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Weyl type theorem and its perturbations for bounded linear operators

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Abstract. As two variations of Weyl's theorem, a-Weyl's theorem and property (ω) are introduced by Rakočević. In this paper, we study a-Weyl's theorem and property (ω) for functions of bounded linear operators. And concrete examples are given to show that the two properties are independent of each other. We give the necessary and sufficient condition for a bounded linear operator with both a-Weyl's theorem and property (ω) utilizing the induced spectrum of topological uniform descent. Also, we investigate the perturbations of operator functions satisfying both a-Weyl's theorem and property (ω).

1. Introduction

Weyl examined the spectra of all compact perturbations of a Hermitian operator on a Hilbert space and found that their intersection consisted precisely of those points of the range that were not isolated eigenvalues of finite multiplicity ([7]). This observation was later called "Weyl's theorem". After that, a lot of variants of this theorem appeared. Among them are properties (w) and the a-Weyl's theorem. And many researchers have done a lot of work on property (ω) and a-Weyl's theorem, respectively. ([2, 4, 8]). The study of spectral structures of operators and their functional calculus obeying these variants of the Weyl's theorem is also a common research direction in spectral theory. The preservation of the Weyl-type theorem under certain classes of perturbations has been studied initially in a series of papers. ([9, 13, 18]).

2. Preliminary Definitions and Facts

Throughout this note, Let $\mathcal{H}$ denote a complex infinite dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ be the two-sided closed ideal of $\mathcal{B}(\mathcal{H})$ which consists of all compact operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, we denote by $N(T)$ and $R(T)$ the kernel and the range of T, respectively. Put $n(T) = dimN(T)$ and $d(T) = codimR(T)$. If $n(T) < \infty$ and $R(T)$ is closed, we say T is an upper semi-Fredholm operator, while T is called a lower semi-Fredholm operator if $d(T) < \infty$. Especially if $n(T) = 0$ and $R(T)$ is closed, then T is called bounded from below. T is said to be a semi-Fredholm operator if T is upper semi-Fredholm or lower semi-Fredholm. Now, the index of T is written as $ind(T) = n(T) - d(T)$. In particular, if $-\infty < ind(T) < \infty$, T is called a Fredholm operator. An operator, T, is considered Weyl if it

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is Fredholm with $ind(T) = 0$. Let $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of non-negative integers, respectively. The ascent and descent of T are defined respectively by

$$asc(T) := \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$$

and

$$des(T) := \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}.$$

As we all know, if both $asc(T)$ and $des(T)$ are finite, then $asc(T) = des(T)$[5], and now T is said to be Drazin invertible. If $T - \lambda I$ is Drazin invertible and $\lambda \in \sigma(T)$, λ is called a pole of resolvent([12]), where $\sigma(T)$ denotes the usual spectrum of T. If T is Fredholm of finite ascent and descent, it is called a Browder operator. It is easily shown that T is Browder if it is Drazin invertible and $n(T) < \infty$.

We use $\sigma_{SF_+}(T)$, $\sigma_w(T)$, $\sigma_b(T)$, $\sigma_D(T)$ and $\sigma_a(T)$ denote the upper semi-Fredholm spectrum, the Weyl spectrum, the Browder spectrum, the Drazin invertible spectrum and the approximate point spectrum of T respectively. Define $\rho_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is upper semi-Fredholm and } ind(T - \lambda I) \leq 0\}$ and $\rho_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is upper semi-Fredholm and } asc(T - \lambda I) < \infty\}$, the Weyl essential approximate point spectrum $\sigma_{ea}(T) = \mathbb{C}\backslash\rho_{ea}(T)$ and the Browder essential approximate point spectrum $\sigma_{ab}(T) = \mathbb{C}\backslash\rho_{ab}(T)$.

Put $\rho_{SF_+}(T) = \mathbb{C}\backslash\sigma_{SF+}(T)$, $\rho_a(T) = \mathbb{C}\backslash\sigma_a(T)$ and $\rho_b(T) = \mathbb{C}\backslash\sigma_b(T)$. Denote

$$\rho^+_{SF}(T) = \{\lambda \in \rho_{SF}(T) : ind(T - \lambda I) > 0\}$$

and

$$\rho^-_{SF}(T) = \{\lambda \in \rho_{SF}(T) : ind(T - \lambda I) < 0\},$$

where $\rho_{SF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is semi-Fredholm}\}$.

We use ∂E, $intE$, $isoE$, and $accE$ to represent the boundary points, interior points, isolated points, and accumulation points of a subset $E \subseteq \mathbb{C}$, respectively.

If all the curves forming the boundary of a Cauchy domain Ω([3]) are regular analytic Jordan curves, then Ω referred to is an analytical Cauchy domain. For $T \in \mathcal{B}(\mathcal{H})$, if σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$, where $\overline{\Omega}$ is the closure of Ω. We denote by $E(\sigma; T)$ the Riesz idempotent of T corresponding to σ, i.e.,

$$E(\sigma; T) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - T)^{-1} d\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we have $H(\sigma; T) = R(E(\sigma; T))$. Clearly, if $\lambda \in iso\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$. We write $H(\lambda; T)$ instead of $H(\{\lambda\}; T)$; if in addition, $H(\lambda; T) < \infty$, then $\lambda \in \sigma(T) \setminus \sigma_b(T)$.

The operator range topology on $R(T^n)$ is defined by the norm $\|\cdot\|_n$ such that for all $y \in R(T^n)$, $\|y\|_n := \inf\{\|x\| : x \in \mathcal{H},\ y = T^n x\}$. If M is a subspace of $R(T^n)$, then $cl(M)$ is the closure of M in the operator range topology on $R(T^n)$.

Let $T \in \mathcal{B}(\mathcal{H})$, and $d \in \mathbb{N}$, T is said to have uniform descent d, if $R(T) + N(T^n) = R(T) + N(T^d)$ for $n \geq d$. If, in addition, $R(T^n)$ is closed in the operator range topology on $R(T^d)$, for $n \geq d$, then T is said to have topological uniform descent (TUD for brevity) d. Grabiner introduces operators with TUD in ([12]).

The topological uniform descent spectrum of T is defined as

$$\sigma_\tau(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not have TUD}\}.$$

Put $\rho_\tau(T) = \mathbb{C}\backslash\sigma_\tau(T)$. Clearly, $\rho_{SF}(T) \subseteq \rho_\tau(T)$. The reader could refer to [17] for more details.

For $T \in \mathcal{B}(\mathcal{H})$, $\mathrm{Hol}(\sigma(T))$ denotes the set of all functions which are analytic on a neighborhood of $\sigma(T)$ and are not constant on any component of $\sigma(T)$. Given $f \in \mathrm{Hol}(\sigma(T))$, we let $f(T)$ denote the Riesz-Dounford functional calculus of T with respect to f ([10]).

Next, the definitions of the central notions studied in this article will be given. Let

$$\pi_{00}(T) := \{\lambda \in iso\sigma(T) : 0 < dimN(T - \lambda I) < \infty\}$$

and

$$\pi_{00}^a(T) := \{\lambda \in iso\sigma_a(T) : 0 < dimN(T - \lambda I) < \infty\}.$$

We say that Weyl's theorem holds for $T \in \mathcal{B}(\mathcal{H})$ if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. The following two variants of Weyl's theorem have been introduced by Rakočević [15, 16].

Definition 2.1. *A bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ is said to satisfy property (ω), denoted by $T \in (\omega)$ if*

$$\sigma_a(T)\backslash\sigma_{ea}(T) = \pi_{00}(T),$$

and is said to satisfy a-Weyl's theorem, if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T),$$

and that a-Browder's theorem holds for T if $\sigma_{ea}(T) = \sigma_{ab}(T)$.

More recent papers have investigated property(ω) and a-Weyl's theorem([7][10]). From the definitions of those Weyl type theorem, if $T \in \mathcal{B}(\mathcal{H})$, we see that [11, 14]

$$\text{either a-Weyl's theorem or property } (\omega) \Rightarrow \sigma_{ea}(T) = \sigma_{ab}(T).$$

This paper will focus on the characteristics of operators and their functional calculus that satisfies a-Weyl's theorem and property (ω) simultaneously. In Section 2, the preliminary definition and the main terminology are given. In Section 3, using the spectrum derived from topological uniform descent, we provide the necessary and sufficient condition for bounded linear operators with both a-Weyl's theorem and property (ω). Moreover, we will provide the necessary and sufficient conditions for the simultaneous existence of the two properties for the first time. Section 4 investigates how property (ω) and a-Weyl's theorem are simultaneously stable under compact perturbations.

3. A-Weyl's theorem and property (ω) for operators

In this section, before giving the main topic of this note, we find that there is no inevitable connection between a-Weyl's theorem and property(ω) in general.

Remark 3.1.

(i) Property(ω) cannot induce a-Weyl's theorem.

Example 3.2. *Let $A, B \in \mathcal{B}(\ell^2)$ be defined by*

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots) \text{ and } B(x_1, x_2, x_3, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Put $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ Then we have $\sigma_a(T) = \sigma_{ea}(T) = \{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\pi_{00}^a(T) = \{0\}$ and $\pi_{00}(T) = \emptyset$. This means that property(ω) holds for T, but a-Weyl's theorem fails for T.

(ii) A-Weyl's theorem cannot induce property (ω).

Example 3.3. *Let $A,\ B \in \mathcal{B}(\ell^2)$ be defined by*

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots) \text{ and } B(x_1, x_2, x_3, \cdots) = (x_1, 0, x_3, x_4, \cdots).$$

Let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$Then we have $\sigma_a(T) = \{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\pi_{00}^a(T) = \{0\}$ and $\pi_{00}(T) = \emptyset$. T satisfies a-Weyl's theorem, but not property (ω).

(iii) If T satisfies a-Weyl's theorem as well as property (ω), then $\pi_{00}(T) = \pi_{00}^a(T)$. A-Weyl's theorem and property (ω) are equivalent in this case.

Clearly, from Remark 3.1, there is no inevitable connection between a-Weyl's theorem and property (ω). It remains interesting for us to discuss the simultaneous existence of the two properties. For convenience, set

$$\mathcal{N} := \{T \in \mathcal{B}(\mathcal{H}) : T \text{ satisfies a-Weyl's theorem as well as property}(\omega)\}.$$

Theorem 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$.Then, the following properties are equivalent:*

(i) $T \in \mathcal{N}$;

(ii) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

Proof. $(i) \Rightarrow (ii)$, the inclusion "$\supseteq$" is clear. For the opposite inclusion, take arbitrarily $\lambda_0 \notin [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Without loss of generality, we assume that $\lambda_0 \in \sigma(T)$. Then $0 < n(T - \lambda_0 I) < \infty$. Besides, since $\lambda_0 \notin acc\sigma_{ea}(T)$, there exists some $\epsilon > 0$ such that $\lambda \in \rho_{ea}(T)$ if $0 < |\lambda - \lambda_0| < \epsilon$. There are two cases to consider:

Case 1 $\lambda_0 \notin \sigma_\tau(T)$.

(a) If there exists λ_1 satisfying $0 < |\lambda_1 - \lambda_0|$ small enough and $\lambda_1 \in \sigma_a(T)\backslash\sigma_{ea}(T)$, using the fact that property (ω) holds for T, we have that $T - \lambda_1 I$ is Browder which means that $\lambda_0 \in \partial\sigma(T)$. From Theorem 4.9 in [15], we get that $\lambda_0 \in \rho_D(T) = \mathbb{C}\backslash\sigma_D(T)$. Since $0 < n(T - \lambda_0 I) < \infty$, it follows that $\lambda_0 \in \rho_b(T)$. Hence $\lambda_0 \notin \sigma_b(T)$.

(b) Now, we suppose that $\lambda_2 \in \rho_a(T)$ if $0 < |\lambda_2 - \lambda_0|$ small enough. Then $\lambda_0 \in \pi^a_{00}(T)$. Since T satisfies a-Weyl's theorem and property (ω), it follows that $T - \lambda_0 I$ is Browder. Again, we get that $\lambda_0 \notin \sigma_b(T)$.

Case 2 $\lambda_0 \notin acc\sigma_a(T)$. In this case, $\lambda_0 \in iso\sigma_a(T)$ and $0 < n(T - \lambda_0 I) < \infty$ which implies that $\lambda_0 \in \pi^a_{00}(T)$. From the assumption (i) of this theorem, yields that $\lambda_0 \in \sigma(T) \setminus \sigma_b(T)$. Thus, $\lambda_0 \notin \sigma_b(T)$

$(ii) \Rightarrow (i)$, since $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \pi^a_{00}(T)\} \cap [\sigma_\tau(T) \cap acc\sigma_a(T)] = \{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \pi^a_{00}(T)\} \cap [acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}] = \emptyset$, we get that $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \pi^a_{00}(T)\} \cap \sigma_b(T) = \emptyset$, $\sigma_a(T)\backslash\sigma_{ea}(T) \subseteq \rho_b(T)$. Also since $\pi_{00}(T) \subseteq \pi^a_{00}(T) \subseteq \rho_b(T)$, $\sigma_a(T)\backslash\sigma_{ea}(T) = \pi_{00}(T) = \pi^a_{00}(T)$, it follows that T satisfies a-Weyl's theorem and property (ω). □

Remark 3.5. *If T satisfies a-Weyl's theorem and property (ω), then the four parts of $\sigma_b(T)$ in Theorem 3.4 are essential. Let us see the following examples.*

(i) Let A, $B \in \mathcal{B}(\ell^2)$ be defined by $A(x_1, x_2, x_3, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4} \cdots)$ and $B(x_1, x_2, x_3, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$. Put $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ Then a-Weyl's theorem and property (ω) hold for T. But $\sigma_b(T) \neq acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

(ii) Let $T \in \mathcal{B}(\ell^2)$ be the left shift operator, defined as $T(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, x_5 \cdots)$. It is evident that $T \in \mathcal{N}$. However, $\sigma_b(T) \neq [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

(iii) Let $T \in \mathcal{B}(\ell^2)$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, x_4, \cdots)$. Then T satisfies a-Weyl's theorem and property (ω). But $\sigma_b(T) \neq [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$.

(iv) Let $T \in \mathcal{B}(\ell^2)$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$. Also, a straightforward calculation shows that T obeys a-Weyl's theorem and property (ω). However, $\sigma_b(T) \neq [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$.

Corollary 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{N}$ if and only if $\pi^a_{00}(T) \subseteq \rho_\tau(T) \subseteq acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \rho_b(T)$.*

Proof. $\Rightarrow$ Note that according to $T \in \mathcal{N}$, $\pi^a_{00}(T) \subseteq [\sigma(T) \setminus \sigma_b(T)] \cup \rho_\tau(T)$. For $\lambda_0 \in \rho_\tau(T)$, if $\lambda_0 \notin acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Then $\lambda_0 \notin [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Theorem 3.4 yields that $\lambda_0 \notin \sigma_b(T)$ which means that $\lambda_0 \in \rho_b(T)$.

$\Leftarrow$ By the part (ii) of Theorem 3.4, we need only to prove that $\sigma_b(T) \subseteq [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. If $\lambda_0 \notin [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$, assume that $\lambda_0 \in \sigma(T)$. Then $0 < n(T - \lambda_0 I) < \infty$. Since $\lambda_0 \notin \sigma_\tau(T) \cap acc\sigma_a(T)$, we know that $\lambda_0 \in \pi^a_{00}(T)$ or $\lambda_0 \in \rho_\tau(T) \setminus [acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}]$. Thus, $\lambda_0 \in \pi^a_{00}(T)$ or $\lambda_0 \in \rho_b(T)$. By the assumption and since clearly $\pi^a_{00}(T) \cap [acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup \rho_b(T)] = \emptyset$, we infer that $\pi^a_{00}(T) \subseteq \rho_b(T)$. Therefore $\lambda_0 \in \rho_b(T)$ that is $\lambda \notin \sigma_b(T)$. □

Applying $\{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} = \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [\rho_a(T) \cap \sigma(T)]$, we obtain the following result.

Corollary 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{N}$ if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}\{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0\} \cup [\rho_a(T) \cap \sigma(T)]$.*

Recall that $T \in \mathcal{B}(\mathcal{H})$ is called a-isoloid if $iso\sigma_a(T) \subseteq \sigma_p(T)$, where $\sigma_p(T)$ denotes the point spectrum of T. For $T \in \mathcal{B}(\mathcal{H})$, it is well known that $\rho_a(T) \cap \sigma(T) \subseteq \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. From Theorem3.4, we obtain the following result:

Corollary 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{N}$ is a-isoloid if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [\rho_a(T) \cap \sigma(T)]$.*

Proof. $\Rightarrow$.Clearly, Theorem 3.4 implies that $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Now, when $\lambda_0 \notin acc\sigma_{ea}(T)$, we have that $\lambda_0 \in iso\sigma_a(T) \cup \rho_a(T)$ or $\lambda_0 \in \partial\sigma(T)$. Then by hypothesis T is a-isoloid, this means that $\{\lambda \in iso\sigma_a(T) : n(T - \lambda I) = 0\} = \emptyset$, so that $\{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} = [\rho_a(T) \cap \sigma(T)] \cup \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0 \text{ and } R(T - \lambda I) \text{ is not closed}\}$. Write $\Gamma = \{\lambda \in \sigma_a(T) : n(T - \lambda I) = 0 \text{ and} R(T - \lambda I) \text{ is not closed}\}$. Since $\{\lambda \in iso\sigma_a(T) : n(T - \lambda I) = 0\} = \emptyset$ it is known that $\Gamma \subseteq acc\sigma_a(T)$. When $\lambda \notin \sigma_\tau(T) \cup acc\sigma_{ea}(T)$, together with $\lambda_0 \in iso\sigma_a(T) \cup \rho_a(T)$ or $\lambda_0 \in \partial\sigma(T)$ ensure that $\lambda_0 \in iso\sigma_a(T) \cup \rho_a(T)$ or $\lambda_0 \in \rho_D(T)$. Hence $\lambda_0 \notin \Gamma$ hold for such λ_0, that is $\Gamma \subseteq \sigma_\tau(T) \cup acc\sigma_{ea}(T)$. Then it follows from a combination of $\Gamma \subseteq acc\sigma_a(T)$ and $acc\sigma_{ea}(T) \subseteq acc\sigma_a(T)$ that $\Gamma \subseteq [\sigma_\tau(T) \cup acc\sigma_{ea}(T)] \cap acc\sigma_a(T)$= $[\sigma_\tau(T) \cap acc\sigma_a(T)] \cup [acc\sigma_{ea}(T) \cap acc\sigma_a(T)]$= $[\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T)$, which implies that $\{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \subseteq [\rho_a(T) \cap \sigma(T)] \cup [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T)$. Thus $\sigma_b(T) \subseteq [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [\rho_a(T) \cap \sigma(T)]$ and the reverse inclusion is clear.

$\Leftarrow$ First note that $\rho_a(T) \cap \sigma(T) \subseteq \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$ which means that $\sigma_b(T) \subseteq [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$ and since the reverse inclusion is always authentic, combined with Theorem 3.4, we then conclude that $T \in \mathcal{N}$. Observe that $\{\lambda \in iso\sigma_a(T) : n(T - \lambda I) = 0\} \cap \Big\{[\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [\rho_a(T) \cap \sigma(T)]\Big\} = \emptyset$, and thus $\{\lambda \in iso\sigma_a(T) : n(T - \lambda I) = 0\} \cap \sigma_b(T) = \emptyset$ implies that $\{\lambda \in iso\sigma_a(T) : n(T - \lambda I) = 0\} \subseteq \rho_b(T)$. Since $n(T - \lambda I) = 0$, we obtain that $\{\lambda \in iso\sigma_a(T) : n(T - \lambda I) = 0\} \subseteq \rho(T)$. But $\{\lambda \in iso\sigma_a(T) : n(T - \lambda I) = 0\} \subseteq \sigma(T)$ is equivalent to saying that $\{\lambda \in iso\sigma_a(T) : n(T - \lambda I) = 0\} = \emptyset$, that is, T is a-isoloid. □

Obviously $\sigma_{ab}(T) \subseteq \sigma_a(T)$, then we replace $acc\sigma_a(T)$ by $acc\sigma_{ab}(T)$. Let $\sigma_c(T) := \{\lambda \in \mathbb{C} : R(T - \lambda I) \text{is not closed}\}$. Then, the following corollaries are easily obtained.

Corollary 3.9. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, the following statements are equivalent.*

(i) $T \in \mathcal{N}$;

(ii) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_{ab}(T)] \cup [\rho_\tau(T) \cap acc\sigma_{ea}(T)] \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acciso\sigma_a(T) \cap \sigma_c(T)]$.

Proof. $\Rightarrow$Follows immediately from Theorem 3.4, noting that $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\}$. Then $\sigma_\tau(T) \cap acc\sigma_a(T)$ can be decomposed into: $\sigma_\tau(T) \cap acc\sigma_a(T) = [\sigma_\tau(T) \cap acc\sigma_a(T) \cap acc\sigma_{ab}(T)] \cup \Big\{\sigma_\tau(T) \cap [acc\sigma_a(T) \setminus acc\sigma_{ab}(T)]\Big\} = [\sigma_\tau(T) \cap acc\sigma_{ab}(T)] \cup \Big\{\sigma_\tau(T) \cap [acc\sigma_a(T) \setminus acc\sigma_{ab}(T)]\Big\}$. For convenience, write $\Delta := \sigma_\tau(T) \cap [acc\sigma_a(T) \setminus acc\sigma_{ab}(T)]$, we get that $\Delta \subseteq \sigma_\tau(T) \cap acciso\sigma_a(T)$= $[\sigma_\tau(T) \cap acciso\sigma_a(T) \cap \sigma_c(T)] \cup [\sigma_\tau(T) \cap acciso\sigma_a(T) \cap \rho_c(T)] \subseteq [acciso\sigma_a(T) \cap \sigma_c(T)] \cup [\sigma_\tau(T) \cap \rho_c(T)]$. However,since $\sigma_\tau(T) \cap \rho_c(T) \subseteq \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$ it follows that $\Delta \subseteq [acciso\sigma_a(T) \cap \sigma_c(T)] \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$. Hence, $\sigma_\tau(T) \cap acc\sigma_a(T) \subseteq [\sigma_\tau(T) \cap acc\sigma_{ab}(T)] \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [acciso\sigma_a(T) \cap \sigma_c(T)]$. Note that $acc\sigma_{ea}(T) = [\sigma_\tau(T) \cap acc\sigma_{ea}(T)] \cup [\rho_\tau(T) \cap acc\sigma_{ea}(T)]$, together with $acc\sigma_{ea}(T) \subseteq acc\sigma_{ab}(T)$, we know that

$$acc\sigma_{ea}(T) \subseteq [\sigma_\tau(T) \cap acc\sigma_{ab}(T)] \cup [\rho_\tau(T) \cap acc\sigma_{ea}(T)]$$

. Therefore $\sigma_b(T) \subseteq [\sigma_\tau(T) \cap acc\sigma_{ab}(T)] \cup [\rho_\tau(T) \cap acc\sigma_{ea}(T)] \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acciso\sigma_a(T) \cap \sigma_c(T)]$. The converse inclusion is obvious.

$\Leftarrow$Note that $[\sigma_a(T) \setminus \sigma_{ea}(T)] \cap \Big\{[\sigma_\tau(T) \cap acc\sigma_{ab}(T)] \cup [\rho_\tau(T) \cap acc\sigma_{ea}(T)] \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acciso\sigma_a(T) \cap \sigma_C(T)]\Big\} = \emptyset$ and

$\pi_{00}^a(T) \cap \Big\{[\sigma_\tau(T) \cap acc\sigma_{ab}(T)] \cup [\rho_\tau(T) \cap acc\sigma_{ea}(T)] \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acciso\sigma_a(T) \cap \sigma_C(T)]\Big\} = \emptyset$. Hence, $\sigma_a(T) \setminus \sigma_{ea}(T) \subseteq \sigma_0(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T)$,$\pi_{00}(T) \subseteq \pi_{00}^a(T) \subseteq \sigma_0(T) \subseteq \sigma_a(T) \setminus \sigma_{ea}(T)$. Therefore $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}(T)$ and $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$ which implies that $T \in \mathcal{N}$. □

It is known that $acciso\sigma_a(T) \cap \sigma_c(T) \subseteq acciso\sigma_a(T)$ and $\rho_{ea}(T) \cap acciso\sigma_a(T) = \emptyset$. We can improve the previous result.

Corollary 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{N}$ if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_{ab}(T)] \cup [\rho_\tau(T) \cap acc\sigma_{ea}(T)] \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup acciso\sigma_a(T)$.*

Let $\pi_{00}^{af}(T) := \{\lambda \in iso\sigma_a(T) : n(T-\lambda I) < \infty\}$. Then, similar to Corollary 3.6, we get:

Corollary 3.11. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{N}$ is a-isoloid if and only if $\pi_{00}^{af}(T) \subseteq \rho_\tau(T) \subseteq acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup [\rho_a(T) \cap \sigma(T)] \cup \rho_b(T)$.*

Furthermore, replacing in the above corollaries the assumption $acc\sigma_{ea}(T)$ by $int\sigma_{ea}(T)$. Since $int\sigma_{ea}(T) \subseteq acc\sigma_{ea}(T)$, we get:

Corollary 3.12. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, the following statements are equivalent.*

(i) $T \in \mathcal{N}$ *and* $\sigma(T) = \sigma_a(T)$;

(ii) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$;

(iii) $\pi_{00}^a(T) \subseteq \rho_\tau(T) \subseteq acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} \cup \rho_b(T)$;

(iv) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_{ab}(T)] \cup [\rho_\tau(T) \cap acc\sigma_{ea}(T)] \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} \cup acciso\sigma_a(T)$;

(v) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup int\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\}$;

(vi) $\pi_{00}^a(T) \subseteq \rho_\tau(T) \subseteq int\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} \cup \rho_b(T)$.

Corollary 3.13. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, the following statements are equivalent.*

(i) $T \in \mathcal{N}$ *is a-isoloid and* $\sigma(T) = \sigma_a(T)$;

(ii) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup acc\sigma_{ea}(T)$;

(iii) $\pi_{00}^{af}(T) \subseteq \rho_\tau(T) \subseteq acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \rho_b(T)$;

(iv) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup int\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}$;

(v) $\pi_{00}^{af}(T) \subseteq \rho_\tau(T) \subseteq int\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \rho_b(T)$.

We will provide another characterization of the operators that satisfy property (ω) and a-Weyl's theorem below.

Theorem 3.14. *Let $T \in \mathcal{B}(\mathcal{H})$. Then, the following statements are equivalent.*

(i) $T \in \mathcal{N}$;

(ii) $\sigma_{ab}(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup int\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}$ *and* $\pi_{00}(T) = \pi_{00}^a(T)$.

4. Property (ω) and a-Weyl's theorem for functions of operators

Next, the property (ω) and a-Weyl's theorem for functions of operators will then be explored via the topological uniform descent spectrum for operators. To begin with, we have the following facts. The following lemma directly results from [14], Lemma 2.9; its proof is omitted here.

Lemma 4.1 ([14],Lemma 2.9). *Let $T \in \mathcal{B}(\mathcal{H})$ and $f \in Hol(\sigma(T))$. Suppose that $f(T) - \mu_0 I = a(T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k}$, where $a \in \mathbb{C}$ and $\lambda_i \neq \lambda_j (i \neq j)$. If $\lambda_i \in iso\sigma_a(T)$, then $\mu_0 \in iso\sigma_a(f(T))$.*

Initially, we consider the a-Weyl's theorem for functions of operators.

Theorem 4.2 ([14],Theorem 1.1). *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T)$ satisfies a-Weyl's theorem for all $f \in Hol(\sigma(T))$ if and only if the following conditions are fulfilled:*

(i) T obeys a-Weyl's theorem;

(ii) $ind(T - \lambda I) \cdot ind(T - \mu I) \geq 0$ for any pairs of $\lambda, \ \mu \in \rho_{SF_+}(T)$;

(iii) If $\sigma_a(T) \backslash \sigma_{ab}(T) \neq \emptyset$, then $\sigma_{SF_+}(T) = acc\sigma_a(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$.

Let $\sigma_0(T)$ denote the set of all normal eigenvalues of T. Furthermore, $\sigma_0(T) := \sigma(T) \backslash \sigma_b(T)$. Similar to Theorem 1.2 in [14], we obtain the property (ω) for functions of operators by using the topological uniform descent spectrum.

Theorem 4.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T) \in (\omega)$ for all $f \in Hol(\sigma(T))$ if and only if the following conditions hold:*

(i) $T \in (\omega)$;

(ii) $ind(T - \lambda I) \cdot ind(T - \mu I) \geq 0$ for any pairs of $\lambda, \mu \in \rho_{SF_+}(T)$;

(iii) If $\sigma_0(T) \neq \emptyset$, then $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$.

An immediate consequence of the previous theorem is the following result.

Corollary 4.4. *Let $T \in \mathcal{B}(\mathcal{H})$. If $\sigma_0(T) \neq \emptyset$, then $f(T) \in (\omega)$ for all $f \in Hol(\sigma(T))$ if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$.*

Using the above corollaries, we have the following result.

Corollary 4.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T) \in \mathcal{N}$ for all $f \in Hol(\sigma(T))$ if and only if the following conditions hold:*

(i) $T \in \mathcal{N}$;

(ii) If $\rho_{SF}^-(T) \neq \emptyset$, then there exists no $\lambda \in \rho_{SF}(T)$ such that $0 < ind(T - \lambda I) < \infty$;

(iii) If $\sigma_0(T) \neq \emptyset$, then $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$.

Combining this fact with the previous corollary, we arrive at one of the main results of this section.

Theorem 4.6. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $f(T) \in \mathcal{N}$ is a-isoloid for all $f \in Hol(\sigma(T))$ if and only if the following conditions hold:*

(i) If $\rho_{SF}^-(T) \neq \emptyset$, then there exists no $\lambda \in \rho_{SF}(T)$ such that $0 < ind(T - \lambda I) < \infty$ and $\sigma_0(T) = \emptyset$;

(ii) If $\rho_{SF}^-(T) = \emptyset$, then $\sigma_{ea}(T) = \sigma_b(T)$;

(iii) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [\rho_a(T) \cap \sigma(T)]$.

Proof. "$\Rightarrow$"

(i) By virtue of Corollary 4.5, the first assertion is obvious.

(ii) If $\rho_{SF}^-(T) = \emptyset$, then $\sigma_{ea}(T) = \sigma_w(T)$. A-Weyl's theorem hold for T leads to $\sigma_{ea}(T) = \sigma_w(T) = \sigma_b(T)$.

(iii) We will distinguish two cases for $\sigma_0(T)$:

Case 1 $\sigma_0(T) \neq \emptyset$. Then Corollary 4.5 ensure that $\sigma_a(T) = \sigma(T)$, this implies that $\rho_a(T) \cap \sigma(T) = \emptyset$. Applying Corollary 4.5 again, then $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [\rho_a(T) \cap \sigma(T)]$.

Case 2 $\sigma_0(T) = \emptyset$. It easily follows from $T \in (\omega)$ that $\sigma_{ea}(T) = \sigma_a(T)$. Then $\sigma_\tau(T) \cap acc\sigma_a(T) = \sigma_\tau(T) \cap acc\sigma_{ea}(T) \subseteq acc\sigma_{ea}(T)$ and $\pi_{00}^a(T) = \emptyset$. Indeed, it only remains to see that $\sigma_b(T) \subseteq acc\sigma_a(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup [\rho_a(T) \cap \sigma(T)]$. For all $\lambda_0 \notin acc\sigma_a(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [\rho_a(T) \cap \sigma(T)]$. From $\lambda_0 \notin \rho_a(T) \cap \sigma(T)$, we may as well assume that $\lambda_0 \in \sigma(T)$. So $\lambda_0 \in \sigma_a(T)$. Hence $\lambda_0 \in iso\sigma_a(T)$ and $n(T - \lambda_0 I) < \infty$. Since T is a-isoloid, we have that $\lambda_0 \in \pi_{00}^a(T)$. Combining with $T \in (\omega)$ satisfies a-Weyl's theorem, we obtain that $\lambda_0 \notin \sigma_b(T)$.

Next, we show "$\Leftarrow$". By the hypothesis *(iii)*, we know that T is a-isoloid. On the other hand, by the assumption *(i)*, we infer that $f(T)$ is a-isoloid for all $f \in \mathrm{Hol}(\sigma(T))$. Let us consider two cases for $\rho_{SF}^-(T)$.

Case 1 Suppose that $\rho_{SF}^-(T) \neq \emptyset$.

In this situation, we have that $\sigma_0(T) = \emptyset$ and $ind(T - \lambda I) \leq 0$ for arbitrary $\lambda \in \rho_{SF}(T)$. Now, it follows from Corollary 4.5 that $f(T) \in (\omega)$ satisfies a-Weyl's theorem for all $f \in \mathrm{Hol}(\sigma(T))$.

Case 2 Suppose that $\rho_{SF}^-(T) = \emptyset$.

In this case, $ind(T - \lambda_0 I) \geq 0$ for all $\lambda_0 \in \rho_{SF}(T)$ which means that $\sigma_{ea}(T) = \sigma_b(T)$. Hence $\sigma_a(T) = \sigma(T)$, we get that $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cup [\rho_a(T) \cap \sigma(T)]$. Now, the use of Corollary 4.5 again allowed us to conclude that a-Weyl's theorem and property (ω) hold for all $f \in \mathrm{Hol}(\sigma(T))$. □

The other primary purpose of this subsection is to provide the necessary and sufficient condition for which the functions of the perturbation of $T \in \mathcal{B}(\mathcal{H})$ to obey both a-Weyl's theorem and property (ω). To this end, the following result will be crucial.

Lemma 4.7. *If $\sigma(T) = \sigma_{SF}(T) \cup \sigma_0(T)$, then given $\epsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ such that $\sigma(T + K) = \sigma_{SF}(T + K) \cup \sigma_0(T + K)$ and $iso\sigma(T + K) = \sigma_0(T + K)$.*

Proof. Put $\Lambda = iso\sigma(T) \cap \sigma_{SF}(T)$. Without loss of generality, we may assume that $\Lambda := \{\lambda_i : i = 1, 2, \cdots\} \neq \emptyset$. Indeed, according to Lemma 3.2.6 in [1] there exists a compact K_0 with $\|K_0\| < \frac{\epsilon}{2}$ such that

$$T + K_0 = \begin{pmatrix} \lambda_1 I_1 & * & * & \cdots & * \\ 0 & \lambda_2 I_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 I_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \vdots \\ \mathcal{H}_0 \end{matrix},$$

where $\sigma(T) = \sigma(A)$, $\sigma_{SF}(T) = \sigma_{SF}(A)$ and $ind(A - \lambda I) = ind(T - \lambda I)$ for all $\lambda \in \rho_{SF}(T)$. And I_i is the identity operator on H_i for $i \geq 1$. Since $\lambda_i \in iso\sigma(T)$, there is $\{\mu_{i,j}\}_{i,j\geq 1} \subseteq \rho(T)$ such that $\sup\limits_{j\to\infty} | \mu_{i,j} - \lambda_i | < \frac{\epsilon}{2^i}$ and $\mu_{i,j} \to \lambda_i$ as $j \to \infty$.

Put $C_i = diag\{\mu_{i,1} - \lambda_i, \mu_{i,2} - \lambda_i, \cdots\}$. Then C_i is compact and $\| C_i \| < \frac{\epsilon}{2^i}$.

Define

$$K_1 = \begin{pmatrix} C_1 & 0 & 0 & \cdots & 0 \\ 0 & C_2 & 0 & \cdots & 0 \\ 0 & 0 & C_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \vdots \\ \mathcal{H}_0 \end{matrix},$$

It is obvious that $K_1 \in \mathcal{K}(\mathcal{H})$ and $\| K_1 \| < \frac{\epsilon}{2}$. Denote $K = K_0 + K_1$. Then $K \in \mathcal{K}(\mathcal{H})$, $\| K \| < \epsilon$ and

$$T + K = \begin{pmatrix} E & * \\ 0 & A \end{pmatrix} \begin{matrix} \bigoplus_{i\geq 1} \mathcal{H}_i \\ \mathcal{H}_0 \end{matrix}$$

where E is diagonal with $\sigma_p(E) = \{\mu_{i,j}\}_{i,j\geq 1}$.

Claim $\sigma(T + K) = \sigma(T) \cup \{\mu_{i,j}\}_{i,j\geq 1}$.

Assume that $\lambda_0 \notin \sigma(T+K)$. Then $E-\lambda_0 I$ is bounded below. It follows that $E-\lambda_0 I$ is upper semi-Fredholm with $n(E-\lambda_0 I) = 0$. So, $\lambda_0 \notin \{\mu_{i,j}\}_{i,j\geq 1}$. Since E is normal, $E-\lambda_0 I$ is invertible. Thus, $A-\lambda_0 I$ is invertible. Also since $\sigma(T) = \sigma_(A)$, we have that $\lambda_0 \in \rho(T)$ and hence $\lambda_0 \notin \sigma(T) \cup \{\mu_{i,j}\}_{i,j\geq 1}$.

Conversely, let $\lambda_0 \notin \sigma(T) \cup \{\mu_{i,j}\}_{i,j\geq 1}$. By the invertibility of $T-\lambda_0 I$ and the perturbation theory of Fredholm operators, it follows that $T+K-\lambda_0 I$ is Weyl. Thus, $E-\lambda_0 I$ is an upper semi-Fredholm operator and $A-\lambda_0 I$ is invertible which means that $\lambda_0 \in \rho_w(E)$. Also since $\lambda_0 \notin \{\mu_{i,j}\}_{i,j\geq 1}$ and $\sigma_p(E) = \{\mu_{i,j}\}_{i,j\geq 1}$, it follows that $N(E-\lambda_0 I) = \{0\}$ and hence $E-\lambda_0 I$ is invertible. Therefore, $T+K-\lambda_0 I$ is invertible. This proves the claim.

It remains to show that $\sigma(T+K) = \sigma_{SF}(T+K) \cup \sigma_0(T+K)$ and $iso\sigma(T+K) = \sigma_0(T+K)$. We first prove that $\sigma(T+K) = \sigma_{SF}(T+K) \cup \sigma_0(T+K)$. Note that $\sigma(T+K) \supseteq \sigma_{SF}(T+K) \cup \sigma_0(T+K)$, we only need to show the conclusion "$\subseteq$". Suppose that $\lambda_0 \notin \sigma_{SF}(T+K) \cup \sigma_0(T+K)$. Then $T+K-\lambda_0 I$ is a semi-Fredholm operator. So is $T-\lambda_0 I$. It follows from $\sigma(T) = \sigma_{SF}(T) \cup \sigma_0(T)$ that $T-\lambda_0 I$ is Browder and hence $A-\lambda_0 I$ is Browder. We know that $T+K-\lambda_0 I$ is Weyl since $\lambda_0 \in \rho_w(T)$. We deduce that $E-\lambda_0 I$ is Weyl, and E is normal, ensuring that $E-\lambda_0 I$ is Browder. Thus $T+K-\lambda_0 I$ is Browder. Also since $\lambda_0 \notin \sigma_0(T+K)$, $\lambda_0 \notin \sigma(T+K)$. This shows that $\sigma(T+K) \subseteq \sigma_{SF}(T+K) \cup \sigma_0(T+K)$.

Now we will show that $iso\sigma(T+K) = \sigma_0(T+K)$. For a proof by contradiction, assume that $\lambda_0 \in iso\sigma(T+K)\backslash\sigma_0(T+K)$ exists. Then $\lambda_0 \in \sigma_{SF}(T)$ and there is $\epsilon > 0$ such that for $0 < |\lambda - \lambda_0| < \epsilon$, we have that $T+K-\lambda I$ is invertible. By the claim above, we derive that $\lambda_0 \in iso\sigma(T) \cap \sigma_{SF}(T)$. Then $\lambda_0 = \lambda_i$ for some $i \geq 1$ and hence there exists $\{\mu_{i,j}\}_{j\geq 1} \subseteq \sigma(T+K)$ satisfying $\mu_{i,j} \to \lambda_0 (j \to \infty)$, a contradiction. The proof is complete. □

Using the conclusion of the Lemma 4.7, we can get another main theorem of this paper.

Theorem 4.8. *Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma_w(T) = \sigma_{SF}(T)$. Then given $\epsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ with $||K|| < \epsilon$ such that $f(T+K) \in \mathcal{N}$ for all $f \in Hol(\sigma(T))$.*

Proof. For given $\epsilon > 0$, set $\sigma_1 = \{\lambda \in \sigma_0(T) : dist[\lambda, \partial\rho_{SF}(T)] \geq \epsilon\}$. Put $\sigma_2 = \sigma(T)\backslash\sigma_1$, then σ_1 is a finite clopen subset of $\sigma(T)$, and also σ_2 is a clopen subset of $\sigma(T)$. Then by [6, Theorem 2.10] T can be represented as

$$T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix} \begin{matrix} \mathcal{H}(\sigma_1, T) \\ \mathcal{H}(\sigma_2, T) \end{matrix},$$

where $\sigma(T_i) = \sigma_i$, $i = 1, 2$. Then $\sigma(T_1) = \sigma_1 = \sigma_0(T_1)$, it follows that $\sigma_p(T_1) = \sigma_1$. Since $max\{dist[\lambda, \partial\rho_{SF}(T_2)] : \lambda \in \sigma_0(T_2)\} < \epsilon$, by Lemma 5.2 in [15], there exists $\widehat{K} \in \mathcal{K}(\mathcal{H}(\sigma_2; T))$ with $||\widehat{K}|| < \frac{\epsilon}{2}$ such that $\sigma_p(T_2 + \widehat{K}) = \rho^+_{SF}(T_2 + \widehat{K})$. Hence $\sigma_{ea}(T_2 + \widehat{K}) = \sigma_a(T_2 + \widehat{K})$ and $\sigma_w(T_2 + \widehat{K}) = \sigma(T_2 + \widehat{K})$.

Denote

$$K_1 = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{K} \end{pmatrix} \begin{matrix} \mathcal{H}(\sigma_1, T) \\ \mathcal{H}(\sigma_2, T) \end{matrix}.$$

It is obvious that $K_1 \in \mathcal{K}(\mathcal{H})$ and $||K_1|| < \frac{\epsilon}{2}$ such that

$$T + K_1 = \begin{pmatrix} T_1 & * \\ 0 & T_2 + \widehat{K} \end{pmatrix} \begin{matrix} \mathcal{H}(\sigma_1, T) \\ \mathcal{H}(\sigma_2, T) \end{matrix}.$$

Claim 1. $\sigma(T + K_1) \subseteq \sigma(T)$.

In fact, suppose that $\lambda_0 \notin \sigma(T)$. Then $T_1 - \lambda_0 I$ is invertible and $T + K_1 - \lambda_0 I$ is Weyl. It follows that $T_2 + \widehat{K} - \lambda_0 I$ is Weyl, and so is invertible. Thus, $T + K_1 - \lambda_0 I$ is invertible. This proves the claim.

It remains to show that $\sigma(T + K_1) = \sigma_{SF}(T + K_1) \cup \sigma_0(T + K_1)$. It is obvious that $\sigma(T + K_1) \supseteq \sigma_{SF}(T + K_1) \cup \sigma_0(T + K_1)$. So it suffices to show the inclusion"$\subseteq$". Suppose that $\lambda_0 \notin \sigma_{SF}(T + K_1) \cup \sigma_0(T + K_1)$. Then $T + K_1 - \lambda_0 I$ is semi-Fredholm, so is $T - \lambda_0 I$. Since $\sigma_w(T) = \sigma_{SF}(T)$, $T - \lambda_0 I$ is Weyl. Then $\lambda_0 \in \rho_w(T + K_1)$. Also since T_1 is an operator in finite dimensional space, we infer that $\lambda_0 \in \rho_b(T)$. Then $\lambda_0 \in \rho_w(T_2 + \widehat{K})$.

It follows from $\sigma_w(T_2 + \widehat{K}) = \sigma(T_2 + \widehat{K})$ that $T_2 + \widehat{K} - \lambda_0 I$ is invertible. So $\lambda_0 \notin \sigma_b(T + K_1)$. By the fact that $\lambda_0 \notin \sigma_0(T + K_1)$, we get that $\lambda_0 \notin \sigma(T + K_1)$. This shows that $\sigma(T + K_1) = \sigma_{SF}(T + K_1) \cup \sigma_0(T + K_1)$. According to Lemma 4.7, there exists $K_2 \in \mathcal{K}(\mathcal{H})$ with $\|K_2\| < \frac{\epsilon}{2}$ such that $\sigma(T + K_1 + K_2) = \sigma_{SF}(T + K_1 + K_2) \cup \sigma_0(T + K_1 + K_2)$ and $iso\sigma(T + K_1 + K_2) = \sigma_0(T + K_1 + K_2)$. Denote $K = K_1 + K_2$. Then $K \in \mathcal{K}(\mathcal{H})$ with $\|K\| < \epsilon$ such that $\sigma(T + K) = \sigma_{SF}(T + K) \cup \sigma_0(T + K)$ and $iso\sigma(T + K) \subseteq \sigma_0(T + K)$.

Claim 2 $\sigma_a(T + K) = \sigma(T + K)$. In fact, the equality $\sigma(T + K) = \sigma_{SF}(T + K) \cup \sigma_0(T + K)$ implies that Claim 2 is proved.

Next, we prove that for the operator $T + K$, the three conditions in Theorem 4.6 are all established, So the conclusion in Theorem 4.8 holds.

(i) From the equation $\sigma(T + K) = \sigma_{SF}(T + K) \cup \sigma_0(T + K)$ and $iso\sigma(T + K) = \sigma_0(T + K)$, we can prove that $\sigma_a(T + K) \backslash \sigma_{ea}(T + K) = \pi_{00}(T + K) = \pi_{00}^a(T + K)$. This means $T + K \in \mathcal{N}$.

(ii) Since $\sigma_w(T) = \sigma_{SF}(T)$, which means that $\rho_w(T + K) = \rho_{SF}(T + K)$. Thus, when $\rho_{SF}^-(T) \neq \emptyset$, then there exists no $\lambda \in \rho_{SF}(T)$ such that $0 < ind(T - \lambda I) < \infty$.

(iii) If $\sigma_0(T + K) \neq \emptyset$, we should prove that $\sigma_b(T + K) = [\sigma_\tau(T + K) \cap acc\sigma_a(T + K)] \cup acc\sigma_{ea}(T + K) \cup \{\lambda \in \mathbb{C} : n(T + K - \lambda I) = \infty\}$.

Since $\sigma(T + K) = \sigma_{SF}(T + K) \cup \sigma_0(T + K)$, it follows that $acc\sigma_{ea}(T + K) = acc\sigma(T + K)$. Also since $iso\sigma(T + K) = \sigma_0(T + K)$, we get that $\sigma_b(T + K) \subseteq acc\sigma(T + K) = acc\sigma_{ea}(T + K)$. Then $\sigma_b(T + K) = acc\sigma_{ea}(T + K)$. The inclusion relation "$\sigma_b(T + K) \supseteq [\sigma_\tau(T + K) \cap \{\lambda \in \mathbb{C} : n(T + K - \lambda I) = \infty\}$" is clear. Then $\sigma_b(T + K) = [\sigma_\tau(T + K) \cap acc\sigma_a(T + K)] \cup acc\sigma_{ea}(T + K) \cup \{\lambda \in \mathbb{C} : n(T + K - \lambda I) = \infty\}$.

Therefore,the use of Theorem 4.6 allowed us to conclude that $f(T + K) \in \mathcal{N}$ for all $f \in \mathrm{Hol}(\sigma(T))$. □

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