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Interpolating numerical radius inequalities for positive semidefinite block matrices

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Abstract. In this article, we derive several numerical radius interpolation inequalities related to positive semidefinite block matrices by employing matrix convex function features. In particular, we show that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$ are such that B is normal and $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$w^{2r}(B) \leq \left\| \int_0^1 \left((1-t)\left(\alpha A^{\frac{r}{\alpha}} + (1-\alpha)C^{\frac{r}{1-\alpha}}\right) + tw^{2r}(B)I \right)^2 dt \right\|^{1/2} \leq \left\| \alpha A^{\frac{r}{\alpha}} + (1-\alpha)C^{\frac{r}{1-\alpha}} \right\|$$

for $0 \leq \alpha \leq 1$, $r \geq 1$.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices. A Hermitian matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called positive semidefinite if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$. To indicate that A is positive semidefinite, we write $A \geq 0$. We write $A \geq B$ to indicate that $A - B$ is positive semidefinite for Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$. A real-valued function $g(t)$ on $[0, \infty)$ is said to be matrix monotone if for all $A, B \in \mathbb{M}_n(\mathbb{C})$, $A \geq B \geq 0$ implies $g(A) \geq g(B)$ and it is said to be matrix convex if

$$g((1-\alpha)A + \alpha B) \leq (1-\alpha)\, g(A) + \alpha g(B),$$

for all Hermitian matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, and for all real numbers $0 \leq \alpha \leq 1$. On the other hand, a function $g : J \to (0, \infty)$, where J is a subinterval of $(0, \infty)$, is said to be geometrically convex if

$$g(a^{1-\alpha} b^{\alpha}) \leq g^{1-\alpha}(a) g^{\alpha}(b)$$

for all real numbers $0 \leq \alpha \leq 1$.

The topic of comparison of matrices has been the subject of current research because of its significance in numerous mathematical fields, such as mathematical analysis, operator theory, and mathematical physics.

A norm $N(.)$ on $\mathbb{M}_n(\mathbb{C})$ is said to be unitarily invariant if it has the basic property $N(UAV) = N(A)$, where $A \in \mathbb{M}_n(\mathbb{C})$ and $U, V \in \mathbb{M}_n(\mathbb{C})$ are unitary, it is called weakly unitarily invariant if $N(UAU^*) = N(A)$, where

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$A \in \mathbb{M}_n(\mathbb{C})$ and $U \in \mathbb{M}_n(\mathbb{C})$ is unitary, and it is called normalized if $N(diag(1, 0, ..., 0)) = 1$. Examples of such norms are the usual operator norm defined by $\|A\| = \max\limits_{\|x\|=1} \|Ax\| = s_1(A)$, where $s_1(A) \geq s_2(A) \geq ... \geq s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity.

For $A \in \mathbb{M}_n(\mathbb{C})$, the numerical radius of A is defined by

$$w(A) = \max \{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \|x\| = 1\}.$$

It is well known that $w(.)$ defines a norm on $\mathbb{M}_n(\mathbb{C})$. In fact, for every $A \in \mathbb{M}_n(\mathbb{C})$, we have

$$w(A) \leq \|A\| \leq 2w(A),$$

which indicates that the numerical radius and the operator norm are equivalent. The norm $w(.)$ is self-adjoint and weakly unitarily invariant, but it is not unitarily invariant.

A useful identity for the numerical radii of matrices was given in [14] as follows:

$$w(A) = \max_{\theta \in R} \left\| \mathrm{Re}(e^{i\theta}A) \right\|.$$

Abu-Omar and Kittaneh [1] defined the generalized numerical radius induced by a norm $N(.)$ on $\mathbb{M}_n(\mathbb{C})$) by

$$w_N(A) = \max_{\theta \in R} N\left(\mathrm{Re}(e^{i\theta}A)\right)$$

for every $A \in \mathbb{M}_n(\mathbb{C})$.

Several generalizations of the numerical radius have been discussed in [2], [3], [5], [13], [15], and references therein.

Established matrix inequalities involving positive semidefinite block matrices of the form $P = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$, where $A, B, C \in \mathbb{M}_n(\mathbb{C})$ is one of the issues that have attracted the interest of scholars in recent years.

An estimation of the numerical radius of the off-diagonal part of P was given by Burqan and Al-Saafin [6] as follows:

$$w(B) \leq \frac{1}{2} \|A + C\|. \tag{1}$$

Burqan and Abu-Rahma [7] generalized the inequality (1) as follows:

$$w^r(B) \leq \frac{1}{2} \|A^r + C^r\| \text{ for } r \geq 1. \tag{2}$$

An interesting generalization of the inequality (2) was introduced by Burqan, Alkhalely, and Conde [8] as follows:

$$w^{2r}(B) \leq \left\| \alpha A^{\frac{r}{\alpha}} + (1-\alpha) C^{\frac{r}{1-\alpha}} \right\| \text{ for } r \geq 1, 0 \leq \alpha \leq 1. \tag{3}$$

Moreover, Al-Naddaf, Burqan, and Kittaneh [9] generalized the inequality (1) for all normalized unitarily invariant norm $N(.)$ as follows:

$$g(2w_N(B)) \leq N\left(g(A) + g(C)\right), \tag{4}$$

where $g(t)$ is a non-negative matrix monotone function on $[0, \infty)$.

The primary objective of this paper is to propose new interpolation inequalities of the aforementioned inequalities via the use of the characteristics of matrix convex functions.

2. Lemmas

The following lemmas are essential to obtain and prove our results. The first lemma is a norm inequality for matrix monotone functions and can be found in [11]. The second lemma has been proved in [4]. Hermite-Hadamard's type inequalities for matrix convex functions of Hermitian matrices is presented in the third lemma (see [10]). The forth lemma is a Cauchy-Schwarz inequality including block positive semidefinite matrices (see [16]). The fifth lemma is derived from Jensen's inequality and the spectral theorem for positive semidefinite matrices (see [12]).

Lemma 2.1. *Let $g(t)$ be a non-negative matrix monotone function on $[0,\infty)$ and let $N(.)$ be a normalized unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$. Then for every $A \in \mathbb{M}_n(\mathbb{C})$,*

$$g(N(A)) \leq N\left(g\left(|A|\right)\right).$$

Lemma 2.2. *Let $g(t)$ be a non-negative matrix monotone function on $[0,\infty)$ and let $N(.)$ be a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$. Then for every positive semidefinite $A, C \in \mathbb{M}_n(\mathbb{C})$,*

$$N(g(A+C)) \leq N(g(A) + g(C)).$$

Lemma 2.3. *Let $g : J \to R$ be a matrix convex function on an interval J. Let $A, C \in \mathbb{M}_n(\mathbb{C})$ be Hermitian matrices with spectra in J. Then*

$$g\left(\frac{A+C}{2}\right) \leq \int_0^1 g\left((1-t)A + tC\right) dt \leq \frac{1}{2}\left(g(A) + g(C)\right).$$

If $g(t)$ is non-negative, then the matrix inequality can be reduced to the following norm inequality

$$N\left(g\left(\frac{A+C}{2}\right)\right) \leq N\left(\int_0^1 g\left((1-t)A + tC\right) dt\right) \leq \frac{1}{2} N\left(g(A) + g(C)\right). \tag{5}$$

Lemma 2.4. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. Then*

$$\left|\langle Bx, y\rangle\right|^2 \leq \langle Ax, x\rangle \langle Cy, y\rangle \text{ for } x, y \in \mathbb{C}^n.$$

Lemma 2.5. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite and $x \in \mathbb{C}^n$ with $\|x\| = 1$. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \text{ for } r \geq 1.$$

3. Main Results

At the beginning of this section, we introduce interpolation and generalization inequalities of Inequality (3) using Hermite-Hadamard's type inequalities for matrix convex functions of Hermitian matrices.

Theorem 3.1. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that B is normal and $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. If $g(t)$ is a non-negative increasing matrix convex function on $[0,\infty)$, then*

$$\begin{aligned} g\left(w^{2r}(B)\right) &\leq \left\| \int_0^1 g\left((1-t)\left(\alpha A^{\frac{r}{\alpha}} + (1-\alpha) C^{\frac{r}{1-\alpha}}\right) + t w^{2r}(B) I\right) dt \right\| \\ &\leq \left\| g\left(\alpha A^{\frac{r}{\alpha}} + (1-\alpha) C^{\frac{r}{1-\alpha}}\right) \right\| \end{aligned}$$

for $r \geq 1, 0 \leq \alpha \leq 1$.

Proof. Using the fact

$$\left\|\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}+w^{2r}(B)I\right\|=\left\|\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}\right\|+w^{2r}(B), \tag{6}$$

Inequality (3) and Equality (6) yield that

$$2w^{2r}(B)\leq\left\|\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}+w^{2r}(B)I\right\|$$

and so,

$$g\left(w^{2r}(B)\right)\leq g\left(\left\|\frac{\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}+w^{2r}(B)I}{2}\right\|\right)$$

for any increasing function $g(t)$ on $[0,\infty)$. Thus, Lemma 2.1 implies that

$$g\left(w^{2r}(B)\right)\leq\left\|g\left(\frac{\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}+w^{2r}(B)I}{2}\right)\right\|.$$

Inequality (5), Equality (6), and Inequality (3), respectively introduce

$$\begin{aligned}
g\left(w^{2r}(B)\right)&\leq\left\|g\left(\frac{\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}+w^{2r}(B)I}{2}\right)\right\|\\
&\leq\left\|\int_0^1 g\left((1-t)\left(\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}\right)+tw^{2r}(B)I\right)dt\right\|\\
&\leq\frac{1}{2}\left\|g\left(\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}\right)+g(w^{2r}(B))I\right\|\\
&=\frac{1}{2}\left(\left\|g\left(\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}\right)\right\|+g(w^{2r}(B))\right)\\
&\leq\left\|g\left(\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}\right)\right\|.
\end{aligned}$$

This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.1 by considering $g(t)=t^2$.

Corollary 3.2. *Let $A,B,C\in\mathbb{M}_n(\mathbb{C})$ be such that B is normal and $\begin{bmatrix} A & B\\ B^* & C\end{bmatrix}\geq 0$. Then*

$$\begin{aligned}
w^{2r}(B)&\leq\left\|\int_0^1\left((1-t)\left(\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}\right)+tw^{2r}(B)I\right)^2dt\right\|^{\frac{1}{2}}\\
&\leq\left\|\alpha A^{\frac{r}{\alpha}}+(1-\alpha)C^{\frac{r}{1-\alpha}}\right\|
\end{aligned}$$

for $r\geq 1, 0\leq\alpha\leq 1$.

Again applying Inequality (5), we get another improvements of Inequality (3).

Theorem 3.3. *Let $A,B,C\in\mathbb{M}_n(\mathbb{C})$ be such that B is normal and $\begin{bmatrix} A & B\\ B^* & C\end{bmatrix}\geq 0$. If $g(t)$ is a non-negative increasing matrix convex function on $[0,\infty)$, then*

$$\begin{aligned}
g\left(w^{2r}(B)\right)&\leq\left\|\int_0^1 g\left(2(1-t)\left(\alpha A^{\frac{r}{\alpha}}\right)+2t(1-\alpha)C^{\frac{r}{1-\alpha}})\right)dt\right\|\\
&\leq\frac{1}{2}\left\|g\left(2\alpha A^{\frac{r}{\alpha}}\right)+g(2(1-\alpha)C^{\frac{r}{1-\alpha}})\right\|
\end{aligned}$$

for $r\geq 1, 0\leq\alpha\leq 1$.

Considering $g(t) = t$ in Theorem 3.3, we get the following corollary.

Corollary 3.4. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that B is normal and $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. Then*

$$w^{2r}(B) \leq 2\left\|\int_0^1 (1-t)\left(\alpha A^{\frac{r}{\alpha}}\right) + t(1-\alpha)C^{\frac{r}{1-\alpha}})dt\right\|$$
$$\leq \left\|\alpha A^{\frac{r}{\alpha}} + (1-\alpha)C^{\frac{r}{1-\alpha}})\right\|$$

for $r \geq 1, 0 \leq \alpha \leq 1$.

In the following theorem, we get a refinement of Inequality (4).

Theorem 3.5. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that B is normal and $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. If $g(t)$ is a non-negative matrix monotone and convex function on $[0, \infty)$, then*

$$g\left(w_N(B)\right) \leq N\left(\int_0^1 g\left((1-t)A + tC\right)dt\right) \leq N\left(\frac{g(A) + g(C)}{2}\right)$$

for every normalized unitarily invariant norm $N(.)$.

Proof. Since $g(t)$ is a non-negative matrix monotone function, from Inequality (4) we get

$$g(w_N(B) \leq g\left(N\left(\frac{A+C}{2}\right)\right).$$

Lemma 2.1 yields that

$$g\left(N\left(\frac{A+C}{2}\right)\right) \leq N\left(g\left(\frac{A+C}{2}\right)\right).$$

Thus, by applying Lemma 2.3, we have

$$g(w_N(B)) \leq N\left(g\left(\frac{A+C}{2}\right)\right) \leq N\left(\int_0^1 g\left((1-t)A + tC\right)dt\right)$$
$$\leq N\left(\frac{g(A) + g(C)}{2}\right).$$

□

Using the fact $\begin{bmatrix} A & A^{1/2}B^{1/2} \\ B^{1/2}A^{1/2} & B \end{bmatrix} = \begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ B^{1/2} & 0 \end{bmatrix}^* \geq 0$ for any positive semidefinite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, Theorem 3.1, Theorem 3.3, and Theorem 3.5 produce the following results.

Corollary 3.6. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices and let $g(t)$ be a non-negative increasing matrix convex function on $[0, \infty)$. Then*

$$g\left(w^{2r}(A^{1/2}B^{1/2})\right) \leq \left\|\int_0^1 g\left((1-t)\left(\alpha A^{\frac{r}{\alpha}} + (1-\alpha)B^{\frac{r}{1-\alpha}}\right) + tw^{2r}(A^{1/2}B^{1/2})I\right)dt\right\|$$
$$\leq \left\|g\left(\alpha A^{\frac{r}{\alpha}} + (1-\alpha)B^{\frac{r}{1-\alpha}}\right)\right\|$$

for $r \geq 1, 0 \leq \alpha \leq 1$.

Corollary 3.7. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices and let $g(t)$ be a non-negative increasing matrix convex function on $[0,\infty)$. Then*

$$g\left(w^{2r}(A^{1/2}B^{1/2})\right) \le \left\| \int_0^1 g\left(2(1-t)\left(\alpha A^{\frac{r}{\alpha}}\right) + 2t(1-\alpha)B^{\frac{r}{1-\alpha}})\right) dt \right\|$$
$$\le \frac{1}{2}\left\| g\left(2\alpha A^{\frac{r}{\alpha}}\right) + g(2(1-\alpha)B^{\frac{r}{1-\alpha}})\right\|$$

for $r \ge 1, 0 \le \alpha \le 1$.

Corollary 3.8. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite matrices and let $g(t)$ be a non-negative matrix monotone and convex function on $[0,\infty)$. Then*

$$g\left(w_N(A^{1/2}B^{1/2})\right) \le N\left(\int_0^1 g\left((1-t)A + tB\right) dt\right) \le N\left(\frac{g(A) + g(B)}{2}\right)$$

for every normalized unitarily invariant norm $N(.)$.

In particular, for $g(t) = t$ and $N(.) = \|.\|$, we have

$$w(A^{1/2}B^{1/2}) \le \left\| \int_0^1 ((1-t)A + tB)\, dt \right\| \le \frac{1}{2}\|A+B\|.$$

Theorem 3.9. *Let $A, B, C \in \mathbb{M}_n(\mathbb{C})$ be such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \ge 0$ and let $g(t)$ be an increasing geometrically convex function on $(0,\infty)$. If in addition g is convex and $g(1) = 1$, then*

$$g^2(w(B)) \le \frac{1}{2}\left\| g\left(A^2\right) + g(C^2)\right\|.$$

Proof. For any unit vector $x \in \mathbb{C}^n$, Lemma 2.4 and the arithmetic-geometric mean inequality imply that

$$|\langle Bx, x\rangle| \le (\langle Ax, x\rangle \langle Cx, x\rangle)^{\frac{1}{2}}$$
$$\le \left(\frac{\langle Ax, x\rangle^2 + \langle Cx, x\rangle^2}{2}\right)^{\frac{1}{2}}.$$

Considering the given assumptions for the function $g(t)$ and Lemma 2.5, we have

$$g(|\langle Bx, x\rangle|) \le g\left(\left(\frac{\langle Ax, x\rangle^2 + \langle Cx, x\rangle^2}{2}\right)^{\frac{1}{2}}\right)$$
$$\le g^{\frac{1}{2}}\left(\frac{\langle Ax, x\rangle^2 + \langle Cx, x\rangle^2}{2}\right) g^{\frac{1}{2}}(1)$$
$$\le \left(\frac{g\left(\langle Ax, x\rangle^2\right) + g\left(\langle Cx, x\rangle^2\right)}{2}\right)^{\frac{1}{2}}$$
$$\le \left(\frac{g\left(\left\langle A^2x, x\right\rangle\right) + g\left(\left\langle C^2x, x\right\rangle\right)}{2}\right)^{\frac{1}{2}}$$
$$\le \left(\frac{\left\langle g(A^2)x, x\right\rangle + \left\langle g(C^2)x, x\right\rangle}{2}\right)^{\frac{1}{2}}$$
$$= \left(\frac{\left\langle \left(g(A^2) + g(C^2)\right)x, x\right\rangle}{2}\right)^{\frac{1}{2}}$$

Thus,

$$
\begin{aligned}
g\left(w(B)\right) &= g(\max_{\|x\|=1}|\langle Bx,x\rangle|) = \max_{\|x\|=1} g(|\langle Bx,x\rangle|)\\
&\le \max_{\|x\|=1}\left(\frac{\left\langle \left(g(A^2)+g(C^2)\right)x,x\right\rangle}{2}\right)^{\frac{1}{2}} = \left(\frac{\max_{\|x\|=1}\left\langle \left(g(A^2)+g(C^2)\right)x,x\right\rangle}{2}\right)^{\frac{1}{2}}\\
&= \left(\frac{1}{2}\left\|g\left(A^2\right)+g(C^2)\right\|\right)^{\frac{1}{2}}.
\end{aligned}
$$

This completes the proof. □

Inequality (2) is a special case of Theorem 3.9 since the function $g(t) = t^r, r \geq 1$ satisfies the assumptions of Theorem 3.9.

Declarations

Conflict of interest The author declares that he has no conflict of interest.

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