk^2\lambda_1^{2s+2\theta-4}e^{4kT(M_\varepsilon)^{\theta-1}} + \frac{T\Big(1+4k\lambda_1^{\theta-2}\Big)^2}{4}(M_\varepsilon)^{2s} \tag{160}$$

for $\lambda_n \le M_\varepsilon$. Hence, we have immediately that

$$\Big\|\mathcal{F}_2(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} \le |\overline{C}_3|^2\Big(e^{4kT(M_\varepsilon)^{\theta-1}}+(M_\varepsilon)^{2s}\Big)\Big\|G^\varepsilon - G\Big\|^2_{L^2(0,T;L^2(\Omega))} \tag{161}$$

where $\overline{C}_3 = \overline{C}_3(k,T,\lambda_1,s,\theta)$. This implies that

$$\Big\|\mathcal{F}_2(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le \overline{C}_3\Big(e^{2kT(M_\varepsilon)^{\theta-1}}+(M_\varepsilon)^{s}\Big)\varepsilon. \tag{162}$$

Combining (153), (158) and (162), we infer that

$$\Big\|Y_\varepsilon(.,t)-W_\varepsilon(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le \Big\|\mathcal{F}_1(.,t)\Big\|_{\mathbb{H}^s(\Omega)} + \Big\|\mathcal{F}_2(.,t)\Big\|_{\mathbb{H}^s(\Omega)}$$

$$\le \Big(\overline{C}_2+\overline{C}_3\Big)\Big(e^{2kT(M_\varepsilon)^{\theta-1}}+(M_\varepsilon)^{s}\Big)\varepsilon. \tag{163}$$

Step 2. Estimation of $\Big\|y(.,t)-W_\varepsilon(.,t)\Big\|_{\mathbb{H}^s(\Omega)}$. It is obvious to see that the following equality

$$y(x,t)-W_\varepsilon(x,t) = \sum_{\lambda_n > M_\varepsilon} y_n(t)e_n(x). \tag{164}$$

By using Parseval's equality, we get that

$$\Big\|y(.,t)-W_\varepsilon(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} = \sum_{\lambda_n > M_\varepsilon}\lambda_n^{2s}y_n^2(t) = \sum_{\lambda_n > M_\varepsilon}\lambda_n^{-2\vartheta}\lambda_n^{2s+2\vartheta}y_n^2(t)$$

$$\le (M_\varepsilon)^{-2\vartheta}\Big\|y(.,t)\Big\|^2_{\mathbb{H}^{s+\vartheta}(\Omega)}, \tag{165}$$

for any $\vartheta > 0$. Thus, we deduce that

$$\Big\|y(.,t)-W_\varepsilon(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le (M_\varepsilon)^{-\vartheta}\Big\|y(.,t)\Big\|_{\mathbb{H}^{s+\vartheta}(\Omega)} \le (M_\varepsilon)^{-\vartheta}\Big\|y\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\vartheta}(\Omega))}. \tag{166}$$

Combining (163) and (166), we deduce that

$$\Big\|Y_\varepsilon(.,t)-y(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le \Big\|Y_\varepsilon(.,t)-W_\varepsilon(.,t)\Big\|_{\mathbb{H}^s(\Omega)} + \Big\|y(.,t)-W_\varepsilon(.,t)\Big\|_{\mathbb{H}^s(\Omega)}$$

$$\le \Big(\overline{C}_2+\overline{C}_3\Big)\Big(e^{2kT(M_\varepsilon)^{\theta-1}}+(M_\varepsilon)^{s}\Big)\varepsilon + (M_\varepsilon)^{-\vartheta}\Big\|y\Big\|_{L^\infty(0,T;\mathbb{H}^{s+\vartheta}(\Omega))}. \tag{167}$$

The proof of Theorem 5.2 is completed. □

Theorem 5.4. *Let us assume f^ε is the noisy data for f which satisfies that*

$$\left\|f^\varepsilon - f\right\|_{L^p(\Omega)} + \left\|G^\varepsilon - G\right\|_{L^2(0,T;L^p(\Omega))} \le \varepsilon, \tag{168}$$

for $\varepsilon > 0$ and $1 \le p < 2$.

- *If $s + \theta - 2 < \frac{N(p-2)}{4p}$ and $y \in L^\infty(0,T;\mathbb{H}^{s+\vartheta}(\Omega))$, we get that*

$$\left\|Y_\varepsilon(.,t) - y(.,t)\right\|_{\mathbb{H}^s(\Omega)} \le \left(\overline{\mathbb{C}}_4\mathbb{C}_{1,p,\ell} + \mathbb{C}_{2,p,\ell}\overline{\mathbb{C}}_5\right)\left(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\right)\varepsilon + (M_\varepsilon)^{-\vartheta}\left\|y\right\|_{L^\infty(0,T;\mathbb{H}^{s+\vartheta}(\Omega))} \tag{169}$$

for $s + \theta - 2 < \ell < \frac{N(p-2)}{4p}$. Here M_ε is chosen suitable such that

$$\lim_{\varepsilon\to 0} e^{2kT(M_\varepsilon)^{\theta-1}}\varepsilon = 0, \quad \lim_{\varepsilon\to 0} M_\varepsilon = +\infty. \tag{170}$$

- *If $s + \theta - 2 > \frac{-N}{4}$ then*

$$\left\|Y_\varepsilon(.,t) - y(.,t)\right\|_{\mathbb{H}^s(\Omega)} \le \mathbb{C}_{2,p,\ell}\overline{\mathbb{C}}_6\left((M_\varepsilon)^{s-\ell+\theta-2}e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\right)\varepsilon + (M_\varepsilon)^{-\vartheta}\left\|y\right\|_{L^\infty(0,T;\mathbb{H}^{s+\vartheta}(\Omega))} \tag{171}$$

where $s + \theta - 2 > \ell > \frac{-N}{4}$. Here M_ε is chosen suitable such that

$$\lim_{\varepsilon\to 0}(M_\varepsilon)^{s-\ell+\theta-2}e^{2kT(M_\varepsilon)^{\theta-1}}\varepsilon = 0, \quad \lim_{\varepsilon\to 0} M_\varepsilon = +\infty. \tag{172}$$

Remark 5.5. *In the above theorem, we need the condition of the input data which is in L^p spaces. This is the interesting point of our method.*

Proof. From Sobolev embedding $L^p(\Omega) \hookrightarrow H^\ell(\Omega)$ for $\frac{-N}{4} \le \ell < 0$ and $p \ge \frac{2N}{N-4\ell}$, there exists a positive constant $\mathbb{C}_{1,p,\ell}$ which depends on p, ℓ such that

$$\left\|f^\varepsilon - f\right\|_{\mathbb{H}^\ell(\Omega)} \le C_{1,p,\ell}\left\|f^\varepsilon - f\right\|_{L^p(\Omega)} \le \varepsilon\mathbb{C}_{1,p,\ell} \tag{173}$$

and

$$\left\|G^\varepsilon - G\right\|_{L^2(0,T;\mathbb{H}^\ell(\Omega))} \le C_{2,p,\ell}\left\|G^\varepsilon - G\right\|_{L^2(0,T;L^p(\Omega))} \le \varepsilon\mathbb{C}_{2,p,\ell}. \tag{174}$$

Step 1. Estimation of $\left\|Y_\varepsilon(.,t) - W_\varepsilon(.,t)\right\|_{\mathbb{H}^s(\Omega)}$.

In view of (154), we know that

$$\begin{aligned}\left\|\mathcal{F}_1(.,t)\right\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{\lambda_n\le M_\varepsilon} \lambda_n^{2s}\left(\frac{f_n^\varepsilon - f_n}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\left(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\right)\right)^2 \\ &\le \sum_{\lambda_n\le M_\varepsilon}\left[8k^2\lambda_n^{2s+2\theta-2\ell-4}e^{4kT\lambda_n^{\theta-1}} + \frac{\left(1+4k\lambda_1^{\theta-2}\right)^2}{4}\lambda_n^{2s-2\ell}\right]\lambda_n^{2\ell}\left(f_n^\varepsilon - f_n\right)^2.\end{aligned} \tag{175}$$

Since the assumption $s + \theta \le 2$, one has

$$\lambda_n^{2s+2\theta-2\ell-4}e^{4kT\lambda_n^{\theta-1}} \le \lambda_1^{2s+2\theta-2\ell-4}e^{4kT(M_\varepsilon)^{\theta-1}} \tag{176}$$

and

$$\lambda_n^{2s-2\ell} \le (M_\varepsilon)^{2s-2\ell} \tag{177}$$

for $\lambda_n \le M_\varepsilon$. Hence, we have immediately that

$$\begin{aligned}\Big\|\mathcal{F}_1(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} &\le |\overline{C}_4|^2\Big(e^{4kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{2s}\Big)\sum_{\lambda_n \le M_\varepsilon}\lambda_n^{2\ell}\Big(f_n^\varepsilon - f_n\Big)^2\\ &\le |\overline{C}_4|^2\Big(e^{4kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{2s-2\ell}\Big)\Big\|f^\varepsilon - f\Big\|^2_{\mathbb{H}^\ell(\Omega)}\end{aligned} \tag{178}$$

where $\overline{C}_4 = \overline{C}_4(k, \ell, \lambda_1, s, \theta)$. Thus, one has

$$\Big\|\mathcal{F}_1(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le \overline{C}_4\Big(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\Big)\varepsilon\mathbb{C}_{1,p,\ell}. \tag{179}$$

In view of (159), we know that

$$\begin{aligned}\Big\|\mathcal{F}_2(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{\lambda_n \le M_\varepsilon}\lambda_n^{2s-2\ell}\left[\frac{1}{\sqrt{\lambda_n^2 + 4k\lambda_n^\theta}}\int_0^t \lambda_n^{2\ell}\Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\Big)\Big(G_n^\varepsilon(\tau) - G_n(\tau)\Big)d\tau\right]^2\\ &\le \sum_{\lambda_n \le M_\varepsilon}\left(\int_0^t \frac{\lambda_n^{2s-2\ell}}{\lambda_n^2 + 4k\lambda_n^\theta}\Big(e^{\mathbb{C}_n(t-\tau)} - e^{\mathbb{D}_n(t-\tau)}\Big)^2 d\tau\right)\left(\int_0^t \lambda_n^{2\ell}\Big(G_n^\varepsilon(\tau) - G_n(\tau)\Big)^2 d\tau\right).\end{aligned} \tag{180}$$

Using (44), we find that

$$\frac{\lambda_n^{2s-2\ell}}{\lambda_n^2 + 4k\lambda_n^\theta}\Big(\mathbb{C}_n e^{\mathbb{C}_n t} - \mathbb{D}_n e^{\mathbb{D}_n t}\Big)^2 \le 8k^2\lambda_n^{2s-2\ell+2\theta-4}e^{4kT\lambda_n^{\theta-1}} + \frac{\Big(1 + 4k\lambda_1^{\theta-2}\Big)^2}{4}\lambda_n^{2s-2\ell}. \tag{181}$$

Let us divide two cases.

Case 1. The case $s + \theta - 2 < \ell$.

If ℓ satisfies that $s + \theta - 2 < \ell$ and $\lambda_n \le M_\varepsilon$, we know that

$$\begin{aligned}&8k^2\lambda_n^{2s-2\ell+2\theta-4}e^{4kT\lambda_n^{\theta-1}} + \frac{\Big(1 + 4k\lambda_1^{\theta-2}\Big)^2}{4}\lambda_n^{2s-2\ell}\\ &\qquad\le 8k^2\lambda_1^{2s-2\ell+2\theta-4}e^{4kT(M_\varepsilon)^{\theta-1}} + \frac{\Big(1 + 4k\lambda_1^{\theta-2}\Big)^2}{4}(M_\varepsilon)^{2s-2\ell}.\end{aligned} \tag{182}$$

Thus, we obtain that

$$\Big\|\mathcal{F}_2(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} \le |\overline{C}_5|^2\Big(e^{4kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{2s-2\ell}\Big)\Big\|G^\varepsilon - G\Big\|^2_{L^2(0,T;\mathbb{H}^\ell(\Omega))} \tag{183}$$

where $\overline{C}_5 = \overline{C}_5(k, T, \ell, \lambda_1, s, \theta)$. Hence, we find that

$$\Big\|\mathcal{F}_2(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le \mathbb{C}_{2,p,\ell}\overline{C}_5\Big(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\Big)\varepsilon. \tag{184}$$

Combining (179) and (184), we obtain

$$\begin{aligned}\Big\|Y_\varepsilon(.,t) - W_\varepsilon(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\le \Big\|\mathcal{F}_1(.,t)\Big\|_{\mathbb{H}^s(\Omega)} + \Big\|\mathcal{F}_2(.,t)\Big\|_{\mathbb{H}^s(\Omega)}\\ &\le \overline{C}_4\mathbb{C}_{1,p,\ell}\Big(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\Big)\varepsilon + \mathbb{C}_{2,p,\ell}\overline{C}_5\Big(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\Big)\varepsilon.\end{aligned} \tag{185}$$

Under the assumption $y \in L^\infty(0,T;\mathbb{H}^{s+\vartheta}(\Omega))$, we remind that

$$\left\|y(.,t) - W_\varepsilon(.,t)\right\|_{\mathbb{H}^s(\Omega)} \le (M_\varepsilon)^{-\vartheta}\left\|y\right\|_{L^\infty(0,T;\mathbb{H}^{s+\vartheta}(\Omega))}. \tag{186}$$

By collecting two previous results (185) and (186), we confirm that

$$\begin{aligned}\left\|Y_\varepsilon(.,t) - y(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\le \left\|Y_\varepsilon(.,t) - W_\varepsilon(.,t)\right\|_{\mathbb{H}^s(\Omega)} + \left\|y(.,t) - W_\varepsilon(.,t)\right\|_{\mathbb{H}^s(\Omega)} \\ &\le \left(\overline{C}_4\mathbb{C}_{1,p,\ell} + \mathbb{C}_{2,p,\ell}\overline{C}_5\right)\left(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\right)\varepsilon + (M_\varepsilon)^{-\vartheta}\left\|y\right\|_{L^\infty(0,T;\mathbb{H}^{s+\vartheta}(\Omega))}.\end{aligned} \tag{187}$$

Case 2. The case $s + \theta - 2 > \ell$.

If ℓ satisfies that $s + \theta - 2 \ge \ell$ and $\lambda_n \le M_\varepsilon$, we derive that

$$\begin{aligned}&8k^2\lambda_n^{2s-2\ell+2\theta-4}e^{4kT\lambda_n^{\theta-1}} + \frac{\left(1 + 4k\lambda_1^{\theta-2}\right)^2}{4}\lambda_n^{2s-2\ell} \\ &\le 8k^2(M_\varepsilon)^{2s-2\ell+2\theta-4}e^{4kT(M_\varepsilon)^{\theta-1}} + \frac{\left(1 + 4k\lambda_1^{\theta-2}\right)^2}{4}(M_\varepsilon)^{2s-2\ell}.\end{aligned} \tag{188}$$

In a similar process, we also obtain that

$$\left\|\mathcal{F}_2(.,t)\right\|_{\mathbb{H}^s(\Omega)} \le \mathbb{C}_{2,p,\ell}\overline{C}_6\left((M_\varepsilon)^{s-\ell+\theta-2}e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\right)\varepsilon. \tag{189}$$

where $\overline{C}_6 = \overline{C}_6(k,T,s,\theta)$. This inequality together with (179) yields to

$$\begin{aligned}\left\|Y_\varepsilon(.,t) - W_\varepsilon(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\le \left\|\mathcal{F}_1(.,t)\right\|_{\mathbb{H}^s(\Omega)} + \left\|\mathcal{F}_2(.,t)\right\|_{\mathbb{H}^s(\Omega)} \\ &\le \overline{C}_4\mathbb{C}_{1,p,\ell}\left(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\right)\varepsilon \\ &\quad + \mathbb{C}_{2,p,\ell}\overline{C}_6\left((M_\varepsilon)^{s-\ell+\theta-2}e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\right)\varepsilon.\end{aligned} \tag{190}$$

By collecting (186) and (190), we obtain that

$$\begin{aligned}\left\|Y_\varepsilon(.,t) - y(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\le \left\|Y_\varepsilon(.,t) - W_\varepsilon(.,t)\right\|_{\mathbb{H}^s(\Omega)} + \left\|y(.,t) - W_\varepsilon(.,t)\right\|_{\mathbb{H}^s(\Omega)} \\ &\le \overline{C}_4\mathbb{C}_{1,p,\ell}\left(e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\right)\varepsilon \\ &\quad + \mathbb{C}_{2,p,\ell}\overline{C}_6\left((M_\varepsilon)^{s-\ell+\theta-2}e^{2kT(M_\varepsilon)^{\theta-1}} + (M_\varepsilon)^{s-\ell}\right)\varepsilon. + (M_\varepsilon)^{-\vartheta}\left\|y\right\|_{L^\infty(0,T;\mathbb{H}^{s+\vartheta}(\Omega))}.\end{aligned} \tag{191}$$

The proof is completed. □

6. Conclusion

The focus of this study lies in examining the Cauchy problem for a parabolic equation with a memory term. The memory component incorporates a fractional Laplacian operator. Initially, we express the mild solution using a Fourier series. Subsequently, we investigate the well-posedness of the Cauchy problem when the initial data and source function belong to Gevrey spaces. Additionally, the this presents results demonstrating the ill-posedness in the sense of Hadamard and proposes regularized methods. In the homogeneous case, we applied the quasi-boundary value method to regularize the problem and estimate errors when observing data in L^2. For inhomogeneous source terms, we used the truncation method to approximate the problem with observed data in L^p.

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