Filomat 38:22 (2024), 7935–7940
https://doi.org/10.2298/FIL2422935L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Q-spectral radius and $[a, b]$-factors of graphs

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Abstract. An $[a, b]$-factor of a graph G is a spanning subgraph H in which the degree of each vertex v satisfies $a \leq d_H(v) \leq b$. In particular, when $a = b = k$, it is also called a k-factor. Let $Q(G)$ and $q(G)$ be the Q-matrix and the Q-spectral radius of G, respectively. Motivated by the conjecture of Cho, Hyun, O and Park [Bull. Korean Math. Soc. 58 (2021) 31–46] and the result of the spectral radius obtained by Fan, Lin and Lu [Discrete Math. 345 (2022) 112892], we in this paper consider the Q-spectral version of the above conjecture and present a tight sufficient condition in terms of the Q-spectral radius to guarantee the existence of $[a, b]$-factors in a graph.

1. Introduction

In this paper, we only consider finite, undirected and simple graphs. For undefined notations and terms, one can refer to [1]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of G are denoted by $|V(G)| = n$ and $|E(G)| = e(G)$, respectively. For any vertex v in G, we use $N_G(v)$ to denote the set of vertices adjacent to v and use $d_G(v)$ to denote the degree of vertex v. Let G_1 and G_2 be two disjoint graphs. The disjoint union of G_1 and G_2 is the graph G with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$, denoted by $G = G_1 + G_2$. The joint of G_1 and G_2 is the graph G with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$, denoted by $G = G_1 \vee G_2$.

The adjacency matrix of G is defined to be the matrix $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} \in \{0, 1\}$ and $a_{ij} = 1$ if and only if there is an edge v_iv_j in $E(G)$. The Q-matrix $Q(G)$ of graph G is defined as $Q(G) = A(G) + D(G)$, where $D(G)$ is the diagonal degree matrix of G, i.e., $D(G) = \mathrm{diag}\{d(v_1), d(v_2), \ldots, d(v_n)\}$. We define the largest eigenvalue of $A(G)$ as the spectral radius of graph G, denoted by $\rho(G)$, and the largest eigenvalue of $Q(G)$ as the Q-spectral radius of G, denoted by $q(G)$.

Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all vertex x in $V(G)$. A (g, f)-factor of G is a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for any vertex x in $V(G)$. Let a and b be two positive integers with $b \geq a \geq 1$. A (g, f)-factor is called an $[a, b]$-factor if $g(x) \equiv a$ and $f(x) \equiv b$. In particular, for a positive integer k, a $[k, k]$-factor of a graph G is called a k-factor of G. The study of factors in graphs has a rich history. In 1952, Tutte [9] provided a necessary and sufficient condition for the existence of k-factors in graphs. In 1970, Lovász [7] presented a necessary and sufficient condition

2020 *Mathematics Subject Classification*. Primary 05C50; Secondary 05C35.
Keywords. $[a, b]$-factors, Q-matrix, Q-spectral radius.
Received: 13 January 2024; Revised: 26 February 2024; Accepted: 05 March 2024
Communicated by Paola Bonacini
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for the existence of (g, f)-factors in graphs, which is known as the (g, f)-factor theorem. In 2021, Cho, Hyun, O and Park [2] proposed a conjecture on the existence of an $[a, b]$-factor based on the spectral radius of G.

Conjecture 1.1 ([2]). *Let a, b be two positive integers such that $1 \le a \le b$. Suppose that G is a graph of order $n \ge a + 1$, where $n \cdot a$ is even. If*

$$\rho(G) > \rho(K_{a-1} \vee (K_{n-a} + K_1)),$$

then G contains an $[a, b]$-factor.

In 2022, Fan, Lin and Lu [3] confirmed the above conjecture when $n \ge 3a + b - 1$.

Theorem 1.1 ([3]). *Let a, b be two positive integers such that $1 \le a \le b$. Suppose that G is a graph of order n. If $n \cdot a$ is even, $n \ge 3a + b - 1$ and*

$$\rho(G) > \rho(K_{a-1} \vee (K_{n-a} + K_1)),$$

then G contains an $[a, b]$-factor.

In this paper, we consider the Q-spectral version of the above conjecture and prove a tight sufficient condition based on the Q-spectral radius to assure the existence of $[a, b]$-factors in a graph.

Theorem 1.2. *Let a, b be two positive integers such that $1 \le a \le b$. Suppose that G is a graph of order n. If $n \cdot a$ is even, $n \ge 3a + b - 1$ and*

$$q(G) \ge q(K_{a-1} \vee (K_{n-a} + K_1)),$$

then G contains an $[a, b]$-factor unless $G \cong K_{a-1} \vee (K_{n-a} + K_1)$ or $K_{1,3}$.

By Theorem 1.2, we can directly obtain the following corollary.

Corollary 1.1. *Let G be a graph of order $n \ge 4k - 1$, $n \cdot k$ even and $k \ge 1$. If*

$$q(G) \ge q(K_{k-1} \vee (K_{n-k} + K_1)),$$

then G contains a k-factor unless $G \cong K_{k-1} \vee (K_{n-k} + K_1)$.

2. Preliminaries

In this section, we will present some important lemmas that support our proof. Yu and Fan [10] presented a sufficient condition in terms of the Q-spectral radius to assure the existence of Hamilton paths or Hamilton cycles.

Lemma 2.1 ([6],[10]). *If G is a connected graph of order $n \ge 3$ and*

$$q(G) \ge 2n - 4,$$

then G contains a Hamilton path unless $G \cong K_{1,3}$. If the above inequality holds strictly, then G contains a Hamilton cycle unless $G \cong K_2 \vee 3K_1$ or $K_1 \vee (K_{n-2} + K_1)$.

In 1998, Li and Cai [5] obtained a degree condition for a graph to have $[a, b]$-factors.

Lemma 2.2 ([5]). *Let a, b be two positive integers such that $b > a \ge 1$. Suppose that G is a graph of order n with minimum degree $\delta(G) \ge a$. If $n \ge 2a + b + \frac{a^2-a}{b}$ and*

$$\max\{d_G(u), d_G(v)\} \ge \frac{an}{a+b}$$

for any two nonadjacent vertices u and v of G, then G contains an $[a, b]$-factor.

Nishimura [8] proved a result to guarantee the existence of k-factors in a graph, which can be seen as a special case of the above result.

Lemma 2.3 ([8]). *Let G be a connected graph of order $n \geq 4k-3$ with minimum degree $\delta(G) \geq k$, where k is a positive integer such that $k \geq 3$. If $n \cdot k$ is even and*

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$$

for any two nonadjacent vertices u and v of G, then G contains a k-factor.

Lemma 2.4 ([4]). *Let G be a connected graph and $X = (x_v)_{v \in V(G)}$ be the Perron vector of $Q(G)$. Assume that $u_1 v \notin E(G)$ while $u_2 v \in E(G)$. If $x_{u_1} \geq x_{u_2}$, then $q(G - u_2 v + u_1 v) > q(G)$.*

3. Proof of Theorem 1.2

We first prove an important lemma to support the proof our main result.

Lemma 3.1. *Let G be a connected graph of order n and $t \geq 1$ be an integer. If u and v are two nonadjacent vertices such that*

$$\max\{d_G(u), d_G(v)\} \leq t,$$

then $q(G) \leq q(K_t \vee (K_{n-t-2} + 2K_1))$, with equality if and only if $G \cong K_t \vee (K_{n-t-2} + 2K_1)$.

Proof. Let x be the Perron vector of $Q(G)$ and $V(G) \backslash \{u, v\} = \{v_1, v_2, \ldots, v_{n-2}\}$ with $x_{v_1} \geq x_{v_2} \geq \cdots \geq x_{v_{n-2}}$, where x_{v_i} corresponds to the vertex v_i. Define

$$\widetilde{G} = G - \{uw | w \in N_G(u)\} - \{vw | w \in N_G(v)\} + \{uv_i | 1 \leq i \leq d_G(u)\} + \{vv_i | 1 \leq i \leq d_G(v)\}.$$

Note that $\max\{d_G(u), d_G(v)\} \leq t$. By using repeatedly Lemma 2.4, we have $q(G) \leq q(\widetilde{G})$, where equality holds if and only if $G \cong \widetilde{G}$. Since $\widetilde{G}$ is a spanning subgraph of $K_t \vee (K_{n-t-2} + 2K_1)$, we have $q(\widetilde{G}) \leq q(K_t \vee (K_{n-t-2} + 2K_1))$. Hence we obtain that

$$q(G) \leq q(\widetilde{G}) \leq q(K_t \vee (K_{n-t-2} + 2K_1)),$$

with equality if and only if $G \cong K_t \vee (K_{n-t-2} + 2K_1)$. □

Now we are ready to give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let G be a graph of order $n \geq 3a + b - 1$ with $q(G) \geq q(K_{a-1} \vee (K_{n-a} + K_1))$, where $n \cdot a$ is even and $1 \leq a \leq b$. Assume that $G \ncong K_{a-1} \vee (K_{n-a} + K_1)$ and $K_{1,3}$. Next, it suffices to prove that G contains an $[a, b]$-factor.

Claim 1. *G is a connected graph.*

Proof. Suppose, to the contrary, that G is not connected. Let $G_1, G_2, \ldots, G_s$ ($s \geq 2$) be the connected components of G. Then

$$q(G) = \max\{q(G_1), q(G_2), \ldots, q(G_s)\} \leq q(K_{n-1}) = 2n - 4.$$

If $a \geq 2$, $q(G) \geq q(K_{a-1} \vee (K_{n-a} + K_1)) > q(K_{n-1}) = 2n - 4$, a contradiction. If $a = 1$, $q(G) \geq q(K_{n-1} + K_1) = 2n - 4$. Hence we have $q(G) = q(K_{n-1} + K_1) = 2n - 4$, which implies that $G \cong K_{n-1} + K_1$. This contradicts that $G \ncong K_{a-1} \vee (K_{n-a} + K_1) = K_{n-1} + K_1$. □

Claim 2. *$\delta(G) \geq a$.*

Proof. By Claim 1, we know that G is connected, and hence $\delta(G) \geq 1$. That is to say, Claim 2 holds for $a = 1$. Next we consider the case of $a \geq 2$. By contradiction, assume that $1 \leq \delta(G) \leq a - 1$. Then G is a spanning subgraph of $K_{a-1} \vee (K_{n-a} + K_1)$. Hence we have

$$q(G) \leq q(K_{a-1} \vee (K_{n-a} + K_1)).$$

Note that $q(G) \geq q(K_{a-1} \vee (K_{n-a} + K_1))$. Hence we have $q(G) = q(K_{a-1} \vee (K_{n-a} + K_1))$. This implies that $G \cong K_{a-1} \vee (K_{n-a} + K_1)$, a contradiction. □

Now we divide the following proof into three cases according to different values of a.

Case 1. $a = 1$.

In this case, we have

$$q(G) \geq q(K_{a-1} \vee (K_{n-a} + K_1)) = q(K_{n-1} + K_1) = 2n - 4.$$

By Claim 1 and Lemma 2.1, we know that G contains a Hamilton path unless $G \cong K_{1,3}$. Since $n \cdot a$ is even, n is even. Then we obtain that G contains a 1-factor unless $G \cong K_{1,3}$. Hence the result holds for $a = 1$.

Case 2. $a = 2$.

We have

$$q(G) \geq q(K_{a-1} \vee (K_{n-a} + K_1)) > q(K_{n-1} + K_1) = 2n - 4.$$

According to Claim 1 and Lemma 2.1, we know that G contains a Hamilton cycle unless $G \cong K_2 \vee 3K_1$ or $K_1 \vee (K_{n-2} + K_1)$. Note that $n \geq b + 5 \geq 7$. Then it follows that $G \ncong K_2 \vee 3K_1$. By Claim 2, we have $\delta(G) \geq 2$, which implies that $G \ncong K_1 \vee (K_{n-2} + K_1)$. Hence G contains a 2-factor.

Case 3. $a \geq 3$.

Suppose that G contains no $[a, b]$-factors. According to Lemmas 2.2 and 2.3, there exist two nonadjacent vertices u and v of G such that

$$\max\{d_G(u), d_G(v)\} \leq \lceil \frac{an}{a+b} \rceil - 1.$$

Let $t = \lceil \frac{an}{a+b} \rceil - 1$. Then we have $t \geq \delta(G) \geq a$ and $n \geq 2t + 1$. Define $G' = K_t \vee (K_{n-t-2} + 2K_1)$. By Lemma 3.1, we have

$$q(G) \leq q(G'). \tag{7}$$

Assume that $\boldsymbol{x}$ is the Perron vector of $Q(G')$. By symmetry, we can take x_1, x_2 and x_3 on the vertices of $V(K_t)$, $V(K_{n-t-2})$ and $V(2K_1)$, respectively. It is easy to see that

$$Q(G') = \begin{bmatrix} (J + (n-2)I)_{t\times t} & J_{t\times(n-t-2)} & J_{t\times 2} \\ J_{(n-t-2)\times t} & (J + (n-4)I)_{(n-t-2)\times(n-t-2)} & O_{(n-t-2)\times 2} \\ J_{2\times t} & O_{2\times(n-t-2)} & tI_{2\times 2} \end{bmatrix},$$

where J denotes a matrix of all elements equal to 1, I denotes the identity matrix, and O denotes the matrix of all elements equal to 0. Let $q' = q(G')$. By $Q(G')\boldsymbol{x} = q'\boldsymbol{x}$, we have

$$tx_1 + (2n - t - 6)x_2 = q'x_2, \tag{8}$$

$$tx_1 + tx_3 = q'x_3. \tag{9}$$

Since G' contains K_{n-2} as a proper subgraph and $G' \ncong K_n$, we have $2n-6 < q' < 2n-2$. Then $q' - 2n + t + 6 > 0$. Combining (8) and (9), we can obtain that

$$x_1 = \frac{q' - t}{t}x_3, \quad x_2 = \frac{q' - t}{q' - 2n + t + 6}x_3. \tag{10}$$

Now we let $G'' = K_{a-1} \vee (K_{n-a} + K_1)$. Then the Q-matrix of G'' is

$$Q(G'') = \begin{bmatrix} (J + (n-2)I)_{(a-1)\times(a-1)} & J_{(a-1)\times(n-a)} & J_{(a-1)\times 1} \\ J_{(n-a)\times(a-1)} & (J + (n-3)I)_{(n-a)\times(n-a)} & O_{(n-a)\times 1} \\ J_{1\times(a-1)} & O_{1\times(n-a)} & a - 1 \end{bmatrix}.$$

Let $\boldsymbol{y}$ be the Perron vector of $Q(G'')$. Similarly, by symmetry, $\boldsymbol{y}$ takes y_1, y_2 and y_3 on the vertices of $V(K_{a-1})$, $V(K_{n-a})$ and $V(K_1)$, respectively. Let $q'' = q(G'')$. According to $Q(G'')\boldsymbol{y} = q''\boldsymbol{y}$, we can obtain that

$$(a - 1)y_1 + (2n - a - 3)y_2 = q''y_2, \tag{11}$$

$$(a-1)y_1+(a-1)y_3=q''y_3. \tag{12}$$

Note that K_{n-1} is a proper subgraph of G''. Then $2n-4<q''<2n-2$. Since $n\geq 2t+1$ and $t\geq a$, we have $q''-a+1>2n-a-3>0$. By (11) and (12), we obtain that

$$y_3=\frac{q''-2n+a+3}{q''-a+1}y_2. \tag{13}$$

Combining (10) and (13), we have

$$\begin{aligned}
\boldsymbol{y}^T(q''-q')\boldsymbol{x} &= \boldsymbol{y}^T(Q(G'')-Q(G'))\boldsymbol{x}\\
&= 2(n-t-2)x_2y_2+(2n-3t+a-5)x_3y_2-(t-a+1)(x_1y_2+x_1y_3+x_3y_3)\\
&= x_3y_2\left[\frac{2(n-t-2)(q'-t)}{q'-2n+t+6}+(2n-3t+a-5)-(t-a+1)\left(\frac{q'-t}{t}\right.\right.\\
&\quad\left.\left.+\frac{(q'-t)(q''-2n+a+3)}{t(q''-a+1)}+\frac{q''-2n+a+3}{q''-a+1}\right)\right]\\
&= x_3y_2\left[\frac{2(n-t-2)(q'-t)}{q'-2n+t+6}+2(n-t-2)-(t-a+1)-(t-a+1)\cdot\right.\\
&\quad\left.\left(\frac{q'-t}{t}+\frac{(q'-t)(q''-2n+a+3)}{t(q''-a+1)}+\frac{q''-2n+a+3}{q''-a+1}\right)\right]\\
&= x_3y_2\left[(n-t-2)\left(4+\frac{4n-4t-12}{q'-2n+t+6}\right)-\frac{q'(2q''-2n+4)(t-a+1)}{t(q''-a+1)}\right].
\end{aligned}$$

Since $q'<2n-2$ and $2n-4<q''<2n-2$, we can obtain that

$$\begin{aligned}
\boldsymbol{y}^T(q''-q')\boldsymbol{x} &> x_3y_2\left[(n-t-2)\left(4+\frac{4n-4t-12}{(2n-2)-2n+t+6}\right)\right.\\
&\quad\left.-\frac{(2n-2)[2(2n-2)-2n+4](t-a+1)}{t(q''-a+1)}\right]\\
&= x_3y_2\left[\frac{4(n+1)(n-t-2)}{t+4}-\frac{4n(n-1)(t-a+1)}{t(q''-a+1)}\right]\\
&\geq x_3y_2\left[\frac{4(n+1)(n-t-2)}{t+4}-\frac{4n(n-1)(t-a+1)}{t[(2n-4)-a+1]}\right]\\
&= x_3y_2\left[\frac{4(n+1)(n-t-2)}{t+4}-\frac{4n(n-1)(t-a+1)}{t(2n-a-3)}\right]\\
&= \frac{4(n+1)x_3y_2}{t+4}\left[n-t-2-\frac{n(n-1)(t+4)}{t(n+1)}\cdot\frac{t-a+1}{2n-a-3}\right].
\end{aligned}$$

Recall that $n\geq 2t+1$. Then we have $\frac{t-a+1}{2n-a-3}<1$. Note that $a\geq 3$. Then $\frac{t-a+1}{2n-a-3}\leq\frac{t-a+1+(a-3)}{2n-a-3+(a-3)}=\frac{t-2}{2n-6}$. Hence we obtain that

$$\begin{aligned}
\boldsymbol{y}^T(q''-q')\boldsymbol{x} &\geq \frac{4(n+1)x_3y_2}{t+4}\left[n-t-2-\frac{n(n-1)(t+4)}{t(n+1)}\cdot\frac{t-2}{2n-6}\right]\\
&= \frac{4(n+1)x_3y_2}{t+4}\left[n-t-2-\frac{(t+4)(t-2)}{t}\cdot\frac{n(n-1)}{(n+1)(2n-6)}\right]\\
&> \frac{4(n+1)x_3y_2}{t+4}\left[n-t-2-\frac{(t+4)(t-2)}{t}\cdot\frac{7}{12}\right]\\
&> \frac{4(n+1)x_3y_2}{t+4}\left[2t+1-t-2-\frac{7(t+4)(t-2)}{12t}\right]
\end{aligned}$$

$$= \quad \frac{(n+1)x_3y_2}{3(t+4)}\left(5t-26+\frac{56}{t}\right) > 0.$$

This implies that

$$q(G') < q(G'').$$

Combining (7), we have

$$q(G) \le q(G') < q(G'') = q(K_{a-1} \vee (K_{n-a} + K_1)),$$

a contradiction. Hence G contains an $[a, b]$-factor. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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