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The clique partition edge-fault numbers of some networks

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Abstract. In a graph G, a clique partition of G is a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_q\}$ of $V(G)$ such that the induced subgraph $G[V_i]$ is a clique (called a clique of $\mathcal{P}$) for each $i \in [q]$. If a clique partition $\mathcal{P}$ also satisfies that $|G[V_i]| = t$ for each $i \in [q]$, the graph G is called a K_t-partitionable graph. A K_t-partition edge-fault set of G is a subset F of $E(G)$ such that the deletion of F results in a graph where no K_t-partitions exist. The K_t-partition edge-fault number of G, denoted by $f_t(G)$, is the smallest size among all K_t-partition edge-fault sets of G. The K_t-preclusion number of G, denoted by $g_t(G)$, is the minimum size of an edge subset A such that there exists at least one vertex in G not contained in any clique $K_t \subseteq G - A$. In this paper, we prove that arrangement graphs and data center networks are clique partitionable. Furthermore, arrangement graphs are shown to be clique decomposable. We determine the exact value of f_{n-k+1} for the arrangement graphs $A_{n,k}$ and establish bounds for $f_t(A_{n,k})$ and $f_t(K_n)$ for specific values of t. Additionally, we derive the exact values of g_3 for maximal planar graphs, g_r for Turán graphs $T(n, r)$, and f_t for graphs obtained from the arrangement graphs $A_{n,k}$ by shrinking a partition $\mathcal{R}$, for specific values of t.

1. Introduction

Clustering is an important aspect for designing interconnection networks. The idea is to partition the vertex set of a given graph (interconnection network) into groups of vertices (processors) such that each group of vertices is assigned an important task. Therefore, the vertices (processors) in each group will communicate with each other frequently, but less frequently with vertices (processors) outside of their group. With this assumption, one may want the subgraph induced by each group to be dense, perhaps a complete graph. So one may question the existence of such partitioning if there are vertex or edge faults. The most basic type is when each group consists of two vertices. For this special case, many problems (including NP-Hard problems) were introduced with many results obtained in the past decade. It is time to go beyond this basic setting and consider the case where each group has more than two vertices.

All graphs considered in this paper are undirected, finite and loopless. A *clique partition* of a graph G, denoted by $\mathcal{P}$, is a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_q\}$ of $V(G)$ such that the induced subgraph $G[V_i]$ is a clique (called a clique of $\mathcal{P}$) for each $i \in [q]$. If a clique partition $\mathcal{P}$ also satisfies that $|G[V_i]| = t$ for each $i \in [q]$, then the graph G is called *K_t-partitionable*. Given the focus on clique partitions where all cliques are of the same size, we call such partitions *uniform clique partitions*. A K_t-partitionable graph contains qt vertices for

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some positive integer q and at least $q\binom{t}{2}$ edges. In this paper, uniform clique partitions are often of a given size t. If $t = 2$, then G has a perfect matching. A *matching preclusion set* of a graph G is a set of edges whose deletion leaves the resulting graph with neither perfect matchings nor almost perfect matchings (matchings that cover all but one vertex in a graph with an odd number of vertices). The *matching preclusion number* of G, denoted by $mp(G)$, is the size of a smallest matching preclusion set of G. A matching preclusion set of minimum cardinality is called *optimal*. For graphs with an even number of vertices, clearly the set of edges incident to a single vertex is a matching preclusion set and such a set is called a *trivial matching preclusion set*. Moreover, as for an r-regular graph G, if G is *r-edge-colorable* which implies that G can be decomposed into r edge-disjoint perfect matchings, a matching preclusion set of G must intersect each of these r perfect matchings. We refer the readers to [8, 18, 19, 23, 26] for details and references.

In what follows we are going to give a generalization of perfect matchings. Let G be a simple graph with tq vertices for positive integers t and q, where $q \geqslant 2$. Let qK_t denote the disjoint union of q copies of complete graph on t vertices, K_t.

A graph G is *K_t-decomposable* if there is a partition of $E(G)$ into edge-disjoint spanning subgraphs, each of them isomorphic to qK_t, for some q. The necessary condition of a graph G being K_t-decomposable is that G contains qt vertices for some positive integer q and $mq\binom{t}{2}$ edges for some positive integer m. The existence of a K_t-decomposition for K_n is equivalent to the existence of a resolvable combinatorial design on n vertices with block size t. This problem has been extensively studied, with notable results such as the existence of Kirkman triple systems, which are resolvable Steiner triple systems decomposing K_n into triangles. In this context, each clique is akin to a “block”, and a clique partition is analogous to a “parallel class” (a partition of points into blocks of the same size), with their union referred to as a “resolution” (a partition of edges of K_n into clique partitions) [9, 16]. Although some networks considered in this paper are vertex-transitive, K_t-decomposable graphs are not essentially vertex-transitive. For example, each 3-edge-colorable cubic (3-regular) graph is K_2-decomposable but a few of them are vertex-transitive.

Clearly, if $t = 2$, a K_t-partitionable graph is a graph with a perfect matching. In other words, if a graph G has a perfect matching, then it is K_2-partitionable. Also, if t is a divisor of an integer k with $2 \leqslant t \leqslant k-1$, then the k-dimensional complete graph K_k is K_t-partitionable.

As a measure of the robustness of a graph G, the matching preclusion number is defined as the minimum number of edges that, upon removal, result in the absence of all perfect matchings (if the number of vertices, $|V(G)|$, is even) or almost perfect matchings (matchings that cover all but one vertex in a graph with an odd number of vertices). This concept can be generalized to K_t-partition edge-fault set. *A K_t-partition edge-fault set* of G is a subset of $E(G)$ whose deletion leaves the resulting graph with no K_t-partitions. *The K_t-partition edge-fault number* of G is the smallest size among all K_t-partition edge-fault sets of G, denoted by $f_t(G)$. In the following, we provide a definition for the K_t-preclusion number.

Definition 1. *Let t be a positive integer greater than 1, and let $G = (V, E)$ be a graph. For a vertex $v \in V$, if there exists $F \subset E$ such that v is not contained in any clique of order t in $G - F$, then F is called a K_t-preclusion set for v. The K_t-preclusion number of v in G, denoted by $g_{tv}(G)$, is defined as the minimum size among all K_t-preclusion sets for v in G. The K_t-preclusion number of G, denoted by $g_t(G)$, is the minimum among the K_t-preclusion numbers of all vertices in $V(G)$ i.e. $g_t(G) = min\{g_{tv}(G) : v \in V(G)\}$.*

If G is not K_t-partitionable, then $f_t(G) = 0$. By the definitions, $f_t(G) \leqslant g_t(G)$. Later, in Example 2, we will provide an illustration of graphs where $f_t(G) < g_t(G)$. Let G be a vertex-transitive graph. Then G is r-regular for a positive integer r and $g_t(G) = g_{tv}(G)$ for each vertex $v \in V(G)$. If $f_2(G) = g_2(G)$, then there exists a trivial matching preclusion set of G which isolates a single vertex of G. In a similar way, if $f_t(G) = g_t(G)$, there exists a K_t-preclusion set of G called a *trivial K_t-partition edge-fault set* with a minimum size which equals $f_t(G)$.

Plenty of interconnection networks, including fat trees [21] and hypercubes [17], have a recursive structure, that is, they can be constructed by some isomorphic subgraphs. To improve the computational efficiency, a large amount of networks take cliques which is a generalization of matchings as initial graphs, such as arrangement graphs [7] and DCell [11]. The *clique covering number* of a graph G is the minimum number of cliques in G needed to cover the vertex set of G. And the clique cover problem which is to find a minimum clique number was one of Richard Karp’s original 21 problems shown NP-complete [14].

References [15, 22] and [2, 5, 6] are consulted for some results on clique covering and edge clique covering, respectively. Additionally, [4] is employed as an instance to explore the problem within a special network architecture.

A network whose initial graph is a clique and having a partition of its vertex set such that the induced subgraph of each part of this partition is isomorphic to a complete graph with a fixed size will be considered. In [13], Átila A.Jones et al. show some results of edge clique partition. And in [10], Grahame Erskine et al. show the construction of clique-partitioned graphs.

In this paper, we prove that arrangement graphs and data center networks are clique partitionable. Furthermore, arrangement graphs are clique decomposable. We determine the exact values of $f_t(G)$ for specific arrangements, such as $f_{n-k+1}(A_{n,k})$ and establish bounds for $f_t(A_{n,k})$ and $f_t(K_n)$ for certain values of t. Furthermore, we derive the exact values of $g_3(G)$ for maximal planar graphs G, $g_r(T(n,r))$, and $f_t(G)$ for graphs obtained from the arrangement graphs $A_{n,k}$ by shrinking a partition $\mathcal{R}$, for specific values of t.

2. Preliminary

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex u in G, we define the *neighbourhood* of the vertex u in G by $N_G(u)$ (or simply say $N(u)$) and *closed neighbourhood* $N_G[u] = N_G(u) \cup \{u\}$. If $U \subseteq V(G)$, then $N_G(U) = (\bigcup_{v\in U} N_G(v)) \setminus U$. The *degree* of the vertex v in G is denoted by $d_G(v)$. The *induced subgraph* by V' in G, denoted by $G[V']$, is a graph whose vertex set is V' and whose edge set consists of all edges of G which have both ends in V'. If $d_G(v) = k$ for any vertex v of G, then G is a *k-regular graph*. For an edge e and a graph G, $G \cup \{e\}$ is abbreviated to $G \cup e$. Let $I_n^i = \{i, i+1, \ldots, n\}$ be the positive integer set. If $i = 1$, I_n^1 is abbreviated to $[n]$.

Let X and Y be subsets of $V(G)$, $E[X, Y]$ be the subset of $E(G)$ with one end in X and the other end in Y, and $e(X, Y)$ denote its number. If $Y = X$, we simply write $E(X)$ and $e(X)$ for $E[X, X]$ and $e(X, X)$, respectively. When $Y = V\backslash X$, the set $E[X, Y]$ is called the *edge cut* of G associated with X, or the *coboundary* of X, and is denoted by $\partial(X)$. An edge cut $\partial(v)$ associated with a single vertex v is a *trivial edge cut*.

Let X be a proper subset of $V(G)$. The resulting graph of G to *shrink* X, denoted by G/X, is the obtained graph from G by deleting all edges between vertices of X and then identifying the vertices of X into a single vertex. Given a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_p\}$ of $V(G)$, the graph of G to *shrink* $\mathcal{P}$ is the graph obtained from G by shrinking each V_i as a single vertex for each $i \in [p]$, and the resulting p-vertex graph is denoted by $G/\mathcal{P}$. Note that $G/\mathcal{P}$ might have multiple edges even if G is simple, but not any loops. Let $\mathcal{P} = \{V_1, V_2, \ldots, V_p\}$ be a partition of $V(G)$, if the induced subgraph by each part of $\mathcal{P}$ is isomorphic to a subgraph H of G, the partition $\mathcal{P}$ is called a *H-partition* of G. Clearly, H-partition is a generalization of K_t-partition.

The arrangement graphs were introduced by Day and Tripathi in [7]. The *(n, k)-arrangem-ent graph*, denoted by $A_{n,k}$, is defined for positive integers n and k such that $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$. The vertex set of $A_{n,k}$ is the set of k-permutations from $[n]$. Let $u = a_1a_2\cdots a_k$ be a vertex of $A_{n,k}$, then a_i is the ith-position of u for $i \in [k]$. Two vertices corresponding to the permutations $a_1a_2\cdots a_k$ and $b_1b_2\cdots b_k$ are adjacent if and only if they differ in exactly one position. Indeed, the graph $A_{n,n-1}$ is isomorphic to the star graph S_n and $A_{n,n-2}$ is isomorphic to the alternating group graph AG_n. Also, $A_{n,1}$ is isomorphic to the complete graph K_n. The n-dimensional star graph S_n is defined as follows. The n-dimensional star graph S_n has vertex set Sym(n) $= \{v = u_1u_2\cdots u_n | u_1u_2\cdots u_n$ is a permutation on $[n]\}$. Each vertex $v = u_1u_2\cdots u_n$ is adjacent to the following $n-1$ vertices $v_i = u_iu_2\cdots u_{i-1}u_1u_{i+1}\cdots u_n$, where $2 \leqslant i \leqslant n$. In fact, $A_{n,k}$ is a $k(n-k)$-regular vertex transitive graph with $n!/(n-k)!$ vertices [7].

A server-centric data center network called DCell was proposed by Guo et al. in [11], which can support millions of servers with high network capacity by only using commodity switches. For integers $k \geqslant 0$ and $n \geqslant 2$, the data center network of *k-dimensional DCell with n-port switches* is denoted by $D_{k,n}$. For notational convenience, let $t_{k,n} = |V(D_{k,n})|$. Let $D_{0,n}$ be a complete graph on n nodes such that each node has a distinct label chosen from $[n]$. Clearly, $t_{0,n} = n$. For $k \geqslant 0$ and $n \geqslant 2$, $t_{k,n} = t_{k-1,n}(t_{k-1,n} + 1)$. Then, $D_{k,n}$ can be defined as follows:

The data center network $D_{k,n}$ is a graph with the node set $V(D_{k,n}) = \{u_ku_{k-1}\cdots u_0 : u_i \in [t_{i-1,n} + 1]$ for $i \in [k]$ and $u_0 \in [n]\}$. Two nodes $u = u_ku_{k-1}\cdots u_0$ and $v = v_kv_{k-1}\cdots v_0$ in $V(D_{k,n})$ are adjacent if and only if there is an integer l such that

(1) $u_k u_{k-1} \cdots u_l = v_k v_{k-1} \cdots v_l$,

(2) $u_{l-1} \neq v_{l-1}$,

(3) $u_{l-1} = v_0 + \Sigma_{j=1}^{l-2}(v_j \times t_{j-1,n})$ and $v_{l-1} = u_0 + \Sigma_{j=1}^{l-2}(u_j \times t_{j-1,n}) + 1$ with $l > 1$ and $u_{l-1} < v_{l-1}$.

As a matter of fact, the DCell network $D_{k,n}$ can be built from $t_{k-1,n} + 1$ disjoint copies of $D_{k-1,n}$.

The book [1] is referred for notations and terminologies undefined here. For more properties about networks can be referred to [12, 24, 25].

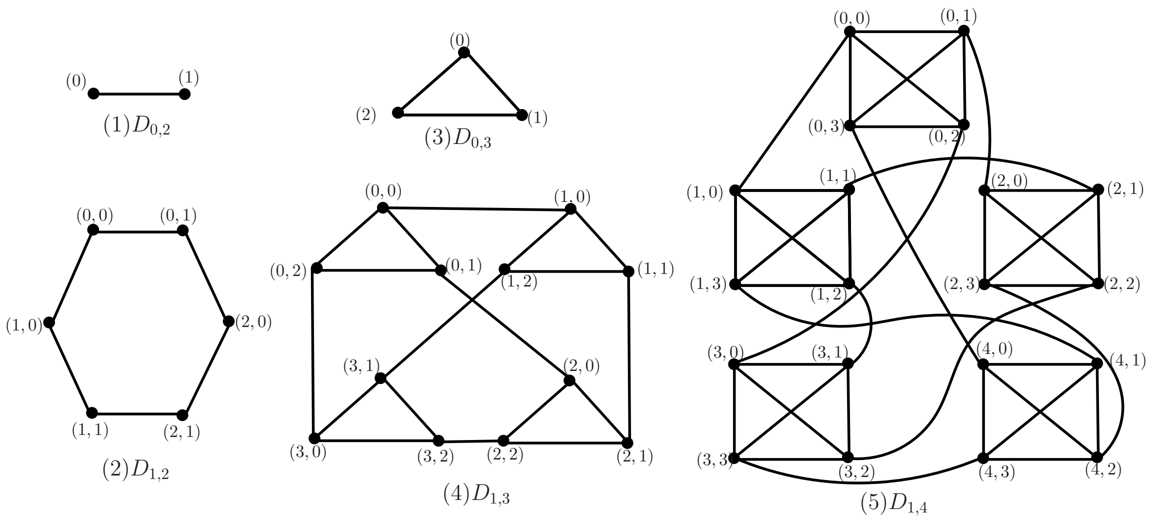

Figure 1: An illustration of data center networks $D_{k,n}$ for some k and n.

3. Main results

Theorem 1. *Let k, n be two integers such that $k \geqslant 1$ and $n \geqslant 2$. The (k, n)-data center networks $D_{k,n}$ is K_n-partitionable.*

Proof. Recall that $D_{0,n}$ is isomorphic to K_n and $V(D_{k,n})$ can be partitioned into $\mathcal{P} = \{V_1, V_2, \ldots, V_\xi\}$ such that $D_i = D_{k,n}[V_i] \cong D_{k-1,n}$ for $i \in [\xi]$, and $\xi = t_{k-1,n} + 1$ while $t_{k,n}$ denotes $|V(D_{k,n})|$. It implies that $D_{k,n}$ has a spanning subgraph K which is isomorphic to $\xi D_{k-1,n}$. By the inductive hypothesis, we assume that D_i has a spanning subgraph G_i which is isomorphic to qK_n for $q \geqslant 2$. Then $D_{k,n}$ has a spanning subgraph $\cup_{i \in [\xi]} G_i$ which is isomorphic to $\xi q K_n$. So $D_{k,n}$ is K_n-partitionable. □

3.1. The exact value of $f_{n-k+1}(A_{n,k})$

Let $A_{n,k}$ be the (n, k)-arrangement graph. Let $x = 12 \cdots k \in V(A_{n,k})$, and $A_k(x) = \{12 \cdots (k-1)s : s \in I_n^{k+1}\} \subset N_{A_{n,k}}(x)$. Indeed, any two vertices from $A_k(x) \cup \{x\}$ are adjacent in $A_{n,k}$. That is, the induced subgraph by $A_k(x) \cup \{x\}$ of $A_{n,k}$ is a $(n-k+1)$-dimensional complete graph K_{n-k+1}, denoted by $S_k(x)$. Note that $A_k(x)$ is the set of all vertices differing only in kth-position from x. Analogously, considering vertices differing exactly in ith-position from x for $i \in [k-1]$, each vertex subset $A_i(x)$ with $|A_i(x)| = n - k$ of $V(A_{n,k})$ and $S_i(x) \cong K_{n-k+1}$ for $i \in [k]$ can be obtained. Since $A_{n,k}$ is a $k(n-k)$-regular vertex transitive graph, for each vertex $u \in V(A_{n,k})$, $A_i(u)$ and $S_i(u)$ can be given directly as those of x. Furthermore, $A(u) = \cup_{i=1}^{k} A_i(u)$ and $S_i(u) \cong K_{n-k+1}$ for $i \in [k]$.

Lemma 1. *Let n, k be two integers such that $n \geqslant 3$ and $2 \leqslant k \leqslant n-1$. Then the (n, k)-arrangement graph $A_{n,k}$ is K_{n-k+1}-decomposable.*

Proof. For each vertex $u \in V(A_{n,k})$, let $A_i(u) = \{v : v$ differs in exactly ith-position from $u\}$, $S_i(u) = A_{n,k}[A_i(u) \cup \{u\}]$ and $A(u) = N_{A_{n,k}}(u) = \{v : v$ differs in exactly one position from $u\}$. Then $A(u) = \cup_{i=1}^{k} A_i(x)$ and $S_i(u) \cong K_{n-k+1}$ for $i \in [k]$. Let $H_i = \{S_i(u) : u \in V(A_{n,k})\}$ for $i \in [k]$. Each H_i is a spanning subgraph of $A_{n,k}$ and $H_i \cong P_n^{k-1} K_{n-k+1}$, in which P_n^{k-1} stands for the number of all $(k-1)$-permutations of n. Moreover, $E(H_i) \cap E(H_j) = \emptyset$ for any $i, j \in [k]$, $i \neq j$ and $\cup_{i=1}^{k} E(H_i) = E(A_{n,k})$. So $A_{n,k}$ is K_{n-k+1}-decomposable. □

Lemma 2. *Let G be a K_t-decomposable graph with m factors (the term "factor" refers to each of the H_i¡¯s), then $f_t(G) \geqslant m$.*

Proof. There exist m spanning subgraphs $H_1, H_2, \ldots, H_m$ of G and each of them is isomorphic to K_t such that $\cup_{i=1}^{m} E(H_i) = E(G)$ and $E(H_i) \cap E(H_j) = \emptyset$ for any $i \neq j$. Let $F \subset E(G)$. If $|F| \leqslant m - 1$, then there exists an $i \in [m]$ such that $F \cap H_i = \emptyset$. It implies that H_i is also a spanning subgraph of $G - F$, so $G - F$ is K_t-partitionable, that is, $f_t(G) \geqslant m$. □

It is straightforward to obtain Lemma 3, by Lemma 2.

Lemma 3. *Let t, q, m be three positive integers with $q \geq 2$ and G be a K_t-decomposable graph with tq vertices and $mq\binom{t}{2}$ edges. Thus $f_t(G) \geq m$.*

Theorem 2. *Let n, k be two integers such that $n \geqslant 3$ and $2 \leqslant k \leqslant n - 1$. If G is the (n, k)-arrangement graph $A_{n,k}$, then $f_{n-k+1}(G) = k$.*

Proof. By Lemma 1, the arrangement graph $A_{n,k}$ is K_{n-k+1}-decomposable with k factors. By Lemma 2, $f_{n-k+1}(A_{n,k}) \geqslant k$. Let F_0 be a subset of $E(G)$ with $|F_0| = k$ such that $|F \cap S_i(u)| = 1$ for each $i \in [k]$, then the vertex u will not be in any subgraph isomorphic to K_{n-k+1} in $G - F$. It implies that $f_{n-k+1}(A_{n,k}) \leqslant k$. Thus $f_{n-k+1}(A_{n,k}) = k$. □

Theorem 3. *Let n, k, l be three integers for $n \geqslant 3$ and $2 \leqslant k \leqslant n - 1$. Let $n - k + 1$ has a divisor t such that $2 \leqslant t \leqslant n - k$. If $f_t(K_{n-k+1}) = l$, then $f_t(A_{n,k}) \geqslant kl$.*

Proof. By Lemma 1, $A_{n,k}$ is K_{n-k+1}-decomposable, that is, there exist k spanning subgraphs $H_1, H_2, \ldots, H_k$ of $A_{n,k}$ and each of them is isomorphic to $P_n^{k-1} K_{n-k+1}$ such that $\cup_{i=1}^{k} E(H_i) = E(A_{n,k})$ and $E(H_i) \cap E(H_j) = \emptyset$ for any $i \neq j$. Let $F \subset E(A_{n,k})$ be any K_t-partition edge-fault set of $A_{n,k}$. If $f_t(K_{n-k+1}) = l$, we claim that $|F| \geqslant kl$. Otherwise, assume that $|F|<kl$. So there exists some $H_i \in \{H_1, H_2, \ldots, H_k\}$ such that $|F \cap H_i| < l$. Since $H_i \cong P_n^{k-1} K_{n-k+1}$ and $f_t(K_{n-k+1}) = l$, for any clique K_{n-k+1} in H_i, we have to delete at least l edges to make it not K_t-partitionable. However, since $|F \cap H_i| < l$, no clique K_{n-k+1} is not K_t-partitionable without F. Therefore, $H_i - F$ is K_t-partitionable. Since H_i is a spanning subgraph of $A_{n,k}$, $A_{n,k} - F$ is also K_t-partitionable, contradicting that F is a K_t-partition edge-fault set of $A_{n,k}$. So $|F| \geq kl$. By the arbitrariness of F, we have that $f_t(A_{n,k}) \geqslant kl$. □

Note that plenty of networks such as $A_{n,k}$ and $D_{k,n}$ are K_t-partitionable for some t. From Theorem 3, we find that some related results of $f_t(A_{n,k})$ will be obtained by exploring $f_t(K_n)$. However, it maybe difficult to determine the exact values of $f_t(K_n)$. Therefore, we give some bounds of $f_t(K_n)$ in the next subsection.

3.2. The bounds of $f_t(K_n)$

Let n and t be two positive integers with $n \geqslant t$. We can verify that $g_t(K_n) = n - t$ as K_n is a vertex-transitive graph and $g_t(K_n) = g_t v(K_n)$ for each vertex $v \in V(K_n)$. However, determining the exact value of $f_t(K_n)$ is not straightforward. We will now consider the bounds of $f_t(K_n)$. Using similar proof techniques as in Theorem 2, we derive Lemma 3.

Lemma 4. *Let G be a K_t-partitionable graph with minimum degree δ. Then $f_t(G) \leq \delta - t + 2$.*

Proof. Let v be a vertex with minimum degree δ of G. Let $F \subset \partial(v) \subset E(G)$ and $|F| = \delta - t + 2$. Since the vertex v has only $t - 2$ neighbours in $G - F$, implying that v is not included in any K_t subgraph of $G - F$, F is therefore a K_t-partition edge-fault set of G. Since $f_t(G)$ is the smallest size of all such sets, $f_t(G) \leqslant \delta - t + 2$. □

Theorem 4. *Let t be a divisor of a positive integer n with $2 \leqslant t \leqslant n - 1$. Then $f_t(K_n) \leqslant min\{n - t + 1, \binom{\frac{n}{t}+1}{2}\}$.*

Proof. Let $F \subset E(K_n)$. By Lemma 4, $f_t(K_n) \leqslant n-t+1$ since K_n is $(n-1)$-regular. Let $X \subset V(K_n)$ and $|X| = \frac{n}{t}+1$. Let $F_0 = E(K_n[X])$. Then $|F_0| = \binom{\frac{n}{t}+1}{2}$. Assume that $K_n - F_0$ is a K_t-partitionable graph, then $V(K_n - F_0) = V(K_n)$ has a partition $\mathcal{P}$ such that the induced subgraph by each part on $K_n - F$ is isomorphic to K_t. It implies that $|\mathcal{P}|= \frac{n}{t}$. Since X is an independent set in the graph $K_n - F_0$, no two vertices of X can be in the same part of $\mathcal{P}$, which implies that $|\mathcal{P}|\geq \frac{n}{t}+1$, which is a contradiction. So $K_n - F_0$ is not a K_t-partitionable graph which show that F_0 is a K_t-partition edge-fault set. As a result, $f_t(K_n) \leqslant |F| \leq \binom{\frac{n}{t}+1}{2}$. Based on the above discussions, one has that $f_t(K_n) \leqslant \min\{n-t+1, \binom{\frac{n}{t}+1}{2}\}$. □

Note. In most cases, especially as n increases relative to t, $\binom{\frac{n}{t}+1}{2}$ is larger than $n-t+1$. But sometimes $\binom{\frac{n}{t}+1}{2}$ is not larger than $n-t+1$. We give two examples with $\binom{\frac{n}{t}+1}{2} \leqslant n-t+1$ as following.

Example 1. For $n = 4$, $t = 2$, $n-t+1 = \binom{\frac{n}{t}+1}{2}$=3. We can check that $f_2(K_4) = 3$.

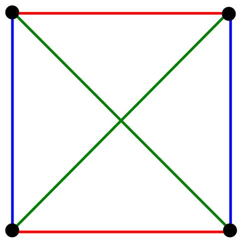

Figure 2: Example 1.

Example 2. For $n = 8$, $t = 4$, $n-t+1 = 5 \geq 3 = \binom{\frac{n}{t}+1}{2}$. We can check that $f_4(K_8) = 3$.

In this subsection, we mainly determine the upper bound of $f_t(K_n)$ which is equal to $\min\{n-t+1, \binom{\frac{n}{t}+1}{2}\}$. Does there exist a lower bound close to such a value? Nevertheless, we still believe that the exact value of $f_t(K_n)$ is equal to this upper bound and put it as a conjecture.

Conjecture. Let t be a divisor of a positive integer n with $2 \leqslant t \leqslant n-1$. Then $f_t(K_n) = \min\{n-t+1, \binom{\frac{n}{t}+1}{2}\}$.

3.3. The exact value of $g_3(G)$ for maximal planar graph G and $g_r(T(n,r))$

A *maximal planar graph* is a planar graph whose all faces are triangles. *Rotation systems* encode embeddings of graphs onto orientable surfaces by describing the circular ordering of a graph's edges around each vertex.

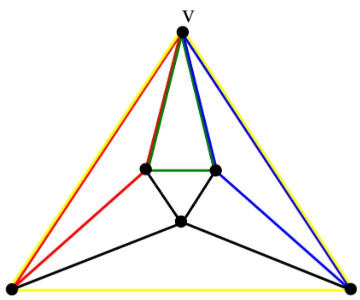

Figure 3: Triangles containing the vertex v in a maximal planar graph.

Theorem 5. *Let G be a maximal planar graph with minimum degree δ. If $v \in V(G)$ is a degree d vertex, then $g_3(v) = \lceil \frac{d}{2} \rceil$. Consequently, $g_3(G) = \lceil \frac{\delta}{2} \rceil$.*

Proof. Let $\tilde{G}$ be a planar embedding of the maximal planar graph G with a rotation system ϕ. Since $d(v) = d$, thus v is in d distinct triangles in $\tilde{G}$. By deleting one edge incident to v (resp. not incident to v), at most two triangles (resp. at most one triangles) containing v are destroyed. Thus $g_3(v) \geq \lceil \frac{d}{2} \rceil$. In the rotation ϕ, we label all edges incident to v clockwise by $e_1, e_2, \ldots, e_d$, respectively. If d is even, deleting all e_i with even $i \in [d]$ can destroy all triangles containing v. If d is odd, deleting all e_i with even $i \in [d]$ can destroy all triangles containing v except one, so we need to delete one more edge. Then $g_3(v) \leq \lceil \frac{d}{2} \rceil$. As a result, $g_3(v) = \lceil \frac{d}{2} \rceil$. By the minimum of $g_3(G)$, one has that $g_3(G) = \lceil \frac{\delta}{2} \rceil$. □

A *Turán graph*, denoted by $T(n, r)$ is a complete r-partite graph with n vertices whose each part has the size $\lfloor \frac{n}{r} \rfloor$ or $\lfloor \frac{n}{r} \rfloor + 1$. The following result is given.

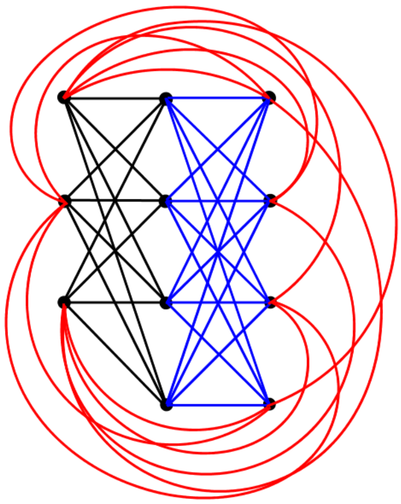

Figure 4: A Turán graph $T(11, 3)$.

Theorem 6. *Let n and r be two positive integers for $n \geqslant r$. Let $T(n, r)$ be a Turán graph. Then the value of $g_{rv}(T(n, r))$ is either $\lfloor \frac{n}{r} \rfloor$ or $\lfloor \frac{n}{r} \rfloor + 1$. Therefore $g_r(T(n, r)) = \lfloor \frac{n}{r} \rfloor$.*

Note. Let $\mathcal{P} = \{V_1, V_2, \ldots, V_r\}$ be a partition of $V(T(n, r))$ such that each part is an independent set with size $\lfloor \frac{n}{r} \rfloor$ or $\lfloor \frac{n}{r} \rfloor + 1$ of $T(n, r)$ with $n_i = |V_i|$. There exists the following equivalent representation of $g_{rv}(T(n, r))$: $g_{rv}(T(n, r)) = \lfloor \frac{n - n_i}{r - 1} \rfloor$ for each $v \in V_i$ and $i \in [r]$, that is, $g_{rv}(T(n, r))$ is the cardinality of the smallest part among all parts except that contains a vertex v.

Proof. We will prove that $g_{rv}(T(n, r))$ equals either $\lfloor \frac{n}{r} \rfloor$ or $\lfloor \frac{n}{r} \rfloor + 1$, that is $g_{rv}(T(n, r)) = \lfloor \frac{n - n_i}{r - 1} \rfloor$ for each $v \in V_i$ and $i \in [r]$, where $|V_i| = n_i$.

For each vertex $v \in V(T(n, r))$, choose one vertex from each part of $T(n, r)$ except the part containing v, then, these vertices together with v induce a clique with order r. We have to delete some edges in the clique. So we only need to consider the minimum part which is not containing v. Since the difference of the size of each part is at most one, one has that $g_{rv}(T(n, r)) \geq \lfloor \frac{n - n_i}{r - 1} \rfloor$. On the other hand, we can find the part V_j with $\lfloor \frac{n - n_i}{r - 1} \rfloor$ vertices, and let $F = E[v, V_j]$. Then $|F| = \lfloor \frac{n - n_i}{r - 1} \rfloor$ and F is a K_t-preclusion set of v in $T(n, r)$. Thus, $g_{rv}(T(n, r)) \leq \lfloor \frac{n - n_i}{r - 1} \rfloor$. As a result, $g_{rv}(T(n, r)) = \lfloor \frac{n - n_i}{r - 1} \rfloor$. Further more, $g_r(T(n, r)) = \min\{g_{rv}(T(n, r)) : v \in V(T(n, r))\} = \min\{\lfloor \frac{n - n_i}{r - 1} \rfloor : i \in [r]\} = \lfloor \frac{n}{r} \rfloor$. □

4. To shrink a partition for $A_{n,k}$

In this subsection, we focus on the graphs obtained from the arrangement graphs $A_{n,k}$ by shrinking a partition $\mathcal{R}$. We give two recursive decomposition approaches of $A_{n,k}$. And the K_t-partition edge-fault numbers of resulting graphs for some special t are determined.

By Lemma 1, we know that $A_{n,k}$ is K_{n-k+1}-decomposable and has a spanning subgraph H_i isomorphic to $P_n^{k-1}K_{n-k+1}$. Let $\mathcal{P}_i = \{V_1^i, V_2^i, \ldots, V_{P_n^{k-1}}^i\}$ be the partition of $V(A_{n,k})$ for $i \in [k]$ such that $A_{n,k}[V_j^i] \cong K_{n-k+1}$ for $j \in [P_n^{k-1}]$ and $H_i = \cup_{j\in[P_n^{k-1}]} A_{n,k}[V_j^i]$. For any vertex $u = a_1a_2\cdots a_k \in V(A_{n,k})$, $V(S_i(u))$ containing the vertex u is shrunk into a single vertex, denoted by the $(k-1)$-permutations $a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_k$. If $u, v \in V(A_{n,k})$ are adjacent and $v \notin S_i(u)$, then v differs in exactly jth-position from u for some $j \neq i$. Let $v = a_1a_2\cdots a_{i-1}a_ia_{i+1}\cdots a_{j-1}b_ja_{j+1}\cdots a_k$ and let $B = \{a_1a_2\cdots a_{i-1}ta_{i+1}\cdots a_{j-1}b_ja_{j+1}\cdots a_k : t \in [n]\backslash\{a_1, a_2, \ldots, a_k, b_j\}\} \subset S_i(v)$. Note that $|B| = n-k-1$ and each vertex of $B \cup \{v\}$ is adjacent to a vertex in $S_i(u)$, as they are different in exactly two positions i and j. It implies that there exist $n-k$ crossing edges between $S_i(u)$ and $S_i(v)$. The vertex set of $A_{n,k}/\mathcal{P}_i$ is the set of $(k-1)$-permutations from $[n]$. And the adjacency of two vertices in $A_{n,k}/\mathcal{P}_i$ corresponds to the edges between $S_i(u)$ and $S_i(v)$. Therefore, the simple subgraph obtained from $A_{n,k}/\mathcal{P}_i$ by deleting extra multiple edges is isomorphic to $A_{n,k-1}$ which is $(k-1)(n-k+1)$-regular. There are $n-k$ multiple edges between any two adjacent vertices in $A_{n,k}/\mathcal{P}_i$. Then $A_{n,k}/\mathcal{P}_i$ is $(k-1)(n-k+1)(n-k)$-regular.

By repeating this process, $A_{n,k-2}$ is a spanning subgraph of the graph obtained from $A_{n,k-1}$ by shrinking a K_{n-k+2}-partition, and $A_{n,k-3}$ is a spanning subgraph of the graph obtained from $A_{n,k-2}$ by shrinking a K_{n-k+3}-partition. Then by shrinking a series of clique partitions, $A_{n,1}$ can be obtained. The resulting graph is $(n-k)!$-regular with $(n-k)!$ multiple edges between any two adjacent vertices. Recall that $A_{n,1} \cong K_n$.

Let $J_{i,j}$ be the subgraph of $A_{n,k}$ induced by the vertices whose ith-position is j for $i \in [k]$ and $j \in [n]$. It implies that $J_{i,j} \cong A_{n-1,k-1}$. For a given integer $i \in [k]$, $J_{i,1}$, $J_{i,2}$, …, $J_{i,n}$ are n disjoint subgraphs of $A_{n,k}$. Let $J_i = \cup_{j\in[n]} J_{i,j}$. Then J_i is a spanning subgraph of $A_{n,k}$. For any $i, l \in [k]$, $E(J_i) \cap E(J_l) = \emptyset$ and $\cup_{i=1}^{k} E(J_i) = E(A_{n,k})$.

Let $\mathcal{Q}_i = \{X_1^i, X_2^i, \ldots, X_n^i\}$ be the partition of $V(A_{n,k})$ for $i \in [k]$ such that $A_{n,k}[X_j^i] = J_{i,j}$ for $j \in [n]$. For any $m, s \in [n]$, there are $\frac{(n-2)!}{(n-k-1)!}$ crossing edges between $J_{i,m}$ and $J_{i,s}$. So $A_{n,k}/\mathcal{Q}_i$ contains the simple graph isomorphic to $A_{n,1} \cong K_n$.

In fact, X_j^i is the set of vertices with i-th position is j (only one position is fixed). Generally, for a given $\gamma \in [k]$, consider γ positions being fixed. Without loss of generality, assume that these γ positions has ζ possible choices. Let $\mathcal{R} = \{V_1, V_2, \ldots, V_\zeta\}$ be a partition of $V(A_{n,k})$, where V_τ is the set corresponding to τ-th choice of γ positions and the subgraph induced by V_τ is isomorphic to $A_{n-\gamma,k-\gamma}$ for $\tau \in [\zeta]$ and $\gamma \in [k-1]$. The partition is called *a recursive partition* of $V(A_{n,k})$.

Each V_τ is shrunk into a vertex in $A_{n,k}/\mathcal{R}$. Since γ positions are fixed in $\mathcal{R}$, the vertex set of $A_{n,k}/\mathcal{R}$ is the set of γ-permutations from $[n]$. And the adjacency of two vertices in $A_{n,k}/\mathcal{R}$ corresponds to the edges between two copies of $A_{n-\gamma,k-\gamma}$. Therefore, the simple subgraph obtained from $A_{n,k}/\mathcal{R}$ by deleting extra multiple edges is isomorphic to $A_{n,\gamma}$ and there are $\frac{(n-\gamma-1)!}{(n-k-1)!}$ multiple edges between any two adjacent vertices in $A_{n,k}/\mathcal{R}$. Please see Figure 6 and Figure 7 as examples.¡¡ (For example, there are $\frac{(n-2)!}{(n-k-1)!}$ multiple edges between any two adjacent vertices in $A_{n,k}/\mathcal{Q}_i$.) Based one the above analysis, the graph $A_{n-\gamma+1,k-\gamma+1}$ can be decomposed into $A_{n-\gamma,k-\gamma}$ for $\gamma \in [k-1]$. For the graph $A_{n,k}$, shrinking $\mathcal{P}_i$ and shrinking $\mathcal{Q}_i$ are two distinct methods. Since $A_{n,k}$ has a recursive decomposition of $A_{n-1,k-1}$, repeating the process, $A_{n,k}$ has a recursive decomposition of $A_{n-\gamma,k-\gamma}$.

Theorem 7. *Let n, k be two integers such that $n \geqslant 3$ and $2 \leqslant k \leqslant n-1$. If $\mathcal{R} = \{V_1, V_2, \ldots, V_\zeta\}$ is a recursive partition of $V(A_{n,k})$, where the subgraph induced by V_i in $A_{n,k}$ is isomorphic to $A_{n-\gamma,k-\gamma}$ for $i \in [\zeta]$ and $\gamma \in [k-1]$, then $A_{n,k}/\mathcal{R}$ is $K_{n-\gamma+1}$-decomposable and $f_{n-\gamma+1}(A_{n,k}/\mathcal{R}) = \gamma \cdot \frac{(n-\gamma-1)!}{(n-k-1)!}$.*

Proof. Recall that $V(A_{n,k})$ is the set of k-permutations from $[n]$. Fix γ positions of the k-permutations by using γ numbers from $[n]$. When the γ numbers are placed in the fixed γ positions, then the k-permutations are obtained by arranging $k-\gamma$ member from $n-\gamma$ numbers, thus the set of these vertices induces a subgraph isomorphic to $A_{n-\gamma,k-\gamma}$.

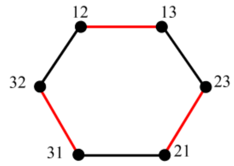

Figure 5: Partitions of $V(A_{3,2})$ such that the induced subgraph of each part is isomorphic to K_2 (red edges corresponding the partition of fixing the first position and black edges corresponding the partition of fixing the second position).

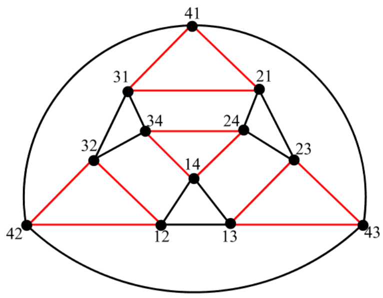

Figure 6: The subgraph J_2 containing $J_{2,1}$, $J_{2,2}$, $J_{2,3}$ and $J_{2,4}$ (red triangles) and the subgraph J_1 containing $J_{1,1}$, $J_{1,2}$, $J_{1,3}$ and $J_{1,4}$ (black triangles) in $A_{4,2}$.

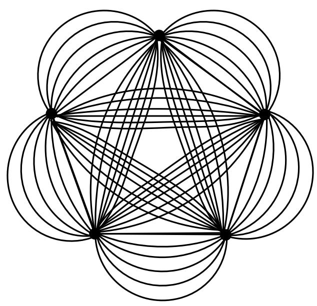

Figure 7: The graph $A_{5,3}/\mathcal{Q}_i$, where $\mathcal{Q}_i$ is a $A_{4,2}$-partition of $A_{5,3}$.

The γ given numbers are arbitrary, so there are P_n^{γ} disjoint subgraphs isomorphic to $A_{n-\gamma,k-\gamma}$ in $A_{n,k}$.

Without loss of generality, we assume that $\mathcal{R}$ is obtained by arranging γ numbers from $[n]$ into the first γ positions, that is, vertices with the same numbers in the first γ positions are in the same part of $\mathcal{R}$. If two adjacent vertices of $A_{n,k}$ are in different parts of $\mathcal{R}$, then they differ in exactly one of the first γ positions. Recall that $\mathcal{R} = \{V_1, V_2, \ldots, V_\zeta\}$. For $i \in [\zeta]$, we denote the vertex $\bar{V}_i$ by the vertex in $A_{n,k}/\mathcal{R}$ corresponding to V_i.

Assume that $u = a_1a_2\cdots a_k \in V_x$ and $v = b_1b_2\cdots b_k \in V_y$ with $x \neq y$ of $\mathcal{R}$. Vertices u and v are adjacent in $A_{n,k}$ if and only if there is an integer $t \in [\gamma]$ such that $a_i \neq b_i$ while $a_j = b_j$ for any $j \in [k]$ and $j \neq i$ which implies that $\bar{V}_x$ and $\bar{V}_y$ is adjacent in $A_{n,k}/\mathcal{R}$.

To determine the number of crossing edges between any two adjacent parts V_x and V_y, count the vertices in both parts that share exactly the same $k-\gamma$ numbers in the last $k-\gamma$ positions. So there are $P_{n-\gamma-1}^{k-\gamma}$ crossing edges between two adjacent parts of $\mathcal{R}$. The graph obtained from $A_{n,k}/\mathcal{R}$ by deleting extra multiple edges is isomorphic to $A_{n,\gamma}$. And there are $P^{k-\gamma}_{n-\gamma-1}$ multiple edges between any two adjacent vertices $\bar{V}_x$ and $\bar{V}_y$ in $A_{n,k}/\mathcal{R}$.

By Lemma 1, $A_{n,\gamma}$ is $K_{n-\gamma+1}$-decomposable. There are exactly $P^{k-\gamma}_{n-\gamma-1}$ edge-disjoint subgraphs isomorphic to $A_{n,\gamma}$ in $A_{n,k}/\mathcal{R}$. So $A_{n,k}/\mathcal{R}$ is $K_{n-\gamma+1}$-decomposable. By Theorem 3, $f_{n-\gamma+1}(A_{n,k}/\mathcal{R}) \geqslant \gamma \cdot \frac{(n-\gamma-1)!}{(n-k-1)!}$. On the other hand, by Theorem 2, $f_{n-\gamma+1}(A_{n,\gamma}) = \gamma$. We have to delete all $P^{k-\gamma}_{n-\gamma-1}$ multiple edges between two adjacent vertices to destroy the adjacency in $A_{n,k}/R$ compared with in $A_{n,\gamma}$. So $f_{n-\gamma+1}(A_{n,k}/\mathcal{R}) \leqslant \gamma \cdot \frac{(n-\gamma-1)!}{(n-k-1)!}$. To sum up, $f_{n-\gamma+1}(A_{n,k}/\mathcal{R}) = \gamma \cdot \frac{(n-\gamma-1)!}{(n-k-1)!}$. □

5. Conclusion

In this paper, we introduce several new concepts, including K_t-partitionable graph, K_t-partition edge-fault number $f_t(G)$ of G and K_t-preclusion number $g_t(G)$ of G which generalize perfect matching and matching preclusion number. Then for specific values of t and for various networks such as arrangement graphs, the complete graphs and data center graphs, we determine either the exact value or the bounds of $f_t(G)$ and $g_t(G)$. Clique partitions play a crucial role in maintaining the overall structure of networks and can help increase their diameters [20]. We hope that the clique partition structure can be applied to the base graphs of blockchain and neural networks. Exploring the values of $f_t(G)$ and $g_t(G)$ for various networks G would be an interesting area for further investigation. Additionally, further research into applications of K_t-partitions would be valuable.

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References

[1] J.A. Bondy, U.S.R Murty Theory, Springer, New York, 2007.
[2] Csilla Bujtás, Akbar Davoodi, Ervin Györi, Zsolt Tuza, Clique coverings and claw-free graphs, European J. Combin. **88** (2020) 103114.
[3] Michael Cavers, Jacques Verstraëte, Clique partitions of complements of forests and bounded degree graphs, Discrete Math. **308** (2008) 2011-2017.
[4] A. Davoodi, D. Gerbner, A. Methuku, M. Vizer, On clique coverings of complete multipartite graphs, Discrete Appl. Math. **276** (2020) 19-23.
[5] A. Davoodi, R. Javadi, B. Omoomi, Pairwise balanced designs and sigma clique partitions, Discrete Math. 339(5) (2016) 1450-1458.
[6] A. Davoodi, R. Javadi, B. Omoomi, Edge clique covering sum of graphs. Acta Mathematica Hungarica, **149** (2016) 82-91.
[7] K. Day, A. Tripathi, Arrangement graphs: a class of generalized star graphs, Inform. Process. Lett. **42** (1992) 235-241.
[8] Q. Ding, H. Zhang, H. Zhou, Matching preclusion for n-grid graphs, Discrete Appl. Math. **243** (2018) 194-206.
[9] J.H. Dinitz, Handbook of combinatorial designs, Chapman and Hall/CRC, 2007.
[10] Grahame Erskine, Terry Griggs, Jozef Širáň, Clique-partitioned graphs, Discrete Appl. Math. **314** (2022) 238-248.
[11] C. Guo, H. Wu, K. Tan, L. Shi, Y. Zhang, S. Lu, DCell: a scalable and fault-tolerant network structure for data centers, in: Special Interest Group on Data Communication, SIGCOMM, 2008, pp. 75-86.
[12] S.-Y. Hsien, Y.-H. Chang, Extraconnectivity of k-ary n-cube networks, Theoret. Comput. Sci. **443**(2012), 63-69.
[13] Átila A. Jones, Fábio Protti, Renata R.Del-Vecchio, Edge clique partition in (k, l)-graphs, Discrete Appl. Math. **306** (2022) 89-97.
[14] R.M. Karp, Reducibility among combinatorial problems, Complexity of Computer Computations (R.E. Miller and J.W. Thatcher, eds.), Plenum Press, 1972, pp. 85-103.
[15] Bo-Jr Li, Gerard J. Chang, Clique coverings and partitions of line graphs, Discrete Math. **308** (2008) 2075-2079.
[16] C.C. Lindner, C.A. Rodger, Design theory, Chapman and Hall/CRC, 2017.
[17] M. Ma, X. Li, G. Wang, Y. Zan, Generalized fault-tolerance for enhanced hypercubes, Discrete Appl. Math. **342** (2024) 244-252.
[18] T. Ma, Y. Mao, E. Cheng, P. Han, A note on the strong matching preclusion problem for data center networks, Inform. Process. Lett. **164** (2020) 106007.
[19] T. Ma, Y. Mao, E. Cheng, J. Wang, Fractional matching preclusion for arrangement graphs, Discrete Appl. Math. **270** (2019) 181-189.

[20] W. Martino, M. Quaintance, S. Popejoy, Chainweb : A proof-of-work parallel-chain architecture for massive throughput, 2018.
[21] S Prabhu, V Manimozhi, A Davoodi, JLG Guirao, Fault-tolerant basis of generalized fat trees and perfect binary tree derived architectures, The Journal of Supercomputing (2024).
[22] N.J. Pullman, Clique coverings of graphs - a survey, In: Casse, L.R.A. (eds) Combinatorial Mathematics X. Lecture Notes in Mathematics, 1983, vol 1036. Springer, Berlin, Heidelberg.
[23] J.-S. Yang, J.-M. Chang, K.-J. Pai, H.-C. Chan, Parallel Construction of Independent Spanning Trees on Enhanced Hypercubes, IEEE Trans. Parallel Distrib. Syst., 26(11) (2015) 3090–3097.
[24] H. Zhang, R.-X. Hao, X.-W. Qin, C.-K. Lin, S.-Y. Hsieh, The high faulty tolerant capability of the alternating group graphs, IEEE Transactions on Parallel and Distributed Systems 34(1) (2023) 225-233.
[25] S.-L. Zhao, R.-X. Hao, J. Wu, The generalized 3-connectivity of some Regular Networks, J. Parallel Distrib. Comput. **133**(2019), 18-29.
[26] B. Zhu, T. Ma, S. Zhang, H. Zhang, Fractional matching preclusion for data center networks, Parallel Process. Lett. **30** (2020) 2050010.