Filomat 38:22 (2024), 7971–7990
https://doi.org/10.2298/FIL2422971N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An analysis of the convergence problem of a function in functional norms by applying the product Hausdorff operator

H. K. Nigam[a], Swagata Nandy[a]

[a] *Department of Mathematics, Central University of South Bihar, Gaya, Bihar (India)*

Abstract. In this study, we have obtained results regarding the convergence properties of a function of derived conjugate Fourier series in weighted Lipschitz and generalized Hölder spaces. These results were obtained by utilizing the product Hausdorff operator. Several intriguing applications of our findings are also examined to observe the convergence behaviour of the function in the aforementioned spaces.

1. Introduction

Several researchers, such as [8], [9], [10], [11], [15–20], [21], [22–24] and others, have conducted investigations on the approximation properties of conjugate Fourier series in Lipschitz, Sobolev and generalized Hölder spaces using summability means.
The paper comprises an analysis of the convergence problems of a function $\tilde{f}'$ of C.D.F.S. in Lipschitz and generalized Hölder norms utilizing product Hausdorff means. Our convergence results are compared using applications. We observe that product Hausdorff means to yield a broader class of summability matrices, encompassing product Cesàro and product Euler means as special cases.
Srivastava et al. [27] have proved the existence of solutions for an infinite system of n^{th} order boundary value problems in Banach spaces, such as c_0 and l_1. Further, Srivastava et al. [28] discussed the existence and Hyers-Ulam stability of solutions for nonlinear differential equations involving the Liouville-Caputo fractional derivative with multi-point boundary conditions and the p-Laplacian operator in Banach spaces. Furthermore, Eidinejad et al. [5] introduced a new class of fuzzy control functions to approximate a Cauchy additive mapping in a fuzzy Banach space (FBS), exploring isomorphisms in unital FBS (UFBS) and evaluating Cauchy-Optimal stability (C-O-stability) for all defined mappings by selecting the optimal control function from several specific functions. Moreover, AlBaidani et al. [2] introduced a space based on a general Riesz sequence space, establishing its completeness and linear isomorphism with $l(p)$. They also derived the K$\ddot{o}$the-dual property of this space. Recently, Srivastava et al. [29] introduced and investigated the q-Cesàro matrix $C(q) = c^q_{uv}$ with $q \in (0,1)$, focusing on the sequence spaces $X^q(p), X^q_0(p), X^q_c(p),$ and $X^q_\infty(p)$, which arise from the domain of the matrix $C(q)$ in the Maddox spaces $l(p), c_0(p), c(p)$ and $l_\infty(p)$ respectively. They established the Schauder basis and the alpha-beta- and gamma-duals of these spaces.
The paper is structured in the following manner: In the second section, we provide significant definitions

2020 *Mathematics Subject Classification*. Primary 41A10; Secondary 42A10, 42B05.
Keywords. Degree of convergence, weighted Lipschitz class of function, generalized Hölder spaces, product Hausdorff operator, conjugate derived Fourier series (C.D.F.S).
Received: 15 December 2023; Revised: 26 February 2024; Accepted: 14 April 2024
Communicated by Miodrag Spalević
Email addresses: hknigam@cusb.ac.in (H. K. Nigam), swagata@cusb.ac.in (Swagata Nandy)

of our study. Section 3 presents auxiliary findings utilized in the proofs of our primary outcomes. In the fourth section, we establish the convergence properties of the function $\tilde{f}'$ within Lipschitz spaces and compare these findings using various applications. In this part, we establish convergence results for the function $\tilde{f}'$ in generalized Hölder spaces and analyze these results through various applications. Section 6 encompasses the concluding remarks.

2. Preliminaries

2.1. Conjugate derived Fourier series

The series

$$\tilde{f}'(y) := \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny) \tag{1}$$

is said to be the conjugate derived series of Fourier series, which is also said to be conjugate derived Fourier series (C.D.F.S.).

We denote the n^{th} partial sum of conjugate derived Fourier series (C.D.F.S.) as $\tilde{s}'(\tilde{f}'; y)$ and is given by

$$\tilde{s}_n'(y) - \tilde{f}'(y) = -\frac{2}{\pi}\int_0^{\pi} \frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\left(n+\frac{1}{2}\right)\sin\left(n+\frac{1}{2}\right)l dl - \frac{1}{\pi}\int_0^{\pi} \frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\frac{\cos(n+\frac{1}{2})l}{\tan\frac{l}{2}}dl, [13] \tag{2}$$

where $\tilde{f}'$ is the conjugate derived function of 2π-periodic function f, which is expressed as

$$\tilde{f}'(y) = \frac{1}{4\pi}\int_0^{\pi} \varrho_{(y)}(l)\csc^2\frac{l}{2}dl,$$

where

$$\varrho_{(y)}(l) = f(y+l) + f(y-l).$$

Note: Detailed work on Fourier series and its allied series can be found in [1].

2.2. Product Hausdorff operator

If

$$g_{n,r} \equiv \begin{cases} {}^nC_r\Delta^{n-r}\mu_r, & 0 \le r \le n; \\ 0, & r > n, \end{cases}$$

where $(g_{n,r})$ is an infinite lower triangular matrix and Δ is a forward operator defined as $\Delta\mu_n \equiv \mu_n - \mu_{n+1}$ and $\Delta^{r+1}\mu_n \equiv \Delta^r(\Delta\mu_n)$, then $(g_{n,r})$ is said to be a Hausdorff matrix [7]).

If $(g_{n,r})$ is regular, then μ_n, known as moment sequence, has the representation

$$\mu_n = \int_0^1 t^n d\gamma(t),$$

where $\gamma(t)$, known as mass function, is continuous at $t = 0$ and belongs to BV $[0,1]$ such that $\gamma(0+) = 0, \gamma(1) = 1$, and $\gamma(t) = \frac{\gamma(t-0)+\gamma(t+0)}{2}$ for $0 < t < 1$ [6, 23].

The product Hausdorff operator of the series (1) is denoted by $\tilde{\mathcal{H}}_{gh}'(\tilde{f}'; y)$ as follows

$$\tilde{\mathcal{H}}_{gh}'(\tilde{f}'; y) := \sum_{r=0}^{n} g_{n,r}\sum_{k=0}^{r} h_{r,k}\tilde{s}_k(\tilde{f}'; y), \quad n, r = 0, 1, 2, \ldots \tag{3}$$

The C.D.F.S. (1) is said to be summable to s by product Hausdorff operator if $\tilde{\mathcal{H}}_{gh}'(\tilde{f}'; y) \to s$ as $n, r \to \infty$.

Remark 2.1. *The product Hausdorff matrix* $\tilde{\mathcal{H}}'_{gh}(\tilde{f}'; y)$ *reduces to*

- $(C, \lambda_1, \lambda_2)$ *means if the mass function is given by*

$$\theta(v,w) = \lambda_1\lambda_2 \int_0^v \int_0^w (1-l)^{\lambda_1-1}(1-z)^{\lambda_2-1} dldz$$

- (E, q_n, q_r) *means if the mass function is given by*

$$\theta(v,w) = \begin{cases} 0 & \text{If } v \in [0,\beta], w \in [0,\alpha] \\ 1 & \text{If } v \in [\beta,1], w \in [\alpha,1], \end{cases}$$

where $\beta = \frac{1}{1+q_n}, \alpha = \frac{1}{1+q_r}, q_n, q_r > 0.$

2.3. Lipschitz Spaces

Let $L^p[0,2\pi] = \{f : [0,2\pi] \to \mathbb{R} : \int_0^{2\pi} |f(y)|^p dy, p \geq 1\}$ be the space of 2π- periodic and integrable function. The L^p norm of $f \in L^p[0,2\pi]$ is given by

$$\|f\|_p := \begin{cases} \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(y)|^p dy\right\}^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty; \\ \underset{0<y<2\pi}{\text{ess sup}}|f(y)|, & \text{for } p = \infty. \end{cases}$$

$W(L^p, \zeta(l))$ class of a function:
A function $f \in W(L^p, \zeta(l))$ if

$$\left(\int_0^{2\pi} \left|f(y+l) - f(y)) \sin^\sigma\left(\frac{y}{2}\right)\right|^p dy\right)^{\frac{1}{p}} = O(\zeta(l)), \ \text{for } \sigma \geq 0, p \geq 1,$$

where $\zeta(l)$ is a positive increasing function of l.

Note: A detailed study on other class of function such as $Lip\ \varphi, Lip(\varphi, p)$ and $Lip(\zeta(l), p)$ can be found in [26]. Following [26] we write

if $p \to \infty$, then $Lip\ \varphi \subseteq Lip(\varphi, p)$;

if $\zeta(l) = l^\varphi\ (0 < \varphi \leq 1)$, then $Lip(\varphi, p) \subseteq Lip(\zeta(l), p)$;

if $\sigma = 0$, then $Lip(\zeta(l), p) \subseteq W(L^p, \zeta(l))$.

2.4. Generalized Hölder Spaces

As defined in[1], $\xi : [0,2\pi] \to \mathbb{R}$ is an arbitrary function such that $\xi(l) > 0, 0 < l \leq 2\pi$ and $\lim_{l\to 0^+} \xi(l) = \xi(0) = 0$.
Now, the class of function $H_p^{(\xi)}$ is defined by

$$H_p^{(\xi)} := \left\{f \in L^p[0,2\pi] : \sup_{l\neq 0} \frac{\|f(\cdot,+l) - f(\cdot)\|_p}{\xi(t)} < \infty, p \geq 1\right\}.$$

The norm of class of function $H_p^{(\xi)}$ is given by

$$\|f\|_p^{(\xi)} = \|f\|_p + \sup_{l\neq 0} \frac{\|f(\cdot,+l) - f(\cdot)\|_p}{\xi(l)}; p \geq 1.$$

We note that $\xi(l)$ and $\eta(l)$ denote the moduli of continuity of order two (moduli of smoothness) such that $\frac{\xi(l)}{\eta(l)}$ is positive, non-decreasing and

$$\|f\|_{(p)}^{(\eta)} \leq \max\Big(1, \frac{\xi(2\pi)}{\eta(2\pi)}\Big)\|f\|_p^{(\xi)} < \infty.$$

Now it can be easily verified that $H_p^{(\xi)}$ is a complete normed linear space. We also observe that

$$H_p^{(\xi)} \subset H_p^{(\eta)} \subset L^p\,[0, 2\pi].$$

2.5. Degree of convergence

The degree of convergence of a function $\tilde{f}$ gives how speedily $\tilde{\mathcal{H}}'_{gh}$ converges to the function $\tilde{f}$, i.e.

$$\|\tilde{f}(y) - \tilde{\mathcal{H}}'_{gh}\| = O\Big(\frac{1}{\varphi_{nr}}\Big),$$

where $\varphi_{nr} \to \infty$ as $n, r \to \infty$[1].

2.6. Notations

$$\tilde{M}'_1 = -\frac{r}{2\pi}\int_0^1\Big(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t)\Big)\int_0^1\Big(\sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k}\frac{\sin(k+\frac{1}{2})l}{\sin\frac{l}{2}} d\gamma_2(t)\Big); \tag{4}$$

$$\tilde{M}'_2 = -\frac{1}{4\pi}\int_0^1\Big(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t)\Big)\int_0^1\Big(\sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k}\frac{\cos(kl)}{\sin^2\frac{l}{2}} d\gamma_2(t)\Big). \tag{5}$$

3. Auxiliary Results

The following auxiliary results are required for the proof of our main theorems.

Lemma 3.1. $\left|\tilde{M}'_1\right| = O(n+1)$, *for* $0 < l \leq \frac{1}{n+1}$.

Proof. For $0 < t < 1$, $0 < l \leq \frac{1}{n+1}$, $\sin kl \leq kl$, $\sup\limits_{0\leq t\leq 1}\frac{d\gamma_1(t)}{dt} = M$, $\sup\limits_{0\leq t\leq 1}\frac{d\gamma_2(t)}{dt} = N$ and $|\sin(\frac{l}{2})| \geq \frac{l}{\pi}$, from (4) we have

$$\begin{aligned}\left|\tilde{M}'_1\right| &\leq \left|\frac{r}{2\pi}\int_0^1\Big(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t)\Big)\int_0^1\Big(\sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k}\frac{(2k+1)\frac{l}{2}}{\frac{l}{\pi}} d\gamma_2(t)\Big)\right| \\ &\leq \frac{Nr}{4}\left|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t)\int_0^1\Big(2\sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k}\cdot k + \sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k}\Big)dt\right|.\end{aligned} \tag{6}$$

First, we solve

$$\sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k} k = (1-t)^r\sum_{k=0}^{r} {}^rC_k\Big(\frac{t}{1-t}\Big)^k k. \tag{7}$$

Taking $\frac{t}{1-t} = j$ in (7), we have

$$\sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k} k = (1-t)^r\sum_{k=0}^{r} {}^rC_k\, j^k k,$$

$$\sum_{k=0}^{r} {}^{r}C_k\, j^k \cdot k = {}^{r}C_1\, j + 2\, {}^{r}C_2\, j^2 + \cdots + r\, {}^{r}C_r\, j^r. \tag{8}$$

We know that

$$(1+j)^r = {}^{r}C_0 + {}^{r}C_1 \cdot j + {}^{r}C_2 \cdot j^2 + \cdots + {}^{r}C_r \cdot j^r \tag{9}$$

Differentiating (9) with respect to j, we get

$$r(1+j)^{r-1} = 0 + {}^{r}C_1 + {}^{r}C_2 \cdot 2j + \cdots + {}^{r}C_r \cdot r j^{r-1}. \tag{10}$$

Multiplying (10) by j on both sides, we get

$$rj(1+j)^{r-1} = {}^{r}C_1 \cdot j + 2\, {}^{r}C_2 \cdot j^2 + \cdots + r\, {}^{r}C_r \cdot j^r. \tag{11}$$

Now from (8) and (11), we have

$$\begin{aligned}
\sum_{k=0}^{r} {}^{r}C_k (j)^k \cdot k =& rj(1+j)^{r-1} \\
(1-t)^r \sum_{k=0}^{r} {}^{r}C_k \left(\frac{t}{1-t}\right)^k \cdot k =& (1-t)^r (rj(1+j)^{r-1}) \\
\sum_{k=0}^{r} {}^{r}C_k\, t^k (1-t)^{r-k} \cdot k =& r(1-t)^r \left(\frac{t}{(1-t)} \frac{1}{(1-t)^{r-1}}\right) \\
\sum_{k=0}^{r} {}^{r}C_k\, t^k (1-t)^{r-k} \cdot k =& rt.
\end{aligned} \tag{12}$$

Now solving

$$\begin{aligned}
\sum_{k=0}^{r} {}^{r}C_k\, t^k (1-t)^{r-k} =& (1-t+t)^r \\
=& 1
\end{aligned} \tag{13}$$

Using (12) and (13) in (6), we have

$$\begin{aligned}
\left|\tilde{M}_1'\right| \leq & \frac{Nr}{4} \left| \int_0^1 \sum_{r=0}^{n} {}^{n}C_r t^r (1-t)^{n-r} d\gamma_1(t) \int_0^1 (2rt+1) dt \right| \\
=& \frac{NMr}{4} \left| \int_0^1 \sum_{r=0}^{n} {}^{n}C_r t^r (1-t)^{n-r} (r+1) dt \right| \\
=& \frac{NMr}{4} \left| \int_0^1 \left[\sum_{r=0}^{n} {}^{n}C_r t^r (1-t)^{n-r} r + \sum_{r=0}^{n} {}^{n}C_r t^r (1-t)^{n-r} \right] dt \right| \\
=& \frac{NMr}{4} \left| \int_0^1 (nt+1) dt \right| \\
=& \frac{NMr}{2} (n+1) \\
=& O(n+1).
\end{aligned}$$

□

Lemma 3.2. $\left|\tilde{M}_2'\right| = O\Big(\frac{1}{l^2}\Big)$, *for* $0 < l \leq \frac{1}{n+1}$.

Proof. For $0 < t < 1,\ 0 < l \leq \frac{1}{n+1},\ |\cos(kl)| \leq 1,\ \sup\limits_{0\leq t\leq 1} \frac{d\gamma_1(t)}{dt} = M,\ \sup\limits_{0\leq t\leq 1} \frac{d\gamma_2(t)}{dt} = N$ and $|\sin(\frac{l}{2})| \geq \frac{l}{\pi}$, from (5) we have

$$\begin{aligned}
\left|\tilde{M}_2'\right| &\leq \frac{N\pi^2}{l^2}\frac{1}{4\pi}\Big|\int_0^1 \Big(\sum_{r=0}^{n} {}^nC_r t^r (1-t)^{n-r} d\gamma_1(t)\Big)\int_0^1 \Big(\sum_{k=0}^{r} {}^rC_k t^k (1-t)^{r-k} dt\Big|\Big) \\
&\leq \frac{N\pi}{4l^2}\Big|\int_0^1 \Big(\sum_{r=0}^{n} {}^nC_r t^r (1-t)^{n-r} d\gamma_1(t)\Big)\int_0^1 (1-t+t)^r dt\Big| \\
&\leq \frac{NM\pi}{4l^2}\Big|\int_0^1 \Big(\sum_{r=0}^{n} {}^nC_r t^r (1-t)^{n-r} dt\Big) \\
&\leq \frac{NM\pi}{4l^2}\Big|\int_0^1 (1-t+t)^n dt\Big| \\
&= O\Big(\frac{1}{l^2}\Big).
\end{aligned}$$

□

Lemma 3.3. $\left|\tilde{M}_1'\right| = O\Big(\frac{1}{(n+1)l^2}\Big)$, *for* $\frac{1}{n+1} < l \leq \pi$.

Proof. For $0 < t < 1,\ \frac{1}{n+1} < l \leq \pi,\ \sin^2(\frac{r+1}{2})l \leq 1,\ \sup\limits_{0\leq t\leq 1} \frac{d\gamma_1(t)}{dt} = M,\ \sup\limits_{0\leq t\leq 1} \frac{d\gamma_2(t)}{dt} = N$ and $|\sin(\frac{l}{2})| \geq \frac{l}{\pi}$, from (4) we have

$$\begin{aligned}
\left|\tilde{M}_1'\right| &\leq \Big|\frac{r}{2\pi}\int_0^1 \Big(\sum_{r=0}^{n} {}^nC_r t^r (1-t)^{n-r} d\gamma_1(t)\Big)\int_0^1 \Big(\sum_{k=0}^{r} {}^rC_k t^k (1-t)^{r-k} \frac{\mathrm{Im}(e^{i(k+\frac{1}{2})l})}{\frac{l}{\pi}} d\gamma_2(t)\Big)\Big| \\
&\leq \frac{Nr}{2l}\Big|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r (1-t)^{n-r} d\gamma_1(t) \cdot \mathrm{Im}\int_0^1 (1-t)^r e^{\frac{il}{2}} \sum_{k=0}^{r} {}^rC_k \Big(\frac{te^{il}}{1-t}\Big)^k dt\Big| \\
&= \frac{Nr}{2l}\Big|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r (1-t)^{n-r} d\gamma_1(t) \mathrm{Im}\Big[e^{\frac{il}{2}}\int_0^1 (1-t)^r \Big\{{}^rC_0\Big(\frac{te^{il}}{1-t}\Big)^0 + {}^rC_1\Big(\frac{te^{il}}{1-t}\Big)^1 + \cdots + {}^rC_r\Big(\frac{te^{il}}{1-t}\Big)^r\Big\}dt\Big] \\
&= \frac{Nr}{2l}\Big|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r (1-t)^{n-r} d\gamma_1(t) \mathrm{Im}\Big[e^{\frac{il}{2}}\int_0^1 \Big\{{}^rC_0(1-t)^r + {}^rC_1(1-t)^{r-1}\Big(te^{il}\Big) + \cdots + {}^rC_r(1-t)^0\Big(te^{il}\Big)^r\Big\}dt\Big] \\
&= \frac{Nr}{2l}\Big|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r (1-t)^{n-r} d\gamma_1(t) \cdot \mathrm{Im}\Big(e^{\frac{il}{2}}\int_0^1 (1+t(e^{il}-1))^r dt\Big)\Big|. \qquad (14)
\end{aligned}$$

Considering

$$\begin{aligned}
&1 + t(e^{il}-1) = x \\
\Rightarrow\ &(e^{il}-1)dt = dx \\
\Rightarrow\ &dt = \frac{dx}{(e^{il}-1)}.
\end{aligned}$$

$$\left|\tilde{M}_1'\right| = \frac{Nr}{2l}\Big|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r (1-t)^{n-r} d\gamma_1(t) \cdot \mathrm{Im}\Big(e^{\frac{il}{2}}\int x^r \frac{dx}{(e^{il}-1)}\Big)\Big|$$

$$= \frac{Nr}{2l}\left|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t) \cdot \operatorname{Im} e^{\frac{il}{2}}\left(\frac{e^{i(r+1)l}-1}{(r+1)(e^{il}-1)}\right)\right|$$
$$= \frac{Nr}{2l}\left|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t) \cdot \operatorname{Im}\left(\frac{\cos(r+1)l + i\sin(r+1)l - 1}{(r+1)2i\sin\frac{l}{2}}\right)\right|$$
$$= \frac{Nr}{2l}\left|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t) \frac{\sin^2(\frac{(r+1)l}{2})}{(r+1)\sin\frac{l}{2}}\right|$$
$$\leq \frac{NMr\pi}{2l^2}\left|\int_0^1 \left(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} \frac{1}{r+1} dt\right)\right|$$
$$\leq \frac{NMr\pi}{2(n+1)l^2}\left|\int_0^1 \left(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} dt\right)\right|$$
$$= \frac{NMr\pi}{2(n+1)l^2}\left|\int_0^1 (1-t+t)^n dt\right|$$
$$= O\left(\frac{1}{(n+1)l^2}\right).$$

□

Lemma 3.4. $\left|\tilde{M}'_2\right| = O\left(\frac{1}{(n+1)l^3}\right)$, *for* $\frac{1}{n+1} \leq l \leq \pi$.

Proof. For $0 < t < 1$, $\frac{1}{n+1} < l \leq \pi$, $|\cos(kl)| \leq 1$, $|\sin(r+1)l| \leq 1$, $\sup\limits_{0\leq t\leq 1} \frac{d\gamma_1(t)}{dt} = M$, $\sup\limits_{0\leq t\leq 1} \frac{d\gamma_2(t)}{dt} = N$ and $|\sin(\frac{l}{2})| \geq \frac{l}{\pi}$, from (5) we have

$$\left|\tilde{M}'_2\right| \leq \frac{N\pi}{4l^2}\left|\int_0^1 \left(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t)\right) \cdot \operatorname{Re}\int_0^1 \left(\sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k} e^{ikl} dt\right|\right)$$
$$\leq \frac{N\pi}{4l^2}\left|\int_0^1 \left(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t)\right) \cdot \operatorname{Re}\left(\int_0^1 (1-t)^r \sum_{k=0}^{r} {}^rC_k \left(\frac{te^{il}}{1-t}\right)^k\right) dt\right|$$
$$\leq \frac{N\pi}{4l^2}\left|\int_0^1 \left(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t)\right) \cdot \operatorname{Re}\left(\int_0^1 (1-t+te^{il})^r dt\right|\right.$$
$$= \frac{N\pi}{4l^2}\left|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t) \cdot \operatorname{Re}\left(\frac{\cos(r+1)l + i\sin(r+1)l - 1}{(r+1)2i\sin\frac{l}{2}}\right)\right|$$
$$= \frac{N\pi}{4l^2}\left|\int_0^1 \sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} d\gamma_1(t) \frac{\sin(r+1)l}{2(r+1)\sin\frac{l}{2}}\right|$$
$$\leq \frac{NM\pi^2}{4l^3}\left|\int_0^1 \left(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} \frac{1}{r+1} dt\right)\right|$$
$$\leq \frac{NM\pi^2}{4(n+1)l^3}\left|\int_0^1 \left(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r} dt\right)\right|$$
$$\leq \frac{NM\pi^2}{(n+1)l^3}\left|\int_0^1 (1-t+t)^n dt\right|$$
$$= O\left(\frac{1}{(n+1)l^3}\right).$$

□

Lemma 3.5. *[12] Let $f \in H_p^{(\xi)}$, then for $0 < l \leq \pi$*

(i) $\|\varrho(\cdot, l)\|_p = O(\xi(l))$;

(ii) $\|\varrho(\cdot + t, l) - \varrho(\cdot, l)\|_p = \begin{cases} O(\xi(l)), \\ O(\xi(t)); \end{cases}$

(iii) *If $\xi(l)$ and $\eta(l)$ are modulus of continuity, then*
$\|\varrho(\cdot + t, l) - \varrho(\cdot, l)\|_p = O(\eta(|t|)(\frac{\xi(l)}{\eta(l)})$.

4. Analysis of convergence of a function in weighted Lipschitz spaces

4.1. Results and their proofs

Here, we establish a theorem to study the convergence of function $\tilde{f}'$ of derived Fourier series in $W(L^p, \zeta(l)), \sigma \geq 0, p \geq 1$ class using product Hausdorff operator. In fact, we prove the following theorem:

Theorem 4.1. *If $\tilde{f}'$ is a derived conjugate function of 2π periodic and Lebesgue integrable function f, then the degree of convergence of $W(L^p, \zeta(l))$ ($p > 1$ and $0 \leq \sigma \leq 1 - \frac{1}{p}$) class using product Hausdorff operator, is given by*

$$\|\tilde{\mathcal{H}}'_{gh}(\tilde{f}'; y) - \tilde{f}'(y)\|_p = O\Big((n+1)^{\sigma+1+\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)\Big),$$

provided $\zeta(l)$ satisfies

$$\Big\{\frac{\zeta(l)}{l}\Big\} \text{ is non-increasing,} \tag{15}$$

$$\Big(\int_0^{\frac{1}{n+1}} \Big(\frac{|\varrho_y(l)|}{\zeta(l)} \sin^\sigma\Big(\frac{l}{2}\Big)\Big)^p dl\Big)^{\frac{1}{p}} = O\Big(\frac{1}{n+1}\Big)^{\frac{1}{p}}, \tag{16}$$

$$\Big(\int_\epsilon^{\frac{1}{n+1}} \Big(\frac{\zeta(l)}{l \sin^\sigma(\frac{l}{2})}\Big)^\omega dl\Big)^{\frac{1}{\omega}} = O\Big((n+1)^{\sigma+\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)\Big), \tag{17}$$

$$\int_{\frac{1}{n+1}}^{\pi} \Big(\Big(\frac{l^{-\delta}|\varrho_y(l)|}{\zeta(l)}\Big)^p dl\Big)^{\frac{1}{p}} = O\Big((n+1)^\delta\Big), \tag{18}$$

where δ is an arbitrary number, $0 < \delta < \sigma + \frac{1}{p}$, $\frac{1}{p} + \frac{1}{\omega} = 1$ for $p > 1$. The conditions (16) and (18) hold uniformly in y.
The conditions (16) and (18) can be verified by using the fact that $\varrho_y(l) \in W(L^p, \zeta(l))$ and $\frac{\varrho_y(l)}{\zeta(l)}$ is a bounded function. The condition (17) is obvious in the light of the mean value theorem for integrals.

Proof. Following [14], the integral representation of $\tilde{s}'_n(\tilde{f}'; y)$ is given by,

$$\begin{aligned}
\tilde{s}'_n(\tilde{f}'; y) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\frac{\partial}{\partial u}\Big(\sum_{e=1}^{r} \sin e(u-y)\Big)du \\
&= -\frac{1}{\pi}\int_0^\pi \frac{d}{dl}\Big(\frac{\cos(\frac{l}{2}) - \cos(r+\frac{1}{2})l}{2\sin(\frac{l}{2})}\Big)(f(y+l) + f(y-l))dl \\
&= -\frac{1}{\pi}\int_0^\pi \frac{d}{dl}\Big(\frac{\cot(\frac{l}{2})}{2}\Big)\varrho_y(l)dl + \frac{1}{\pi}\int_0^\pi \frac{d}{dl}\Big(\frac{\cos(r+\frac{1}{2})l}{2\sin(\frac{l}{2})}\Big)\varrho_y(l)dl \\
&= \frac{1}{4\pi}\int_0^\pi \csc^2(\frac{l}{2})\varrho_y(l)dl - \frac{2}{\pi}\int_0^\pi \frac{\varrho_y(l)}{4\sin(\frac{l}{2})}\Big(r+\frac{1}{2}\Big)\sin\Big(r+\frac{1}{2}\Big)ldl - \frac{1}{\pi}\int_0^\pi \frac{\varrho_y(l)}{4\sin(\frac{l}{2})}\frac{\cos(r+\frac{1}{2})l}{\tan(\frac{l}{2})}dl.
\end{aligned}$$

$$\tilde{s}_n'(f;y) - \frac{1}{4\pi}\int_0^{\pi} \csc^2\left(\frac{l}{2}\right)\varrho_y(l)dl = -\frac{2}{\pi}\left(r+\frac{1}{2}\right)\int_0^{\pi} \frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\sin\left(r+\frac{1}{2}\right)ldl - \frac{1}{\pi}\int_0^{\pi}\frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\frac{\cos(r+\frac{1}{2})l\cos(\frac{l}{2})}{\sin\frac{l}{2}}dl$$

$$\tilde{s}_n'(f;y) - \tilde{f}'(y) = -\frac{2r}{\pi}\int_0^{\pi}\frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\sin\left(r+\frac{1}{2}\right)ldl - \frac{1}{\pi}\int_0^{\pi}\frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\frac{\cos(rl)}{\sin\frac{l}{2}}dl.$$

Now,

$$\begin{aligned}
T_n'(y) &= \tilde{\mathcal{H}}_{gh}'(\tilde{f}';y) - \tilde{f}'(y)\\
&= \sum_{r=0}^{n} m_{n,r}\sum_{k=0}^{r} n_{r,k}(\tilde{s}_n'(f;y) - \tilde{f}'(y))\\
&= -\frac{2r}{\pi}\int_0^{\pi}\frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\sum_{r=0}^{n} m_{n,r}\sum_{k=0}^{r} n_{r,k}\sin(k+\frac{1}{2})ldl - \frac{1}{\pi}\int_0^{\pi}\frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\sum_{r=0}^{n} m_{n,r}\sum_{k=0}^{r} n_{r,k}\frac{\cos(kl)}{\sin\frac{l}{2}}dl\\
&= -\frac{2r}{\pi}\int_0^{\pi}\left[\frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\int_0^1\left(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r}\right)d\gamma_1(t)\int_0^1\left(\sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k}\sin(k+\frac{1}{2})l\right)d\gamma_2(t)\right]dl\\
&\quad - \frac{1}{\pi}\int_0^{\pi}\left[\frac{\varrho_{(y)}(l)}{4\sin\frac{l}{2}}\int_0^1\left(\sum_{r=0}^{n} {}^nC_r t^r(1-t)^{n-r}\right)d\gamma_1\int_0^1\left(\sum_{k=0}^{r} {}^rC_k t^k(1-t)^{r-k}\frac{\cos(kl)}{\sin\frac{l}{2}}\right)d\gamma_2\right]dl\\
&= \int_0^{\pi}\left(\varrho_{(y)}(l)\tilde{M}_1'dl\right) + \int_0^{\pi}\left(\varrho_{(y)}(l)\tilde{M}_2'dl\right)\\
&= \int_0^{\pi}\varrho_{(y)}(l)\left(\tilde{M}_1' + \tilde{M}_2'\right)dl\\
&= \left(\int_0^{\frac{1}{n+1}}\varrho_{(y)}(l)\left(\tilde{M}_1' + \tilde{M}_2'\right) + \int_{\frac{1}{n+1}}^{\pi}\varrho_{(y)}(l)\left(\tilde{M}_1' + \tilde{M}_2'\right)\right)dl\\
&= A + B.
\end{aligned} \tag{19}$$

Now $\varrho_y(l) \in W(L_p, \zeta(l))$, we have

$$|A| \leq \int_0^{\frac{1}{n+1}} |\varrho_y(l)||\tilde{M}_1'|dl + \int_0^{\frac{1}{n+1}} |\varrho_y(l)||\tilde{M}_2'|dl.$$

using the Lemmas 3.1 and 3.2, we write

$$\begin{aligned}
|A| &\leq \int_0^{\frac{1}{n+1}} |\varrho_y(l)|(n+1)dl + \int_0^{\frac{1}{n+1}} |\varrho_y(l)|\frac{1}{l^2}dl\\
&\leq \int_0^{\frac{1}{n+1}} (n+1)|\varrho_y(l)|dl + \int_0^{\frac{1}{n+1}} \frac{|\varrho_y(l)|}{l^2}dl
\end{aligned}$$

Applying Hölder's inequality

$$\begin{aligned}
|A| &= \left(\int_0^{\frac{1}{n+1}}\left(\frac{(n+1)|\varrho_y(l)|\sin^{\sigma}\frac{l}{2}}{\zeta(l)}\right)^p dl\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{n+1}}\left(\frac{\zeta(l)}{\sin^{\sigma}\frac{l}{2}}\right)^{\omega}dl\right)^{\frac{1}{\omega}}\\
&\quad + \left(\int_0^{\frac{1}{n+1}}\left(\frac{|\varrho_y(l)|\sin^{\sigma}\frac{l}{2}}{\zeta(l)}\right)^p dl\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{n+1}}\left(\frac{\zeta(l)}{l^2\sin^{\sigma}\frac{l}{2}}\right)^{\omega}dl\right)^{\frac{1}{\omega}}
\end{aligned}$$

Using the conditions (15), (16) and second mean value theorem for integrals, we have

$$
\begin{aligned}
|A| &= O\Big(\frac{1}{n+1}\Big)^{\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)(n+1)\Big(\int_0^{\frac{1}{n+1}}\Big(\frac{1}{l^\sigma}\Big)^\omega dl\Big)^{\frac{1}{\omega}} + O\Big(\frac{1}{n+1}\Big)^{\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)\Big(\int_0^{\frac{1}{n+1}}\Big(\frac{1}{l^{\sigma+2}}\Big)^\omega dl\Big)^{\frac{1}{\omega}}\\
&= O\Big(\frac{1}{n+1}\Big)^{\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)(n+1)\Big(\lim_{\epsilon\to 0}\int_\epsilon^{\frac{1}{n+1}}\Big(\frac{1}{l^{\sigma\omega}}\Big)dl\Big)^{\frac{1}{\omega}} + O\Big(\frac{1}{n+1}\Big)^{\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)\Big(\lim_{\epsilon\to 0}\int_\epsilon^{\frac{1}{n+1}}\Big(\frac{1}{l^{\sigma\omega+2\omega}}\Big)dl\Big)^{\frac{1}{\omega}}\\
&= O\Big(\frac{1}{n+1}\Big)^{\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)(n+1)\Big(\Big[\frac{l^{-\sigma\omega+1}}{-\sigma\omega+1}\Big]_\epsilon^{\frac{1}{n+1}}\Big)^{\frac{1}{\omega}} + O\Big(\frac{1}{n+1}\Big)^{\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)\Big(\Big[\frac{l^{-\sigma\omega-2\omega+1}}{-\sigma\omega-2\omega+1}\Big]_\epsilon^{\frac{1}{n+1}}\Big)^{\frac{1}{\omega}}\\
&= O\Big(\Big(\frac{1}{n+1}\Big)^{\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)(n+1)^{\sigma+\frac{1}{p}}\Big) + O\Big(\frac{1}{n+1}\Big)^{\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)(n+1)^{\sigma+1+\frac{1}{p}}\Big)\\
&= O\Big((n+1)^{\sigma+1}\zeta\Big(\frac{1}{n+1}\Big)\Big). \qquad (20)
\end{aligned}
$$

Now,

$$|B| \le \int_{\frac{1}{n+1}}^{\pi}|\varrho_y(l)||\tilde{M}_1'|dl + \int_{\frac{1}{n+1}}^{\pi}|\varrho_y(l)||\tilde{M}_2'|dl.$$

Using the Lemmas 3.1 and 3.2, we have

$$|B| \le \int_{\frac{1}{n+1}}^{\pi}|\varrho_y(l)|\frac{1}{(n+1)l^2}dl + \int_{\frac{1}{n+1}}^{\pi}|\varrho_y(l)|\frac{1}{(n+1)l^3}dl$$

Applying Hölder's inequality, we have

$$
\begin{aligned}
|B| \le& \Big(\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{l^{-\delta}|\varrho_y(l)|\sin^\sigma\frac{l}{2}}{\zeta(l)}\Big)^p dl\Big)^{\frac{1}{p}}\Big(\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{\zeta(l)}{(n+1)l^{2-\delta+\sigma}}\Big)^\omega dl\Big)^{\frac{1}{\omega}}\\
&+\Big(\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{l^{-\delta}|\varrho_y(l)|\sin^\sigma\frac{l}{2}}{\zeta(l)}\Big)^p dl\Big)^{\frac{1}{p}}\Big(\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{\zeta(l)}{(n+1)l^{3-\delta+\sigma}}\Big)^\omega dl\Big)^{\frac{1}{\omega}}.
\end{aligned}
$$

Using conditions (15), (18), $|\sin(l)| \le 1$ and second mean value theorem for integrals, we have

$$
\begin{aligned}
|B| =& O(n+1)^\delta\zeta\Big(\frac{1}{n+1}\Big)\Big(\frac{1}{n+1}\Big)\Big(\int_{\frac{1}{\pi}}^{n+1}\Big(g^{2+\sigma-\delta}\Big)^\omega\frac{dg}{g^2}\Big)^{\frac{1}{\omega}}\\
&+O(n+1)^\delta\zeta\Big(\frac{1}{n+1}\Big)\Big(\frac{1}{n+1}\Big)\Big(\int_{\frac{1}{\pi}}^{n+1}\Big(g^{3+\sigma-\delta}\Big)^\omega\frac{dg}{g^2}\Big)^{\frac{1}{\omega}}\Big(\text{putting }\frac{1}{l}=g\Big)\\
=& O(n+1)^\delta\zeta\Big(\frac{1}{n+1}\Big)\Big(\frac{1}{n+1}\Big)(n+1)^{2+\sigma-\delta-\frac{1}{\omega}} + O(n+1)^\delta\zeta\Big(\frac{1}{n+1}\Big)\Big(\frac{1}{n+1}\Big)(n+1)^{3+\sigma-\delta-\frac{1}{\omega}}\\
=& O\Big(\zeta\Big(\frac{1}{n+1}\Big)(n+1)^{\sigma+\frac{1}{p}}\Big) + O\Big(\zeta\Big(\frac{1}{n+1}\Big)(n+1)^{1+\sigma+\frac{1}{p}}\Big)\\
=& O\Big((n+1)^{\sigma+1+\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)\Big). \qquad (21)
\end{aligned}
$$

Combining (19), (20) and (21), we get

$$\|\tilde{\mathcal{H}}_{gh}'(\tilde{f}';y) - \tilde{f}'(y)\|_p = O\Big((n+1)^{\sigma+1+\frac{1}{p}}\zeta\Big(\frac{1}{n+1}\Big)\Big). \qquad (22)$$

□

Remark 4.2. *In view of Remark 2.1, two corollaries can be deduced from the Theorem 4.1.*

Now, we establish a theorem for $p = 1$ in order to study the convergence of function $\tilde{f}'$ of derived Fourier series of a 2π periodic and Lebesgue integrable function f in $W(L^1, \zeta(l))$ class using product Hausdorff operator. In fact, we prove the following theorem:

Theorem 4.3. *If $\tilde{f}'$ is a derived conjugate function of 2π periodic and Lebesgue integrable function f, then the degree of convergence of $W(L^1, \zeta(l))$ $(0 \le \sigma < 1$ and $p = 1)$ class using product Hausdorff operator, is given*

$$\|\tilde{\mathcal{H}}'_{gh}(\tilde{f}'; y) - \tilde{f}'(y)\|_1 = O\Big((n+1)^{(\sigma+2)}\zeta\Big(\frac{1}{n+1}\Big)\Big),$$

provided $\zeta(l)$ satisfies (15) and the following conditions

$$\Big\{\frac{\zeta(l)}{l^{\sigma+\rho}}\Big\} \text{ be non-decreasing,} \tag{23}$$

$$\int_0^{\frac{1}{n+1}} \frac{l^{\rho-1}|\varrho_y(l)|\sin^\sigma(\frac{l}{2})}{\zeta(l)} dl = O\Big(\frac{1}{n+1}\Big)^\rho, \tag{24}$$

for some $\rho > 0, \sigma + \rho < 1$,

$$\int_{\frac{1}{n+1}}^{\pi} \frac{l^{-\delta}|\varrho_y(l)|}{\zeta(l)} dl = O(n+1)^\delta, \tag{25}$$

$$\Big\{\frac{\zeta(l)}{l^{\sigma-\delta+2}}\Big\} \text{is non-increasing,} \tag{26}$$

where $\zeta(l)$ is a positive increasing function, $0 < \delta < \sigma + 1$ and (23) and (24) hold uniformly in y.

Proof. Following the proof of Theorem 4.1 for $p = 1$ i.e. $\omega = \infty$, we have

$$\begin{aligned}|A| \le (n+1)\int_0^{\frac{1}{n+1}} \Big(\frac{l^{\rho-1}|\varrho_y(l)|\sin^\sigma(\frac{l}{2})}{\zeta(l)}\Big) dl \cdot \operatorname*{ess\,sup}_{0<l\le\frac{1}{n+1}} \Big|\frac{\zeta(l)}{l^{\rho-1}\sin^\sigma(\frac{l}{2})}\Big|. \\ + \int_0^{\frac{1}{n+1}} \Big(\frac{l^{\rho-1}|\varrho_y(l)|\sin^\sigma(\frac{l}{2})}{\zeta(l)}\Big) dl \cdot \operatorname*{ess\,sup}_{0<l\le\frac{1}{n+1}} \Big|\frac{\zeta(l)}{l^{\rho+1}\sin^\sigma(\frac{l}{2})}\Big|.\end{aligned}$$

Using the condition (24) and $|\sin\frac{l}{2}| \le \frac{\pi}{l}$, we have

$$\begin{aligned}|A| &= O(n+1)\Big(\frac{1}{n+1}\Big)^\rho \operatorname*{ess\,sup}_{0<l\le\frac{1}{n+1}} \Big|\frac{\zeta(l)}{l^{\sigma+\rho-1}}\Big| + O\Big(\frac{1}{n+1}\Big)^\rho \operatorname*{ess\,sup}_{0<l\le\frac{1}{n+1}} \Big|\frac{\zeta(l)}{l^{\sigma+\rho+1}}\Big| \\ &= O\Big((n+1)^\sigma\zeta\Big(\frac{1}{n+1}\Big)\Big) + O\Big((n+1)^{\sigma+1}\zeta\Big(\frac{1}{n+1}\Big)\Big) \\ &= O\Big((n+1)^{\sigma+1}\zeta\Big(\frac{1}{n+1}\Big)\Big).\end{aligned} \tag{27}$$

Similarly,

$$|B| \le \frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} \frac{l^{-\delta}|\varrho_y(l)|\sin^\sigma(\frac{l}{2})}{\zeta(l)} dl \cdot \operatorname*{ess\,sup}_{\frac{1}{n+1}<l\le\pi} \Big|\frac{\zeta(l)}{l^{-\delta+\sigma+2}}\Big| + \frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} \frac{l^{-\delta}|\varrho_y(l)|\sin^\sigma(\frac{l}{2})}{\zeta(l)} dl \cdot \operatorname*{ess\,sup}_{\frac{1}{n+1}<l\le\pi} \Big|\frac{\zeta(l)}{l^{-\delta+\sigma+3}}\Big|.$$

Using the condition (25), we have

$$|B| = O\Big((n+1)^{\delta-1}\zeta\Big(\frac{1}{n+1}\Big)\Big(\frac{(n+1)^{\sigma-\delta+2}}{\pi^{\sigma-\delta+2}}\Big)\Big) + O\Big((n+1)^{\delta-1}\zeta\Big(\frac{1}{n+1}\Big)\Big(\frac{(n+1)^{\sigma-\delta+3}}{\pi^{\sigma-\delta+3}}\Big)\Big)$$

$$= O\Big((n+1)^{\sigma+2}\zeta\Big(\frac{1}{n+1}\Big)\Big). \quad (28)$$

Combining (19), (27) and (28), we get

$$\|\tilde{\mathcal{H}}'_{gh}(\tilde{f}'; y) - \tilde{f}'(y)\|_1 = O\Big((n+1)^{(\sigma+2)}\zeta\Big(\frac{1}{n+1}\Big)\Big). \quad (29)$$

□

Remark 4.4. *In view of Remark 2.1, two corollaries can be deduced from the Theorem 4.1.*

4.2. Applications

4.2.1. Degree of convergence of a function $\tilde{f}'$ in $W(L^p, \zeta(l))$ class

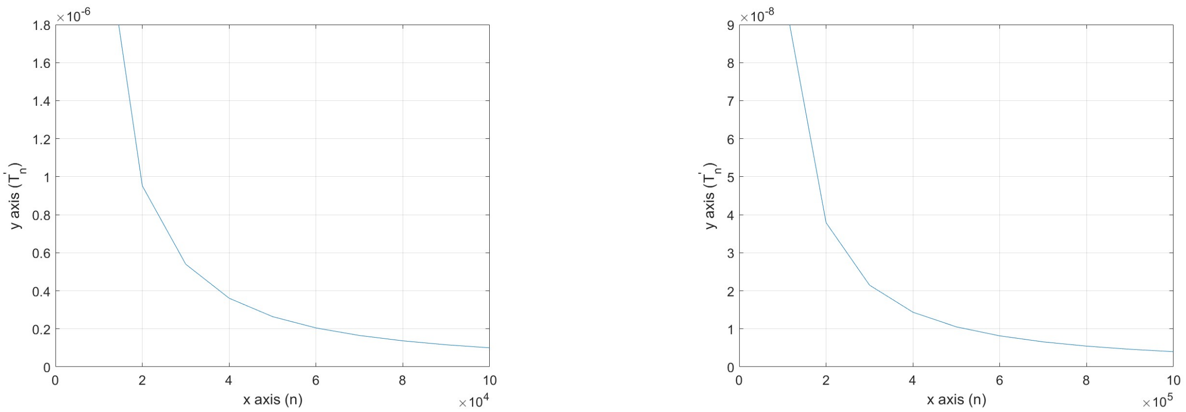
(i) Considering $\zeta\Big(\frac{1}{n+1}\Big) = \Big(\frac{e^{\frac{1}{n+1}}}{(n+1)^3}\Big)$, $\sigma = 0.1$, $p = 2$. From (22), we write

$$\|T'_n(y)\|_2 = \|\tilde{\mathcal{H}}'_{gh}(\tilde{f}'; y) - \tilde{f}'(y)\|_2 = O\Big(\frac{e^{\frac{1}{n+1}}}{(n+1)^{1.4}}\Big).$$

n	$\|T'_n(y)\|_2 = O\Big(\frac{e^{\frac{1}{n+1}}}{(n+1)^{1.4}}\Big)$
10000	0.000002512
100000	0.000000100
1000000	0.000000004
.	.
∞	0.0

Table 1: Values of $\|T'_n(\cdot)\|$ for different values of n.

Now, we draw the following graphs of $T'_n(\cdot)$ for different values of n:



(a) For n=100000

(b) For n=1000000

Figure 1: Graphs of $\|T'_n(\cdot)\|$ for different values of n.

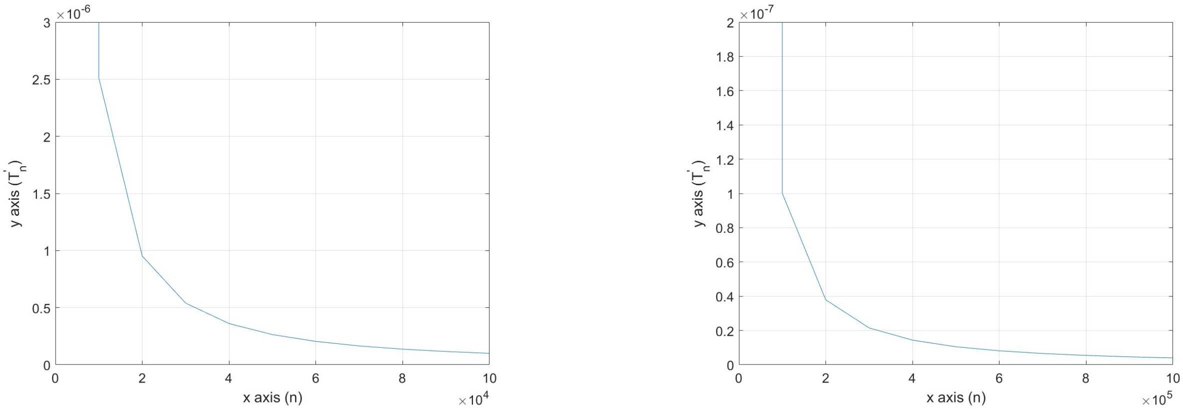
(ii) Considering $\zeta\Big(\frac{1}{n+1}\Big) = \frac{1}{(n+1)^3}$, $\sigma = 0.1$, $p = 2$. From (22), we can write

$$\|T_n^{'}(y)\|_p = \|\tilde{\mathcal{H}}_{gh}^{'}(\tilde{f}'; y) - \tilde{f}'(y)\|_p = O\Big(\frac{1}{(n+1)^{1.4}}\Big).$$

n	$\|T_n^{'}(y)\|_p = O\Big(\frac{1}{(n+1)^{1.4}}\Big)$
10000	0.000002512
100000	0.000000100
1000000	0.000000004
.	.
∞	0.0

Table 2: Values of $\|T_n'(\cdot)\|$ for different values of n.

Now, we draw the following graphs of $T_n^{'}(\cdot)$ for the different values of n:



(a) For n=100000 (b) For n=1000000

Figure 2: Graphs of $\|T_n'(\cdot)\|$ for different values of n.

4.2.2. Degree of convergence of a function $\tilde{f}'$ in $W(L^1, \zeta(l))$ class

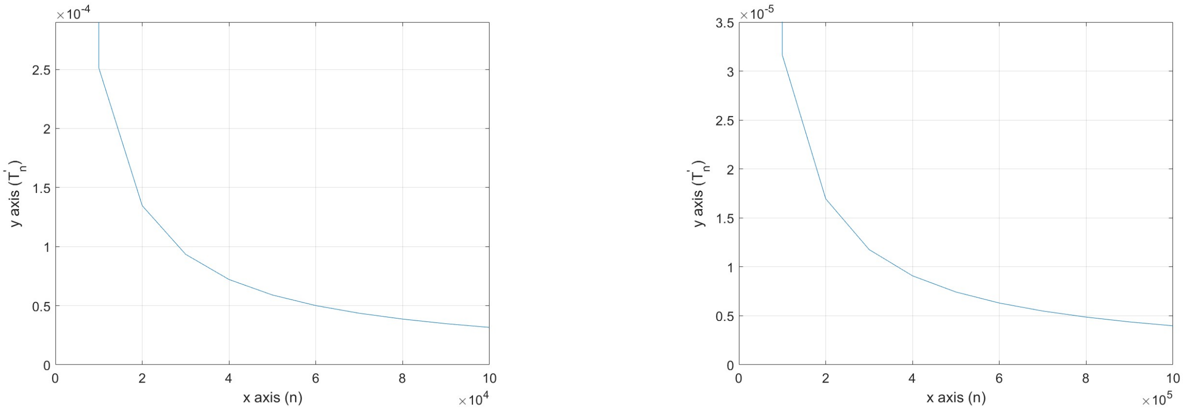
(i) Considering $\zeta\Big(\frac{1}{n+1}\Big) = \Big(\frac{e^{\frac{1}{n+1}}}{(n+1)^3}\Big)$, $\sigma = 0.1$. From (29), we write

$$\|T_n^{'}(y)\|_1 = \|\tilde{\mathcal{H}}_{gh}^{'}(\tilde{f}'; y) - \tilde{f}'(y)\|_1 = O\Big(\frac{e^{\frac{1}{n+1}}}{(n+1)^{0.9}}\Big).$$

n	$\|T_n'(y)\|_1 = O\left(\frac{e^{\frac{1}{n+1}}}{(n+1)^{0.9}}\right)$
10000	0.000251191
100000	0.000031623
1000000	0.000003981
.	.
∞	0.0

Table 3: Values of $\|T_n'(\cdot)\|$ for different values of n.

Now, we draw the following graphs of $T_n'(\cdot)$ for different values of n:



(a) For n=100000 (b) For n=1000000

Figure 3: Graphs of $\|T_n'(\cdot)\|$ for different values of n.

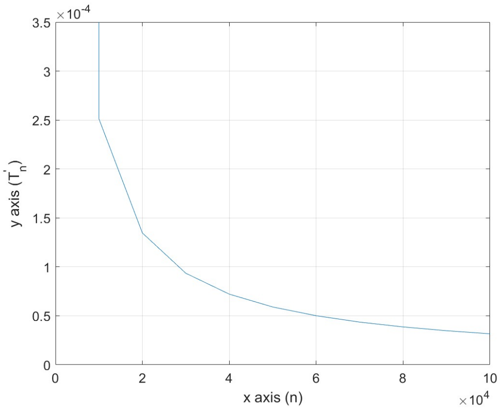
(ii) Consider $\zeta\left(\frac{1}{n+1}\right) = \frac{1}{(n+1)^3}$, $\sigma = 0.1$. From (29), we can write

$$\|T_n'(y)\|_1 = \|\tilde{\mathcal{H}}_{gh}'(\tilde{f}'; y) - \tilde{f}'(y)\|_1 = O\left(\frac{1}{(n+1)^{0.9}}\right).$$

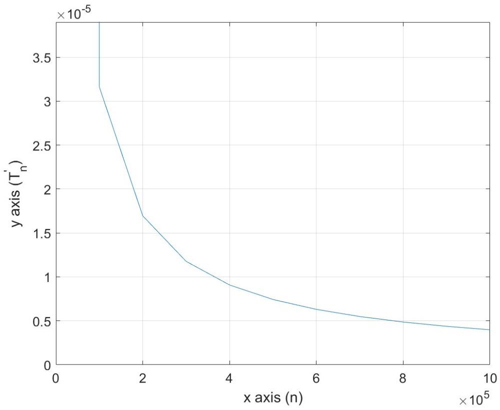
n	$\|T_n'(y)\|_1 = O\left(\frac{1}{(n+1)^{0.9}}\right)$
10000	0.000251166
100000	0.000031622
1000000	0.000003981
.	.
∞	0.0

Table 4: Values of $\|T_n'(\cdot)\|$ for different values of n.

Now, we draw the following graphs of $T_n'(\cdot)$ for the different values of n:



(a) For n=100000



(b) For n=1000000

Figure 4: Graphs of $\|T_n'(\cdot)\|$ for different values of n.

5. Analysis of convergence of a function in generalized Hölder spaces

5.1. Result and its proof

Now, we establish a theorem to study the convergence of function $\tilde{f}'$ of derived Fourier series in generalized Hölder spaces $((H^p_{\xi,\eta}, p \geq 1))$ class using product Hausdorff operator. In fact, we prove the following theorem:

Theorem 5.1. *If $\tilde{f}'$ is a derived conjugate function of 2π periodic and Lebesgue integrable function f, then the degree of convergence of generalized Hölder $(H^p_{\xi,\eta}, p \geq 1)$ class using product Hausdorff operator, is given by*

$$\|\tilde{\mathcal{H}}'_{gh}(\tilde{f}'; y) - \tilde{f}'(y)\|_p = O\Big(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} \frac{\xi(l)}{\eta(l)}\frac{l+1}{l^3}dl\Big).$$

Proof. Let

$$T_n'(y) = \tilde{\mathcal{H}}'_{gh}(\tilde{f}'; y) - \tilde{f}'(y) = \int_0^{\pi} \varrho_y(l)(\tilde{M}_1' + \tilde{M}_2')dl.$$

or

$$T_n'(y) = \int_0^{\pi} \varrho_y(l)(\tilde{M}_1' + \tilde{M}_2')dl.$$

So,

$$T_n'(y+t) - T_n'(y) = \int_0^{\pi} (\varrho(y+t,l) - \varrho(y,l))(\tilde{M}_1' + \tilde{M}_2')dl.$$

Using generalized Minkowski's inequality[1], we have

$$\begin{aligned}
&\|T_n'(\cdot + t) - T_n'(\cdot)\|_p \\
&\leq \int_0^{\pi} \|\varrho(\cdot + t, l) - \varrho(\cdot, l)\|_p(\tilde{M}_1' + \tilde{M}_2')dl \\
&= \int_0^{\frac{1}{n+1}} \|\varrho(\cdot + t, l) - \varrho(\cdot, l)\|_p(\tilde{M}_1' + \tilde{M}_2')dl + \int_{\frac{1}{n+1}}^{\pi} \|\varrho(\cdot + t, l) - \varrho(\cdot, l)\|_p(\tilde{M}_1' + \tilde{M}_2')dl
\end{aligned}$$

$$= U + V \tag{30}$$

Using Lemmas 3.1, 3.2 and 3.5 (iii), we get

$$\begin{aligned}
|U| &= \int_0^{\frac{1}{n+1}} \|\varrho(\cdot+t,l)-\varrho(\cdot,l)\|_p(\tilde{M}_1'+\tilde{M}_2')dl \\
&\leq \Big[\eta(|t|)\int_0^{\frac{1}{n+1}} \frac{\xi(l)}{\eta(l)}\Big(n+1+\frac{1}{l^2}\Big)dl\Big] \\
&= \Big[\eta(|t|)\lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{n+1}} \frac{\xi(l)}{\eta(l)}\Big(n+1+\frac{1}{l^2}\Big)dl\Big] \\
&= \Big[\eta(|t|)\frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\lim_{\epsilon\to 0}\Big(nl+l-\frac{1}{l}\Big)_{\epsilon}^{\frac{1}{n+1}}\Big] \\
&= \Big[\eta(|t|)\frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\Big((n+1)\Big(\frac{1}{n+1}\Big)-(n+1)\Big)\Big] \\
&= O\Big(n\ \eta(|t|)\frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\Big).
\end{aligned} \tag{31}$$

Using Lemmas 3.3, 3.4 and 3.5 (iii), we get

$$\begin{aligned}
|V| &= \int_{\frac{1}{n+1}}^{\pi} \|\varrho(\cdot+t,l)-\varrho(\cdot,l)\|_p(\tilde{M}_1'+\tilde{M}_2')dl \\
&= \Big[\eta(|t|)\int_{\frac{1}{n+1}}^{\pi} \frac{\xi(l)}{\eta(l)}\Big(\frac{1}{(n+1)l^2}+\frac{1}{(n+1)l^3}\Big)dl\Big] \\
&= O\Big(\frac{1}{n+1}\eta(|t|)\int_{\frac{1}{n+1}}^{\pi} \frac{\xi(l)}{\eta(l)}\Big(\frac{l+1}{l^3}\Big)dl\Big).
\end{aligned} \tag{32}$$

From (30), (31) and (32) we get

$$\sup_{t\neq 0}\frac{\|T_n'(\cdot+t)-T_n'(\cdot)\|_p}{\eta(|t|)} = O\Big(n\ \frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\Big)+O\Big(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi(l)}{\eta(l)}\Big(\frac{l+1}{l^3}\Big)dl\Big). \tag{33}$$

Again applying generalized Minkowski's inequality [1] and using Lemma 3.5 (i), we have

$$\begin{aligned}
\|T_n'(\cdot)\|_p &= \|\tilde{\mathcal{H}}_{gh}'(\tilde{f}';y)-\tilde{f}'(y)\|_p \\
&\leq \Big(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\Big)\|\varrho(\cdot,l)\|_p(\tilde{K_1}'+\tilde{K_2}')dl \\
&= O\Big(\int_0^{\frac{1}{n+1}}\Big((n+1)+\frac{1}{l^2}\Big)\xi(l)dl\Big)+O\Big(\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{1}{n+1}\Big(\frac{l+1}{l^3}\Big)\xi(l)\Big)dl\Big) \\
&= O\Big(\xi\Big(\frac{1}{n+1}\Big)n\Big)+O\Big(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\xi(l)\Big(\frac{l+1}{l^3}\Big)dl\Big).
\end{aligned} \tag{34}$$

We know that

$$\|T_n'(\cdot)\|_p^{(\eta)} = \|T_n(\cdot)\|_p+\sup_{t\neq 0}\frac{\|T_n'(\cdot,+t)-T_n'(\cdot)\|_p}{\eta(|t|)} \tag{35}$$

Using (33), (34) and (35) we get

$$\|T_n^{'}\|_p^{(\eta)} = O\Big(\xi\Big(\frac{1}{n+1}\Big)n\Big) + O\Big(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\xi(l)\Big(\frac{l+1}{l^3}\Big)dl\Big) + O\Big(n\,\frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\Big) + O\Big(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi(l)}{\eta(l)}\Big(\frac{l+1}{l^3}\Big)dl\Big). \quad (36)$$

Due to monotonicity of $\eta(l)$,

$$\xi(l) = \frac{\xi(l)}{\eta(l)}\eta(l) \leq \frac{\xi(\pi)}{\eta(\pi)}\eta(\pi)$$

for $0 < l \leq \pi$ we get

$$\|T_n^{'}(\cdot)\|_p^{(\eta)} = O\Big(n\,\frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\Big) + O\Big(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi(l)}{\eta(l)}\Big(\frac{l+1}{l^3}\Big)dl\Big). \quad (37)$$

Now

$$\begin{aligned}\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi(l)}{\eta(l)}\Big(\frac{l+1}{l^3}\Big)dl &\geq \frac{1}{n+1}\frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\int_{\frac{1}{n+1}}^{\pi}\Big(\frac{1}{l^2}+\frac{1}{l^3}\Big)dl\\ &= \frac{1}{n+1}\frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\Big[-\frac{1}{l}-\frac{1}{2l^2}\Big]_{\frac{1}{n+1}}^{\pi}\\ &= \frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\Big(\frac{n+3}{2}\Big)\\ &\geq O\Big(\frac{n}{2}\frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\Big).\end{aligned}$$

Thus

$$O\Big(n\frac{\xi(\frac{1}{n+1})}{\eta(\frac{1}{n+1})}\Big) = O\Big(\frac{2}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi(l)}{\eta(l)}\Big(\frac{l+1}{l^3}\Big)dl\Big). \quad (38)$$

Now from (37) and (38)

$$\|T_n^{'}(\cdot)\|_p^{(\eta)} = O\Big(\frac{3}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{\xi(l)}{\eta(l)}\Big(\frac{l+1}{l^3}\Big)dl\Big) \quad (39)$$

□

5.1.1. Corollary

Corollary 5.2. *Let $\tilde{f}' \in H_{(\gamma,\delta),p}; p \geq 1$ and assume that $\xi(l) = l^{\gamma}, \eta(l) = l^{\delta}$ and $0 \leq \delta < \gamma \leq 1$, then*

$$\|T_n'(\cdot)\|_p^{\eta} = \begin{cases} O\Big((n+1)^{\gamma-\delta-2}, & \text{if } 0 \leq \delta < \gamma < 1,\\ O\Big(\frac{1}{n+1}(\ln(\pi(n+1)) + (n+1))\Big), & \text{if } \delta = 0, \gamma = 1.\end{cases}$$

5.2. Applications

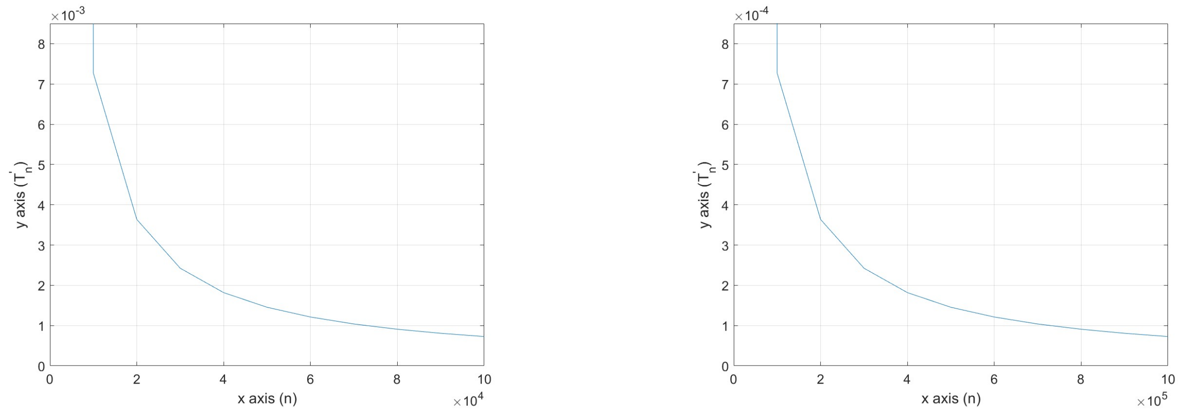
5.2.1. Degree of convergence of a function $\tilde{f}'$ in generalized Hölder spaces

(i) Considering $\frac{\xi(l)}{\eta(l)} = e^l l^3$. From (39), we can write

$$\|T_n^{'}(y)\|_p = \|\tilde{\mathcal{H}}_{gh}^{'}(\tilde{f}'; y) - \tilde{f}'(y)\|_p = O\Big(\frac{1}{n+1}\Big(\pi e^{\pi} - \frac{e^{\frac{1}{n+1}}}{n+1}\Big)\Big).$$

n	$\|T_n^{'}(y)\|_p = O\left(\frac{1}{n+1}\left(\pi e^{\pi} - \frac{e^{\frac{1}{n+1}}}{n+1}\right)\right)$
10000	0.00726
100000	0.00072
1000000	0.00007
.	.
∞	0.0

Table 5: Values of $\|T_n'(\cdot)\|$ for different values of n.



(a) For n=100000 (b) For n=1000000

Figure 5: Graphs of $\|T_n'(\cdot)\|$ for different values of n.

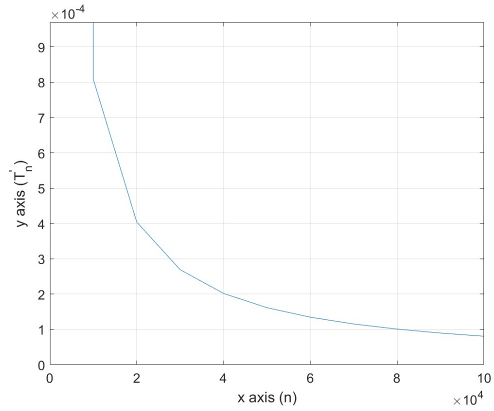
Now, we draw the following graphs of $T_n^{'}(\cdot)$ for different values of n:
(ii) Considering $\frac{\xi(l)}{\eta(l)} = l^3$. From (39), we can write

$$\|T_n^{'}(y)\|_p = \|\tilde{\mathcal{H}}_{gh}^{'}(\tilde{f}'; y) - \tilde{f}'(y)\|_p = O\left(\frac{1}{n+1}\left(\frac{\pi^2}{2} + \pi - \frac{1}{2(n+1)^2} - \frac{1}{n+1}\right)\right).$$

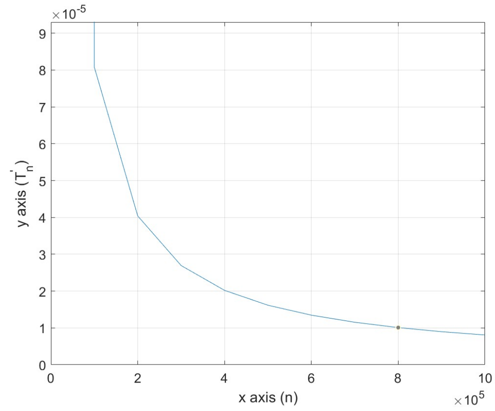
n	$\|T_n^{'}(y)\|_p = O\left(\frac{1}{n+1}\left(\frac{\pi^2}{2} + \pi - \frac{1}{2(n+1)^2} - \frac{1}{n+1}\right)\right)$
10000	0.00080
100000	0.00008
1000000	0.00000
.	.
∞	0.0

Table 6: Values of $\|T_n'(\cdot)\|$ for different values of n.

Now, we draw the following graphs of $T_n^{'}(\cdot)$ for the different values of n:



(a) For n=100000



(b) For n=1000000

Figure 6: Graphs of $\|T_n'(\cdot)\|$ for different values of n.

6. Conclusion

In every individual application, it is evident that the convergence rate of $\tilde{f}'$ grows more rapidly as the value of n increases. This implies that each unique norm of $\tilde{f}'$ offers the most accurate estimate. However, the rate of convergence in Application 4.2.1(i) is faster than that of Application 4.2.1(ii) and the rate of convergence in Application 4.2.2(i) is faster than that of Application 4.2.2(ii). Additionally, it is seen that the rate of convergence in Application 5.2.1(i) is faster than that of Application 5.2.1(ii).

References

[1] A. Zygmund, *Trigonometric series* (Vol. 1). Cambridge university press, 2002.

[2] M. M. Albaidani, H. M. Srivastava, A. H. Ganie, *Notion of non-absolute family of spaces*, International Journal of Nonlinear Analysis and Applications. **14(1)** (2023), 345–354.

[3] J. Boos, F. P. Cass, *Classical and modern methods in summability*, Clarendon Press, 2000.

[4] C. K. Chui, *An introduction to wavelets*, (Vol. 1). Academic press, 1992.

[5] Z. Eidinejad, R. Saadati, H. M. Srivastava, *The Cauchy–optimal stability results for Cauchy–Jensen additive mappings in the Fuzzy Banach space and the unital Fuzzy Banach Space*, Axioms, **12(4)**, (2023) 405, 1–14.

[6] H. L. Garabedian, *Hausdorff matrices*, The American Mathematical Monthly, **46(7)**, (1939), 390–410.

[7] F. Hausdorff, *Summationsmethoden and Momentfolgen* I. Mathematische Zeitschrift, **9(1)**, (1921), 74–109.

[8] S. Keska, *The degree of approximation by Hausdorff means of a conjugate Fourier series*, Annales Universitatis Mariae Curie-Sklodowska, section A–Mathematica, **70(2)**, (2016), 63.

[9] R. Kranz, A. Rzepka, E. Sylwestrzak-Maślanka, *Degrees of the approximations by some special matrix means of conjugate Fourier series*, Demonstratio Mathematica, **52(1)**, (2019), 116–129.

[10] X. Z. Krasniqi, B. Szal, *On the degree of approximation of continuous functions by means of Fourier series in the Hölder metric*, Anal. Theory Appl., **35(4)**, (2019), 392–404.

[11] S. Lal, A. Mishra, *Approximation of signals of generalized Lipschitz class by* $(N, p_n)(E, 1)$ *summability means of Fourier series*, Pacific Journal of Applied Mathematics, **5(1)**, (2013), 69.

[12] S. Lal, A. Mishra, *The method of summation* $(E, 1)(N, p_n)$ *and trigonometric approximation of function in generalized Hölder metric*, J. Indian Math. Soc, **80(1-2)**, (2013), 87-–98.

[13] A. F. Moursund, *On summation of derived series of the conjugate Fourier series*, American Journal of Mathematics, **57(4)**, (1935), 854–860.

[14] M. Mursaleen, A. Alotaibi, *Generalized matrix summability of a conjugate derived Fourier series*, Journal of Inequalities and Applications, (2017), 1–6.

[15] H. K. Nigam, *A study on approximation of conjugate of functions belonging to Lipschitz class and generalized Lipschitz class by product summability means of conjugate series of Fourier series*, Thai Journal of Mathematics, **10(2)**, (2012), 275–287.

[16] H. K. Nigam, H. M. Srivastava, S. Nandy, *Degree of convergence of a function of several variables in generalized Hölder spaces*, Mathematical Methods in the Applied Sciences, **46(8)**, (2023), 9028–9052.

[17] H. K. Nigam, S. Yadav, *Degree of convergence of the functions of trigonometric series in Sobolev spaces and its applications*, Journal of Inequalities and Applications, **2022(1)**, (2022), 59.

[18] H. K. Nigam, M. Hadish, *Best approximation of functions in generalized Hölder class*, Journal of Inequalities and Applications, **2018(1)**, (2018), 1–15.

[19] H. K. Nigam, *Best approximation of a function by product operator*, Electronic Journal of Mathematical Analysis and Applications, **8(1)**, (2020), 199–208.

[20] H. K. Nigam, M. Hadish, *Functional approximation in Besov space using generalized Nörlund–Hausdorff product matrix*, Journal of Inequalities and Applications, **2019(1)**, (2019), 1–17.

[21] H. K. Nigam, M. Hadish, *A study on approximation of a conjugate function using Cèsaro-matix product operator*, Nonlinear Functional Analysis and Applications, **25(4)**, (2020), 697–713.

[22] B. E. Rhoades, *On the degree of approximation of functions belonging to the weighted* $(L^p, \xi(t))$ *class by Hausdorff means*, Tamkang Journal of Mathematics, **32(4)**, (2001), 305–314.

[23] B. E. Rhoades, *On the degree of approximation of functions belonging to a Lipschitz class by Hausdorff means of its Fourier series*, Tamkang Journal of Mathematics, **34(3)**, (2003), 245–247.

[24] B. E. Rhoades, *The degree of approximation of functions and their conjugates belonging to several general Lipschitz classes by Hausdorff matrix means of the Fourier series and conjugate series of a Fourier series*, Tamkang J. Math., **45(4)**, (2014) 389-–395.

[25] B. Singh, U. Singh, *Some characterizations of Hausdorff matrices and their application to Fourier approximation*, Journal of Computational and Applied Mathematics, **367**,(2020), 112450.

[26] U. Singh, S. K. Srivastava, *Approximation of conjugate of functions belonging to weighted Lipschitz class* $W(L_p, \xi(t))$ *by Hausdorff means of conjugate Fourier series*, Journal of Computational and Applied Mathematics, **259**, (2014), 633–640.

[27] H. M. Srivastava, A. Das, B. Hazarika, S. A. Mohiuddine, (2018). *Existence of solutions of infinite systems of differential equations of general order with boundary conditions in the spaces* c_0 *and* l_1 *via the measure of noncompactness*, Mathematical methods in the applied sciences, **41(10)**, (2018), 3558–3569.

[28] H. M. Srivastava, M. I. Abbas, A. Boutiara, B. Hazarika, (2023). *Fractional p-Laplacian differential equations with multi-point boundary conditions in Banach spaces*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, **117(2)**, (2023), 68, 1–16.

[29] H. M. Srivastava, T. Yayingg, B. Hazarika, *A study of the q-analogue of the paranormed Ces'aro sequence spaces*, Filomat, **38(1)**, (2024), 99–117.