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Nonlinear mixed bi-skew Jordan-type derivations on prime $*$-algebras

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Abstract. Let $\mathcal{A}$ be a unite prime $*$-algebra containing a non-trivial projection. Assume that $\phi : \mathcal{A} \to \mathcal{A}$ satisfies $\phi(A_1 \diamond_1 A_2 \diamond_2 \cdots \diamond_n A_{n+1}) = \sum_{h=1}^{n+1} A_1 \diamond_1 \cdots \diamond_{h-2} A_{h-1} \diamond_{h-1} \phi(A_h) \diamond_h A_{h+1} \diamond_{h+1} \cdots \diamond_n A_{n+1} (n \geq 2)$ for any $A_1, A_2, \cdots, A_{n+1} \in \mathcal{A}$ and $\diamond_r$ is $\bullet$ or $\circ$ with $1 \leq r \leq n$, where $A \bullet B = AB^* + BA^*$ and $A \circ B = AB + BA$. In this article, we prove that if n is even and $\diamond_{2u-1} = \bullet$, $\diamond_{2u} = \circ$ with $1 \leq u \leq \frac{n}{2}$, then there exists an element $\lambda \in \mathcal{Z}_S(\mathcal{A})$ such that $\phi(A) = \delta(A) + i\lambda A$, where δ is an additive $*$-derivation. Otherwise, ϕ is an additive $*$-derivation. In particular, the nonlinear mixed bi-skew Jordan-type derivations on factor von Neumann algebras and standard operator algebras are characterized.

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. For any $A, B \in \mathcal{A}$, we say the products $A * B = AB + BA^*$ and $A \bullet B = AB^* + BA^*$ are the $*$-Jordan product and the bi-skew Jordan product, respectively. These two products have been studied by a lot of scholars in many topics, see [1–10]. Recall that an additive map $\phi : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\phi(AB) = \phi(A)B + A\phi(B)$ for all $A, B \in \mathcal{A}$. Besides, if $\phi(A^*) = \phi(A)^*$ for all $A \in \mathcal{A}$, then ϕ is an additive $*$-derivation. Correspondingly, a map (without the additivity assumption) $\phi : \mathcal{A} \to \mathcal{A}$ is called a nonlinear $*$-Jordan derivation if $\phi(A * B) = \phi(A) * B + A * \phi(B)$ for all $A, B \in \mathcal{A}$, and is called a nonlinear bi-skew Jordan derivation if $\phi(A \bullet B) = \phi(A) \bullet B + A \bullet \phi(B)$ for all $A, B \in \mathcal{A}$. Taghavi et al. [11] showed that each nonlinear $*$-Jordan derivation on factor von Neumann algebras is an additive $*$-derivation. Darvish et al. [12] prove that each nonlinear bi-skew Jordan derivation on prime $*$-algebras is an additive $*$-derivation. In addition, Zhao et al. [13] and Khan et al. [14] extended to the cases of nonlinear $*$-Jordan triple derivations on von Neumann algebras with no central summands of type I_1 and nonlinear bi-skew Jordan triple derivations on prime $*$-algebras, respectively. With the nonlinear $*$-Jordan triple derivation and the nonlinear bi-skew Jordan triple derivation. A map (without the additivity assumption) $\phi : \mathcal{A} \to \mathcal{A}$ is called a nonlinear $*$-Jordan-type derivation if

$$\phi(A_1 * A_2 * \cdots * A_{n+1}) = \sum_{h=1}^{n+1} A_1 * \cdots * A_{h-1} * \phi(A_h) * A_{h+1} * \cdots * A_{n+1}$$

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for any $A_1, A_2, \cdots, A_{n+1} \in \mathcal{A}$, where $A_1 * A_2 * \cdots * A_{n+1} = (\cdots((A_1 * A_2) * A_3) \cdots * A_n)$, and is called a nonlinear bi-skew Jordan-type derivation if

$$\phi(A_1 \bullet A_2 \bullet \cdots \bullet A_{n+1}) = \sum_{h=1}^{n+1} A_1 \bullet \cdots \bullet A_{h-1} \bullet \phi(A_h) \bullet A_{h+1} \bullet \cdots \bullet A_{n+1}$$

for any $A_1, A_2, \cdots, A_{n+1} \in \mathcal{A}$. Li et al. [15] proved that any nonlinear $*$-Jordan-type derivation on $*$-algebras is an additive $*$-derivation. Ashraf et al. [16] obtained similar structure of the nonlinear bi-skew Jordan-type derivation on $*$-algebras.

Recently, many researchers have shown great interest in the study of maps related to mixed products comprising skew Jordan products or bi-skew Jordan products, see [17–21]. For instance, a map (without the additivity assumption) $\phi : \mathcal{A} \to \mathcal{A}$ is called a second nonlinear mixed Jordan triple derivation if

$$\phi(A \circ B * C) = \phi(A) \circ B * C + A \circ \phi(B) * C + A \circ B * \phi(C)$$

for all $A, B, C \in \mathcal{A}$, where $A \circ B = AB + BA$. Rehman et al. [22] proved that every second nonlinear mixed Jordan triple derivation on $*$-algebras is an additive $*$-derivation. Let $\phi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption), then ϕ is called a nonlinear mixed Jordan triple derivation on $\mathcal{A}$ if

$$\phi(A * B \circ C) = \phi(A) * B \circ C + A * \phi(B) \circ C + A * B \circ \phi(C)$$

for all $A, B, C \in \mathcal{A}$. Ning and Zhang [23] proved that each nonlinear mixed Jordan triple derivation on factor von Neuamnn algebras is an additive $*$-derivation. Similarly, a map (without the additivity assumption) $\phi : \mathcal{A} \to \mathcal{A}$ is called a second nonlinear mixed bi-skew Jordan triple derivation if

$$\phi(A \circ B \bullet C) = \phi(A) \circ B \bullet C + A \circ \phi(B) \bullet C + A \circ B \bullet \phi(C) \tag{1.1}$$

for all $A, B, C \in \mathcal{A}$, and is called a nonlinear mixed bi-skew Jordan triple derivation if

$$\phi(A \bullet B \circ C) = \phi(A) \bullet B \circ C + A \bullet \phi(B) \circ C + A \bullet B \circ \phi(C) \tag{1.2}$$

for all $A, B, C \in \mathcal{A}$. In [24], Ferreira et al. considered a map $\phi : \mathcal{A} \to \mathcal{A}$ such that

$$\phi(A_1 \circ A_2 \circ \cdots \circ A_n \bullet A_{n+1}) = \sum_{h=1}^{n+1} A_1 \circ \cdots \circ A_{h-1} \circ \phi(A_h) \circ A_{h+1} \circ \cdots \circ A_n \bullet A_{n+1} \tag{1.3}$$

for any $A_1, A_2, \cdots, A_{n+1} \in \mathcal{A}$, which is called a nonlinear mixed $*$-Jordan-type derivation. We can see that if ϕ satisfies Eq. (1.3) with $n = 2$, then ϕ is Eq. (1.1). Also, the authors [24] prove that each nonlinear mixed $*$-Jordan-type derivation on $*$-algebras is an additive $*$-derivation. Define a map $\phi : \mathcal{A} \to \mathcal{A}$ such that $\phi(A) = [A, T] - iA$, where $T^* = -T$. It is easy check that ϕ is a nonlinear mixed bi-skew Jordan triple derivation, but it does not an additive $*$-derivation. Encouraged by the above work, let $\phi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption). If

$$\phi(A_1 \diamond_1 A_2 \diamond_2 \cdots \diamond_n A_{n+1}) = \sum_{h=1}^{n+1} A_1 \diamond_1 \cdots \diamond_{h-2} A_{h-1} \diamond_{h-1} \phi(A_h) \diamond_h A_{h+1} \diamond_{h+1} \cdots \diamond_n A_{n+1} \tag{1.4}$$

for any $A_1, A_2, \cdots, A_{n+1} \in \mathcal{A}$ $(n \geq 2)$, where $\diamond_r$ is $\bullet$ or $\circ$ with $1 \leq r \leq n$, then ϕ is called a nonlinear mixed bi-skew Jordan-type derivation. Obviously, take $\diamond_r = \circ$ with $1 \leq r \leq n-1$ and $\diamond_n = \bullet$ in Eq. (1.4), then ϕ is Eq. (1.3). Meanwhile, if ϕ satisfies Eq. (1.4) with $\diamond_1 = \bullet$, $\diamond_2 = \circ$ and $n = 2$, we can obtain that ϕ is Eq. (1.2). Hence, Eqs. (1.2) and (1.3) are special forms of Eq. (1.4). In this paper, we will give the structure of the nonlinear mixed bi-skew Jordan-type derivation on prime $*$-algebras. Let $\mathcal{A}$ be a prime $*$-algebra, i.e. $A = 0$ or $B = 0$ if $A\mathcal{A}B = 0$, and $\mathcal{A}_{sa} = \{A \in \mathcal{A} : A^* = A\}$. Denote by $\mathcal{Z}(\mathcal{A})$ the central of $\mathcal{A}$ and $\mathcal{Z}_S(\mathcal{A}) = \mathcal{Z}(\mathcal{A}) \cap \mathcal{A}_{sa}$.

2. Additivity

In this section, we will prove the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be a unite prime $*$-algebra containing a non-trivial projection, and let $\phi : \mathcal{A} \to \mathcal{A}$ such that*

$$\phi(A_1 \diamond_1 A_2 \diamond_2 \cdots \diamond_n A_{n+1}) = \sum_{h=1}^{n+1} A_1 \diamond_1 \cdots \diamond_{h-2} A_{h-1} \diamond_{h-1} \phi(A_h) \diamond_h A_{h+1} \diamond_{h+1} \cdots \diamond_n A_{n+1}$$

for any $A_1, A_2, \cdots, A_{n+1} \in \mathcal{A}$ with $n \geq 2$, then ϕ is additive.

To prove Theorem 2.1, we need some lemmas.

Lemma 2.2. $\phi(0) = 0$.

Proof. It is clear that

$$\phi(0) = \sum_{h=1}^{n+1} 0 \diamond_1 \cdots \diamond_{h-2} 0 \diamond_{h-1} \phi(0) \diamond_h 0 \diamond_{h+1} \cdots \diamond_n 0 = 0.$$

The proof is completed.

Let $P_1 \in \mathcal{A}$ be a non-trivial projection and $P_2 = I - P_1$, where I is the unite of this algebra. Put $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$ for $i, j = 1, 2$. Then by Peirce decomposition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Note that any $T \in \mathcal{A}$ can be written as $T = T_{11} + T_{12} + T_{21} + T_{22}$, where $T_{ij} \in \mathcal{A}_{ij}$ for $i, j = 1, 2$. From [24] and [26], we only need to consider the case when at least one of $\diamond_r$ is $\bullet$, where $r \in \{1, 2, 3 \cdots, n-1\}$. Let $\diamond_s = \bullet$ and $\diamond_r = \circ$ with $1 \leq r \leq s-1$.

$$\Gamma\langle A, B, C, D\rangle = \underbrace{A \diamond_1 A \diamond_2 \cdots \diamond_{s-2} A}_{s-1} \diamond_{s-1} B \diamond_s C \diamond_{s+1} D \diamond_{s+2} \underbrace{A \diamond_{s+3} \cdots \diamond_n A}_{n-s-1}$$

and

$$\Gamma_m^{\phi}\langle A, B, C, D\rangle = A \diamond_1 A \diamond_2 \cdots \diamond_{m-1} \phi(A) \diamond_m \cdots \diamond_{s-2} A \diamond_{s-1} B \diamond_s C \diamond_{s+1} D \diamond_{s+2} A \diamond_{s+3} \cdots \diamond_n A$$

for any $A, B, C, D \in \mathcal{A}$, where $1 \leq m \leq s-1$, $s+3 \leq m \leq n+1$.

Lemma 2.3. $\phi(\sum_{i,j=1}^2 A_{ij}) = \sum_{i,j=1}^2 \phi(A_{ij})$ *for all $A_{ij} \in \mathcal{A}_{ij}$ with $1 \leq i, j \leq 2$.*

Proof. Let $T = \phi(\sum_{i,j=1}^2 A_{ij}) - \sum_{i,j=1}^2 \phi(A_{ij})$. For $1 \leq k \neq l \leq 2$, it follows from $\Gamma\langle \frac{I}{2}, P_k, A_{kk}, P_l\rangle = 0$, $\Gamma\langle \frac{I}{2}, P_k, A_{ll}, P_l\rangle = 0$ and $\Gamma\langle \frac{I}{2}, P_k, A_{kl}, P_l\rangle = 0$ that

$$\begin{aligned}
&\phi(\Gamma\langle \frac{I}{2}, P_k, \sum_{i,j=1}^2 A_{ij}, P_l\rangle) = \sum_{i,j=1}^2 \phi(\Gamma\langle \frac{I}{2}, P_k, A_{ij}, P_l\rangle) \\
&= \sum_{m=1}^{s-1} \Gamma_m^{\phi}\langle \frac{I}{2}, P_k, \sum_{i,j=1}^2 A_{ij}, P_l\rangle + \sum_{m=s+3}^{n+1} \Gamma_m^{\phi}\langle \frac{I}{2}, P_k, \sum_{i,j=1}^2 A_{ij}, P_l\rangle \\
&+ \Gamma\langle \frac{I}{2}, \phi(P_k), \sum_{i,j=1}^2 A_{ij}, P_l\rangle + \Gamma\langle \frac{I}{2}, P_k, \sum_{i,j=1}^2 \phi(A_{ij}), P_l\rangle + \Gamma\langle \frac{I}{2}, P_k, \sum_{i,j=1}^2 A_{ij}, \phi(P_l)\rangle.
\end{aligned}$$

On the other hand,

$$\phi(\Gamma\langle\frac{I}{2},P_k,\sum_{i,j=1}^{2}A_{ij},P_l\rangle)=\sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2},P_k,\sum_{i,j=1}^{2}A_{ij},P_l\rangle+\sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2},P_k,\sum_{i,j=1}^{2}A_{ij},P_l\rangle$$
$$+\Gamma\langle\frac{I}{2},\phi(P_k),\sum_{i,j=1}^{2}A_{ij},P_l\rangle+\Gamma\langle\frac{I}{2},P_k,\phi(\sum_{i,j=1}^{2}A_{ij}),P_l\rangle+\Gamma\langle\frac{I}{2},P_k,\sum_{i,j=1}^{2}A_{ij},\phi(P_l)\rangle,$$

which implies that $\Gamma\langle\frac{I}{2},P_k,T,P_l\rangle=0$. Thus $P_kT^*P_l+P_lTP_k=0$, and so $T_{lk}=0$. For any $X_{kl}\in\mathcal{A}_{kl}$, it follows from $\Gamma\langle\frac{I}{2},X_{kl},A_{kk},P_l\rangle=0$, $\Gamma\langle\frac{I}{2},X_{kl},A_{kl},P_l\rangle=0$ and $\Gamma\langle\frac{I}{2},X_{kl},A_{lk},P_l\rangle=0$ that

$$\phi(\Gamma\langle\frac{I}{2},X_{kl},\sum_{i,j=1}^{2}A_{ij},P_l\rangle)=\sum_{i,j=1}^{2}\phi(\Gamma\langle\frac{I}{2},X_{kl},A_{ij},P_l\rangle)$$
$$=\sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{kl},\sum_{i,j=1}^{2}A_{ij},P_l\rangle+\sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{kl},\sum_{i,j=1}^{2}A_{ij},P_l\rangle$$
$$+\Gamma\langle\frac{I}{2},\phi(X_{kl}),\sum_{i,j=1}^{2}A_{ij},P_l\rangle+\Gamma\langle\frac{I}{2},X_{kl},\sum_{i,j=1}^{2}\phi(A_{ij}),P_l\rangle+\Gamma\langle\frac{I}{2},X_{kl},\sum_{i,j=1}^{2}A_{ij},\phi(P_l)\rangle.$$

On the other hand,

$$\phi(\Gamma\langle\frac{I}{2},X_{kl},\sum_{i,j=1}^{2}A_{ij},P_l\rangle)=\sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{kl},\sum_{i,j=1}^{2}A_{ij},P_l\rangle$$
$$+\sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{kl},\sum_{i,j=1}^{2}A_{ij},P_l\rangle+\Gamma\langle\frac{I}{2},\phi(X_{kl}),\sum_{i,j=1}^{2}A_{ij},P_l\rangle$$
$$+\Gamma\langle\frac{I}{2},X_{kl},\phi(\sum_{i,j=1}^{2}A_{ij}),P_l\rangle+\Gamma\langle\frac{I}{2},X_{kl},\sum_{i,j=1}^{2}A_{ij},\phi(P_l)\rangle.$$

This implies that $\Gamma\langle\frac{I}{2},X_{kl},T,P_l\rangle=0$. Thus $X_{kl}T^*P_l+P_lTX_{kl}^*=0$. It follows from the primeness of $\mathcal{A}$ that $T_{ll}=0$. Hence $T=0$. The proof is completed.

Lemma 2.4. *For all $A_{ij},B_{ij}\in\mathcal{A}_{ij}$ with $(i\neq j)$, we have*
(1) $\phi(A_{12}+B_{12})=\phi(A_{12})+\phi(B_{12})$;
(2) $\phi(A_{21}+B_{21})=\phi(A_{21})+\phi(B_{21})$.

Proof. Let $T=\phi(A_{12}+B_{12})-(\phi(A_{12})+\phi(B_{12}))$. For any $X_{kl}\in\mathcal{A}_{kl}$, it follows from $\Gamma\langle\frac{I}{2},X_{kl},A_{12},P_l\rangle=0$ that

$$\phi(\Gamma\langle\frac{I}{2},X_{kl},(A_{12}+B_{12}),P_l\rangle)=\phi(\Gamma\langle\frac{I}{2},X_{kl},A_{12},P_l\rangle)+\phi(\Gamma\langle\frac{I}{2},X_{kl},B_{12},P_l\rangle)$$
$$=\sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{kl},(A_{12}+B_{12}),P_l\rangle+\sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{kl},(A_{12}+B_{12}),P_l\rangle$$
$$+\Gamma\langle\frac{I}{2},\phi(X_{kl}),(A_{12}+B_{12}),P_l\rangle+\Gamma\langle\frac{I}{2},X_{kl},(\phi(A_{12})+\phi(B_{12})),P_l\rangle$$
$$+\Gamma\langle\frac{I}{2},X_{kl},(A_{12}+B_{12}),\phi(P_l)\rangle.$$

On the other hand,

$$\phi(\Gamma\langle \frac{I}{2}, X_{kl}, (A_{12}+B_{12}), P_l\rangle) = \sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle \frac{I}{2}, X_{kl}, (A_{12}+B_{12}), P_l\rangle$$
$$+ \sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle \frac{I}{2}, X_{kl}, (A_{12}+B_{12}), P_l\rangle + \Gamma\langle \frac{I}{2}, \phi(X_{kl}), (A_{12}+B_{12}), P_l\rangle$$
$$+ \Gamma\langle \frac{I}{2}, X_{kl}, \phi(A_{12}+B_{12}), P_l\rangle + \Gamma\langle \frac{I}{2}, X_{kl}, (A_{12}+B_{12}), \phi(P_l)\rangle.$$

This implies that $\Gamma\langle \frac{I}{2}, X_{kl}, T, P_l\rangle = 0$. Thus $X_{kl}T^*P_l + P_lTX_{kl}^* = 0$, and so $T_{ll} = 0$. It follows from $\Gamma\langle \frac{I}{2}, P_1, A_{12}, P_2\rangle = 0$ that

$$\phi(\Gamma\langle \frac{I}{2}, P_1, (A_{12}+B_{12}), P_2\rangle) = \phi(\Gamma\langle \frac{I}{2}, P_1, A_{12}, P_2\rangle) + \phi(\Gamma\langle \frac{I}{2}, P_1, B_{12}, P_2\rangle)$$
$$= \sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle \frac{I}{2}, P_1, (A_{12}+B_{12}), P_2\rangle + \sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle \frac{I}{2}, P_1, (A_{12}+B_{12}), P_2\rangle$$
$$+ \Gamma\langle \frac{I}{2}, \phi(P_1), (A_{12}+B_{12}), P_2\rangle + \Gamma\langle \frac{I}{2}, P_1, (\phi(A_{12})+\phi(B_{12})), P_2\rangle$$
$$+ \Gamma\langle \frac{I}{2}, P_1, (A_{12}+B_{12}), \phi(P_2)\rangle.$$

On the other hand,

$$\phi(\Gamma\langle \frac{I}{2}, P_1, (A_{12}+B_{12}), P_2\rangle) = \sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle \frac{I}{2}, P_1, (A_{12}+B_{12}), P_2\rangle$$
$$+ \sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle \frac{I}{2}, P_1, (A_{12}+B_{12}), P_2\rangle + \Gamma\langle \frac{I}{2}, \phi(P_1), (A_{12}+B_{12}), P_2\rangle$$
$$+ \Gamma\langle \frac{I}{2}, P_1, \phi(A_{12}+B_{12}), P_2\rangle + \Gamma\langle \frac{I}{2}, P_1, (A_{12}+B_{12}), \phi(P_2)\rangle.$$

This implies that $\Gamma\langle \frac{I}{2}, P_1, T, P_2\rangle = 0$. Thus $P_1T^*P_2 + P_2TP_1 = 0$. Hence $T_{21} = 0$.

It follows from the above expression that $T_{12} = \phi(A_{12}+B_{12}) - (\phi(A_{12}) + \phi(B_{12}))$. Meanwhile, there exists $S_{21} \in \mathcal{A}_{21}$ such that $S_{21} = \phi(A_{12}^*+B_{12}^*) - (\phi(A_{12}^*)+\phi(B_{12}^*))$. Since $\Gamma\langle \frac{I}{2}, (P_2+A_{12}^*), (P_1+B_{12}), \frac{I}{2}\rangle = A_{12}+B_{12}+A_{12}^*+B_{12}^*$, it follows from Lemma 2.3 that

$$\phi(A_{12}+B_{12}) + \phi(A_{12}^*+B_{12}^*) = \phi(\Gamma\langle \frac{I}{2}, (P_2+A_{12}^*), (P_1+B_{12}), \frac{I}{2}\rangle)$$
$$= \sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle \frac{I}{2}, (P_2+A_{12}^*), (P_1+B_{12}), \frac{I}{2}\rangle + \sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle \frac{I}{2}, (P_2+A_{12}^*), (P_1+B_{12}), \frac{I}{2}\rangle$$
$$+ \Gamma\langle \frac{I}{2}, (\phi(P_2)+\phi(A_{12}^*)), (P_1+B_{12}), \frac{I}{2}\rangle + \Gamma\langle \frac{I}{2}, (P_2+A_{12}^*), (\phi(P_1)+\phi(B_{12})), \frac{I}{2}\rangle$$
$$+ \Gamma\langle \frac{I}{2}, (P_2+A_{12}^*), (P_1+B_{12}), \phi(\frac{I}{2})\rangle = \phi(\Gamma\langle \frac{I}{2}, P_2, B_{12}, \frac{I}{2}\rangle) + \phi(\Gamma\langle \frac{I}{2}, A_{12}^*, P_1, \frac{I}{2}\rangle)$$
$$= \phi(A_{12}) + \phi(B_{12}) + \phi(A_{12}^*) + \phi(B_{12}^*).$$

This implies that $T_{12} + S_{21} = 0$, and so $T_{12} = 0$. Hence $T = 0$. Similarly, we can show that (2) holds. The proof is completed.

Lemma 2.5. *For all $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ with $i \in \{1, 2\}$, we have*

(1) $\phi(A_{11}+B_{11})=\phi(A_{11})+\phi(B_{11})$;
(2) $\phi(A_{22}+B_{22})=\phi(A_{22})+\phi(B_{22})$.

Proof. Let $T=\phi(A_{11}+B_{11})-(\phi(A_{11})+\phi(B_{11}))$. Since $\Gamma\langle\frac{I}{2},P_k,A_{11},P_l\rangle=0$, we have that

$$\begin{aligned}&\phi(\Gamma\langle\frac{I}{2},P_k,(A_{11}+B_{11}),P_l\rangle)=\phi(\Gamma\langle\frac{I}{2},P_k,A_{11},P_l\rangle)+\phi(\Gamma\langle\frac{I}{2},P_k,B_{11},P_l\rangle)\\&=\sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2},P_k,(A_{11}+B_{11}),P_l\rangle+\sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2},P_k,(A_{11}+B_{11}),P_l\rangle\\&+\Gamma\langle\frac{I}{2},\phi(P_k),(A_{11}+B_{11}),P_l\rangle+\Gamma\langle\frac{I}{2},P_k,(\phi(A_{11})+\phi(B_{11})),P_l\rangle\\&+\Gamma\langle\frac{I}{2},P_k,(A_{11}+B_{11}),\phi(P_l)\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\phi(\Gamma\langle\frac{I}{2},P_k,(A_{11}+B_{11}),P_l\rangle)=\sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2},P_k,(A_{11}+B_{11}),P_l\rangle\\&+\sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2},P_k,(A_{11}+B_{11}),P_l\rangle+\Gamma\langle\frac{I}{2},\phi(P_k),(A_{11}+B_{11}),P_l\rangle\\&+\Gamma\langle\frac{I}{2},P_k,\phi(A_{11}+B_{11}),P_l\rangle+\Gamma\langle\frac{I}{2},P_k,(A_{11}+B_{11}),\phi(P_l)\rangle.\end{aligned}$$

This implies that $\Gamma\langle\frac{I}{2},P_k,T,P_l\rangle=0$. Thus $P_kT^*P_l+P_lTP_k=0$, and so $T_{lk}=0$. For any $X_{12}\in\mathcal{A}_{12}$, it follows from $\Gamma\langle\frac{I}{2},X_{12},A_{11},P_2\rangle=0$ that

$$\begin{aligned}&\phi(\Gamma\langle\frac{I}{2},X_{12},(A_{11}+B_{11}),P_2\rangle)=\phi(\Gamma\langle\frac{I}{2},X_{12},A_{11},P_2\rangle)+\phi(\Gamma\langle\frac{I}{2},X_{12},B_{11},P_2\rangle)\\&=\sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{12},(A_{11}+B_{11}),P_2\rangle+\sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{12},(A_{11}+B_{11}),P_2\rangle\\&+\Gamma\langle\frac{I}{2},\phi(X_{12}),(A_{11}+B_{11}),P_2\rangle+\Gamma\langle\frac{I}{2},X_{12},(\phi(A_{11})+\phi(B_{11})),P_2\rangle\\&+\Gamma\langle\frac{I}{2},X_{12},(A_{11}+B_{11}),\phi(P_2)\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\phi(\Gamma\langle\frac{I}{2},X_{12},(A_{11}+B_{11}),P_2\rangle)=\sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{12},(A_{11}+B_{11}),P_2\rangle\\&+\sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2},X_{12},(A_{11}+B_{11}),P_2\rangle+\Gamma\langle\frac{I}{2},\phi(X_{12}),(A_{11}+B_{11}),P_2\rangle\\&+\Gamma\langle\frac{I}{2},X_{12},\phi(A_{11}+B_{11}),P_2\rangle+\Gamma\langle\frac{I}{2},X_{12},(A_{11}+B_{11}),\phi(P_2)\rangle.\end{aligned}$$

This implies that $\Gamma\langle\frac{I}{2},X_{12},T,P_2\rangle=0$. Thus $X_{12}T^*P_2+P_2TX_{12}^*=0$. Hence $T_{22}=0$. For any $X_{21}\in\mathcal{A}_{21}$, it follows from Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned}
&\phi(\Gamma\langle\frac{I}{2}, X_{21}, (A_{11}+B_{11}), P_1\rangle) = \phi(X_{21}A_{11}^*) + \phi(A_{11}X_{21}^*) + \phi(X_{21}B_{11}^*) + \phi(B_{11}X_{21}^*)\\
&= \phi(X_{21}A_{11}^* + A_{11}X_{21}^*) + \phi(X_{21}B_{11}^* + B_{11}X_{21}^*)\\
&= \phi(\Gamma\langle\frac{I}{2}, X_{21}, A_{11}, P_1\rangle) + \phi(\Gamma\langle\frac{I}{2}, X_{21}, B_{11}, P_1\rangle) = \sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2}, X_{21}, (A_{11}+B_{11}), P_1\rangle\\
&+ \sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2}, X_{21}, (A_{11}+B_{11}), P_1\rangle + \Gamma\langle\frac{I}{2}, \phi(X_{21}), (A_{11}+B_{11}), P_1\rangle\\
&+ \Gamma\langle\frac{I}{2}, X_{21}, (\phi(A_{11})+\phi(B_{11})), P_1\rangle + \Gamma\langle\frac{I}{2}, X_{21}, (A_{11}+B_{11}), \phi(P_1)\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\phi(\Gamma\langle\frac{I}{2}, X_{21}, (A_{11}+B_{11}), P_1\rangle) = \sum_{m=1}^{s-1}\Gamma_m^{\phi}\langle\frac{I}{2}, X_{21}, (A_{11}+B_{11}), P_1\rangle\\
&+ \sum_{m=s+3}^{n+1}\Gamma_m^{\phi}\langle\frac{I}{2}, X_{21}, (A_{11}+B_{11}), P_1\rangle + \Gamma\langle\frac{I}{2}, \phi(X_{21}), (A_{11}+B_{11}), P_1\rangle\\
&+ \Gamma\langle\frac{I}{2}, X_{21}, \phi(A_{11}+B_{11}), P_1\rangle + \Gamma\langle\frac{I}{2}, X_{21}, (A_{11}+B_{11}), \phi(P_1)\rangle.
\end{aligned}$$

This implies that $\Gamma\langle\frac{I}{2}, X_{21}, T, P_1\rangle = 0$. Thus $X_{21}T^*P_1 + P_1TX_{21}^* = 0$, and so $T_{11} = 0$. Hence $T = 0$. Similarly, we can show that (2) holds. The proof is completed.

Lemma 2.6. *ϕ is additive on $\mathcal{A}$.*

Proof. Let $A = \sum_{i,j=1}^{2} A_{ij}$, $B = \sum_{i,j=1}^{2} B_{ij}$, where $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$. It follows from Lemma 2.3-2.5 that

$$\begin{aligned}
\phi(A+B) &= \phi(\sum_{i,j=1}^{2} A_{ij} + \sum_{i,j=1}^{2} B_{ij}) = \phi(\sum_{i,j=1}^{2}(A_{ij}+B_{ij}))\\
&= \sum_{i,j=1}^{2}\phi(A_{ij}+B_{ij}) = \phi(\sum_{i,j=1}^{2} A_{ij}) + \phi(\sum_{i,j=1}^{2} B_{ij}) = \phi(A) + \phi(B).
\end{aligned}$$

Hence ϕ is additive. The proof is completed.

3. Structures

In this section, we will prove the following theorem.

Theorem 3.1. *Let $\mathcal{A}$ be a unite prime $*$-algebra containing a non-trivial projection, and let $\phi : \mathcal{A} \to \mathcal{A}$ such that*

$$\phi(A_1 \diamond_1 A_2 \diamond_2 \cdots \diamond_n A_{n+1}) = \sum_{h=1}^{n+1} A_1 \diamond_1 \cdots \diamond_{h-2} A_{h-1} \diamond_{h-1} \phi(A_h) \diamond_h A_{h+1} \diamond_{h+1} \cdots \diamond_n A_{n+1}$$

for any $A_1, A_2, \cdots, A_{n+1} \in \mathcal{A}$ with $n \geq 2$. If n is even and $\diamond_{2u-1} = \bullet$, $\diamond_{2u} = \circ$ with $1 \leq u \leq \frac{n}{2}$, then there exists an element $\lambda \in \mathcal{Z}_S(\mathcal{A})$ such that $\phi(A) = \delta(A) + \mathrm{i}\lambda A$, where δ is an additive $*$-derivation. Otherwise, ϕ is an additive $*$-derivation.

By the results of [24] and [26], we only need to consider the case when at least one of $\diamond_r$ is $\bullet$, where $r \in \{1, 2, 3 \cdots, n-1\}$.

Lemma 3.2. *If n is even and $\diamond_{2u-1} = \bullet$, $\diamond_{2u} = \circ$ with $1 \leq u \leq \frac{n}{2}$, then $\phi(I)^* = -\phi(I)$ and $\phi(I) \in \mathcal{Z}(\mathcal{A})$. Otherwise, $\phi(I) = 0$.*

Proof. Let $\diamond_{s_p} = \bullet$, $\diamond_{t_q} = \circ$ with $1 \leq s_1 \leq s_p \leq s_{\mu_1} \leq n$, $1 \leq t_1 \leq t_q \leq t_{\mu_2} \leq n$, where $1 \leq p \leq \mu_1$, $1 \leq q \leq \mu_2$ and $\mu_1 + \mu_2 = n$.

If $n \geq 2$ and $s_{\mu_1} = n$, then it follows from Theorem 2.1 and $\phi(I) \bullet I = I \bullet \phi(I) \in \mathcal{A}_{sa}$ that

$$\begin{aligned}2^n\phi(I) = \phi(I \diamond_1 I \diamond_2 \cdots \diamond_n I) &= \sum_{h=1}^{n+1} I \diamond_1 \cdots \diamond_{h-2} I \diamond_{h-1} \phi(I) \diamond_h I \diamond_{h+1} \cdots \diamond_n I \\ &= (n+1)2^{n-1}(\phi(I)^* + \phi(I)).\end{aligned}$$

Moreover, $2^n\phi(I)^* = (n+1)2^{n-1}(\phi(I)^* + \phi(I))$. Hence $\phi(I) = 0$.

If $n \geq 3$ and $1 \leq s_{\mu_1} < n-1$, then

$$\begin{aligned}2^n\phi(I) = \phi(I \diamond_1 I \diamond_2 \cdots \diamond_n I) &= \sum_{h=1}^{n+1} I \diamond_1 \cdots \diamond_{h-2} I \diamond_{h-1} \phi(I) \diamond_h I \diamond_{h+1} \cdots \diamond_n I \\ &= (s_{\mu_1} + 1)2^{n-1}(\phi(I)^* + \phi(I)) + (n - s_{\mu_1})2^n\phi(I).\end{aligned}$$

Moreover,

$$2^n\phi(I)^* = (s_{\mu_1} + 1)2^{n-1}(\phi(I)^* + \phi(I)) + (n - s_{\mu_1})2^n\phi(I)^*.$$

Comparing the above two equations, we can obtain that $\phi(I)^* = \phi(I)$. Hence $\phi(I) = 0$.

If $n \geq 2$, $t_{\mu_2 - r+1} = n - 2(r-1)$ and $s_{\mu_1 - r+1} = n - 2(r-1) - 1$ with $1 \leq r \leq g \leq [\frac{n}{2}]$. Take $A_c = I$ with $1 \leq c \leq n+1$, it follows from Theorem 2.1 and $\phi(I) \bullet I = I \bullet \phi(I) \in \mathcal{A}_{sa}$ that

$$\begin{aligned}2^n\phi(I) = \phi(I \diamond_1 I \diamond_2 \cdots \diamond_n I) &= \sum_{h=1}^{n+1} I \diamond_1 \cdots \diamond_{h-2} I \diamond_{h-1} \phi(I) \diamond_h I \diamond_{h+1} \cdots \diamond_n I \\ &= n2^{n-1}(\phi(I)^* + \phi(I)) + 2^n\phi(I).\end{aligned}$$

It follows that $\phi(I)^* = -\phi(I)$. There are seven further cases:

Case 1: When $n \geq 3$ with n is odd, $s_1 = 1$ and $g = [\frac{n}{2}]$. On the one hand, take $A_1 = I$, $A_2 = I$, $A_3 = \mathrm{i}I$ and $A_c = I$ with $4 \leq c \leq n+1$, it follows from Theorem 2.1 and $\phi(I) \bullet I = I \bullet \phi(I) = 0$ that

$$0 = \phi(I \diamond_1 I \diamond_2 \mathrm{i}I \diamond_3 \cdots \diamond_n I) = 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I)).$$

Thus $\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I) = 0$. On the other hand, take $A_1 = I$, $A_2 = \mathrm{i}I$ and $A_c = I$ with $3 \leq c \leq n+1$, we have that $2^n\mathrm{i}\phi(I) = 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I))$. Hence $\phi(I) = 0$.

Case 2: When $n \geq 3$ with n is odd, $t_1 = 1$ and $g = [\frac{n}{2}]$. Take $A_1 = \mathrm{i}I$ and $A_c = I$ with $2 \leq c \leq n+1$, then

$$\begin{aligned}0 =&\phi(\mathrm{i}I \diamond_1 I \diamond_2 \cdots \diamond_n I) = \phi(\mathrm{i}I) \diamond_1 I \diamond_2 \cdots \diamond_n I + \mathrm{i}I \diamond_1 \phi(I) \diamond_2 \cdots \diamond_n I \\ &+ \mathrm{i}I \diamond_1 I \diamond_2 \phi(I) \diamond_3 \cdots \diamond_n I \\ &= 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I)).\end{aligned}$$

Thus $\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I) = 0$. On the other hand, take $A_1 = I$, $A_2 = I$, $A_3 = \mathrm{i}I$ and $A_c = I$ with $4 \le c \le n+1$, we have that

$$0 = \phi(I \diamond_1 I \diamond_2 \mathrm{i}I \diamond_3 \cdots \diamond_n I) = -2^{n+1}\mathrm{i}\phi(I) + 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I)).$$

Thus $\phi(I) = 0$.

Case 3: When $n \ge 4$, $t_{\mu_2-g} = n-2g-1$ and $s_{\mu_1-g} = n-2g$ with $1 \le g < [\frac{n}{2}]$. Take $A_c = I$ with $1 \le c \le n-2g+1$, $n-2g+3 \le c \le n+1$ and $A_{n-2g+2} = \mathrm{i}I$, it follows from Theorem 2.1 and $\phi(I)\bullet I = I \bullet \phi(I) = 0$ that there exists $\alpha_1 > 0$ such that

$$\begin{aligned} 0 =&\phi(I \diamond_1 I \diamond_2 \cdots \diamond_{n-2g+1} \mathrm{i}I \diamond_{n-2g+2} \cdots \diamond_n I)\\ =& \sum_{h=1}^{n-2g+1} I \diamond_1 \cdots I \diamond_{h-1} \phi(I) \diamond_h I \diamond_{h+1} \cdots \diamond_{n-2g+1} \mathrm{i}I \diamond_{n-2g+2} \cdots \diamond_n I\\ &+ I \diamond_1 I \diamond_2 \cdots \diamond_{n-2g+1} \phi(\mathrm{i}I) \diamond_{n-2g+2} \cdots \diamond_n I\\ =& \alpha_1\phi(I) \diamond_{n-2g} I \diamond_{n-2g+1} \mathrm{i}I \diamond_{n-2g+2} \cdots \diamond_n I + 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I))\\ =& 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I)). \end{aligned}$$

Hence $\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I) = 0$. Take $A_c = I$ with $1 \le c \le n-2g$, $n-2g+2 \le c \le n+1$ and $A_{n-2g+1} = \mathrm{i}I$, then there exists $\alpha_2 > 0$ such that

$$\begin{aligned} 0 =& \phi(I \diamond_1 I \diamond_2 \cdots \diamond_{n-2g} \mathrm{i}I \diamond_{n-2g+1} \cdots \diamond_n I)\\ =& \sum_{h=1}^{n-2g} I \diamond_1 \cdots \diamond_{h-1} \phi(I) \diamond_h I \diamond_{h+1} \cdots \diamond_{n-2g} \mathrm{i}I \diamond_{n-2g+1} \cdots \diamond_n I\\ &+ I \diamond_1 \cdots \diamond_{n-2g} \phi(\mathrm{i}I) \diamond_{n-2g+1} \cdots \diamond_n I\\ =& \alpha_2\phi(I) \diamond_{n-2g} \mathrm{i}I \diamond_{n-2g+1} \cdots I \diamond_n I + 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I))\\ =& -2^{2g+1}\alpha_2\mathrm{i}\phi(I). \end{aligned}$$

Hence $\phi(I) = 0$.

Case 4: When $n \ge 4$, $\mu_1 + g = n$ with $1 \le g < [\frac{n}{2}]$. Similarly Case 3, take $A_c = I$ with $1 \le c \le n-2g+1$, $n-2g+3 \le c \le n+1$ and $A_{n-2g+2} = \mathrm{i}I$, we can easy obtain that $\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I) = 0$. Take $A_1 = I$, $A_2 = \mathrm{i}I$ and $A_c = I$ with $3 \le c \le n+1$, then

$$\begin{aligned} 0 =& \phi(I \diamond_1 \mathrm{i}I \diamond_2 \cdots \diamond_n I) = \phi(I) \diamond_1 \mathrm{i}I \diamond_2 \cdots \diamond_n I + I \diamond_1 \phi(\mathrm{i}I) \diamond_2 \cdots \diamond_n I\\ =& -2^n\mathrm{i}\phi(I) + 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I))\\ =& -2^n\mathrm{i}\phi(I). \end{aligned}$$

Hence $\phi(I) = 0$.

Case 5: When $n \ge 5$, $1 \le t_{\mu_2-g} \le n-2g-2$ with $1 \le g < [\frac{n}{2}]$. Similarly Case 3, take $A_c = I$ with $1 \le c \le n-2g+1$, $n-2g+3 \le c \le n+1$ and $A_{n-2g+2} = \mathrm{i}I$, we have that $\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I) = 0$. Take $A_c = I$ with $1 \le c \le n-2g-1$, $n-2g+1 \le c \le n+1$ and $A_{n-2g} = \mathrm{i}I$, then there exists $\beta > 0$ such that

$$\begin{aligned} 0 =& \phi(I \diamond_1 I \diamond_2 \cdots \mathrm{i}I \diamond_{n-2g} \cdots \diamond_n I)\\ =& \sum_{h=1}^{n-2g-1} I \diamond_1 \cdots I \diamond_{h-1} \phi(I) \diamond_h I \diamond_{h+1} \cdots \diamond_{n-2g-1} \mathrm{i}I \diamond_{n-2g} \cdots I \diamond_n I\\ &+ I \diamond_1 \cdots \diamond_{n-2g-1} \phi(\mathrm{i}I) \diamond_{n-2g} \cdots I \diamond_n I\\ =& \beta\phi(I) \diamond_{n-2g-1} \mathrm{i}I \diamond_{n-2g} \cdots \diamond_n I + 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I))\\ =& -2^{2g+2}\beta\mathrm{i}\phi(I). \end{aligned}$$

Hence $\phi(I) = 0$.

Case 6: When $n \geq 4$, $t_{\mu_2-g-1} = n-2g-1$ and $t_{\mu_2-g} = n-2g$ with $1 \leq g < [\frac{n}{2}]$. On the one hand, take $A_c = I$ with $1 \leq c \leq n-2g+1$, $n-2g+3 \leq c \leq n+1$ and $A_{n-2g+2} = \mathrm{i}I$, it follows from Theorem 2.1 that there exists $\gamma_1 \geq 0$ such that

$$\begin{aligned} 0 &= \phi(I \diamond_1 I \diamond_2 \cdots \diamond_{n-2g+1} \mathrm{i}I \diamond_{n-2g+2} \cdots \diamond_n I) \\ &= \sum_{h=1}^{n-2g-1} I \diamond_1 \cdots \diamond_{h-1} \phi(I) \diamond_h \cdots \diamond_n I + I \diamond_1 \cdots \diamond_{n-2g-1} \phi(I) \diamond_{n-2g} \cdots \diamond_n I \\ &+ I \diamond_1 \cdots \diamond_{n-2g} \phi(I) \diamond_{n-2g+1} \cdots \diamond_n I + I \diamond_1 \cdots \diamond_{n-2g+1} \phi(\mathrm{i}I) \diamond_{n-2g+2} \cdots \diamond_n I \\ &= \gamma_1 \phi(I) \diamond_{n-2g-1} I \diamond_{n-2g} I \diamond_{n-2g+1} \mathrm{i}I \diamond_{n-2g+2} \cdots \diamond_n I - 2^{n+1}\mathrm{i}\phi(I) \\ &+ 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I)). \end{aligned}$$

Thus $(2^{2g+2}\gamma_1 + 2^{n+1})\mathrm{i}\phi(I) = 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I))$. On the other hand, take $A_c = I$ with $1 \leq c \leq n-2g-1$, $n-2g+1 \leq c \leq n+1$ and $A_{n-2g} = \mathrm{i}I$, it follows from Theorem 2.1 that there exists $\gamma_2 \geq 0$ such that

$$\begin{aligned} 0 &= \phi(I \diamond_1 I \diamond_2 \cdots \diamond_{n-2g-1} \mathrm{i}I \diamond_{n-2g} \cdots \diamond_n I) \\ &= \sum_{h=1}^{n-2g-1} I \diamond_1 \cdots \diamond_{h-1} \phi(I) \diamond_h \cdots \diamond_n I + I \diamond_1 \cdots \diamond_{n-2g-1} \phi(\mathrm{i}I) \diamond_{n-2g} \cdots \diamond_n I \\ &= \gamma_2 \phi(I) \diamond_{n-2g-1} \mathrm{i}I \diamond_{n-2g} I \diamond_{n-2g+1} \cdots \diamond_n I + 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I)). \end{aligned}$$

Thus $2^{2g+2}\gamma_2\mathrm{i}\phi(I) + 2^{n-1}(\phi(\mathrm{i}I)^* + \phi(\mathrm{i}I)) = 0$. Hence $\phi(I) = 0$.

Case 7: When $n \geq 2$ with n is even and $g = \frac{n}{2}$. Take $A_1 = A \in \mathcal{A}_{sa}$ and $A_c = I$ with $2 \leq c \leq n+1$, it follows from $(A \bullet \phi(I))^* = A \bullet \phi(I)$ and $(A \circ \phi(I))^* = -(A \circ \phi(I))$ that

$$\begin{aligned} 2^n\phi(A) = \phi(A \diamond_1 I \diamond_2 \cdots I \diamond_n I) = 2^{n-1}(\phi(A)^* + \phi(A)) + 2^{n-1}g(A\phi(I)^* \\ + \phi(I)A) + 2^{n-1}(A\phi(I) + \phi(I)A). \end{aligned}$$

Thus

$$\phi(A) = \phi(A)^* + g(\phi(I)A - A\phi(I)) + A\phi(I) + \phi(I)A.$$

On the other hand,

$$\phi(A)^* = \phi(A) + g(\phi(I)A - A\phi(I)) - A\phi(I) - \phi(I)A.$$

We can get that $\phi(I)A = A\phi(I)$ for all $A \in \mathcal{A}_{sa}$. Hence $\phi(I) \in \mathcal{Z}(\mathcal{A})$. The proof is completed.

Proof of Theorem 3.1. Let $\diamond_s = \bullet$ and $\diamond_h = \circ$ with $1 \leq h \leq s-1$. If $\phi(I) = 0$, Let $A_c = I$ with $1 \leq c \leq s-1$, $s+2 \leq c \leq n+1$, it follows from Theorem 2.1 that

$$\begin{aligned} 2^{n-1}\phi(A_s \diamond_s A_{s+1}) &= \phi(I \diamond_1 I \diamond_2 \cdots \diamond_{s-1} A_s \diamond_s A_{s+1} \diamond_{s+1} \cdots \diamond_n I) \\ &= 2^{n-1}(\phi(A_s) \diamond_s A_{s+1} + A_s \diamond_s \phi(A_{s+1})) \end{aligned}$$

for any $A_s, A_{s+1} \in \mathcal{A}$. Thus

$$\phi(A_s \diamond_s A_{s+1}) = \phi(A_s) \diamond_s A_{s+1} + A_s \diamond_s \phi(A_{s+1}).$$

It follows from [25, Main Theorem] that ϕ is an additive $*$-derivation.

If n is even and $\diamond_{2u-1} = \bullet$, $\diamond_{2u} = \circ$ with $1 \leq u \leq \frac{n}{2}$. Define a map $\delta : \mathcal{A} \to \mathcal{A}$ by $\delta(A) = \phi(A) - \phi(I)A$. It follows from Lemma 3.2 that δ is an additive map and satisfies

$$\delta(A_1 \diamond_1 A_2 \diamond_2 \cdots \diamond_n A_{n+1}) = \sum_{h=1}^{n+1} A_1 \diamond_1 \cdots \diamond_{h-1} \delta(A_h) \diamond_h \cdots \diamond_n A_{n+1}$$

for any $A_1, A_2, \cdots, A_{n+1} \in \mathcal{A}$ and $\delta(I) = 0$. It follows from the above conclusion that

$$\delta(A \diamond_s B) = \delta(A) \diamond_s B + A \diamond_s \delta(B)$$

for any $A, B \in \mathcal{A}$. It follows from [25, Main Theorem] that δ is an additive $*$-derivation. Hence, there exists an element $\lambda \in \mathcal{Z}_S(\mathcal{A})$ such that

$$\phi(A) = \delta(A) + \mathrm{i}\lambda A$$

for any $A \in \mathcal{A}$, where δ is an additive $*$-derivation. The proof is completed.

As a consequences of Theorem 3.1, we have the following corollaries.

Corollary 3.1. *Let $\mathcal{M}$ be a factor von Neumann algebra with $\dim\mathcal{M} > 1$, and let $\phi : \mathcal{M} \to \mathcal{M}$ be a nonlinear mixed bi-skew Jordan-type derivation, that is, ϕ satisfies*

$$\phi(A_1 \diamond_1 A_2 \diamond_2 \cdots \diamond_n A_{n+1}) = \sum_{h=1}^{n+1} A_1 \diamond_1 \cdots \diamond_{h-2} A_{h-1} \diamond_{h-1} \phi(A_h) \diamond_h A_{h+1} \diamond_{h+1} \cdots \diamond_n A_{n+1}$$

for any $A_1, A_2, \cdots, A_{n+1} \in \mathcal{M}$ with $n \geq 2$. If n is even and $\diamond_{2u-1} = \bullet$, $\diamond_{2u} = \circ$ with $1 \leq u \leq \frac{n}{2}$, then there exists an number $\lambda \in \mathbb{R}$ such that $\phi(A) = \delta(A) + \mathrm{i}\lambda A$, where δ is an additive $*$-derivation. Otherwise, ϕ is an additive $*$-derivation.

Corollary 3.2. *Let $\mathcal{A}$ be a standard operator algebra on an infinite-dimensional complex Hilbert space $\mathcal{H}$ containing the identity operator I, which $\mathcal{A}$ is closed under the adjoint operation. Assume that $\phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed bi-skew Jordan-type derivation. It is show that if n is even and $\diamond_{2u-1} = \bullet$, $\diamond_{2u} = \circ$ with $1 \leq u \leq \frac{n}{2}$, then there exist $T, S \in B(\mathcal{H})$ satisfying $T^* + T = 0$, $T - S \in \mathrm{i}\mathbb{R}I$ such that $\phi(A) = AT - SA$. Otherwise, there exists $Y \in B(\mathcal{H})$ such that $\phi(A) = AY - YA$ with $Y^* + Y = 0$.*

References

[1] L. Molnár, *A condition for a subspace of $B(\mathcal{H})$ to be an ideal*, Linear Algebra Appl. **235** (1996), 229–234.
[2] P. Šemrl, *Quadratic functionals and Jordan $*$-derivations*, Studia Math. **97** (1991), 157–165.
[3] P. Šemrl, *Jordan $*$-derivations of standard operator algebras*, Proc. Amer. Math. Soc. **120** (1994), 515–518.
[4] D. Zhao, C. Li, Y. Zhao, *Nonlinear maps preserving bi-skew Jordan triple product on factor von Neumann algebras*, Periodica Mathematica Hungarica. **86** (2023), 578–586.
[5] M. Wang, G. Ji, *A characterization of $*$-isomorphism on factor von Neumann algebras*, Acta Math. Sinica, Chin. Ser. **58** (2015), 71–78.
[6] M. Fošner, *Prime rings with involution equipped with some new product*, Southeast Asian Bull. Math. **26** (2002), 27–31.
[7] T. K. Lee, T. L. Wong, Y. Zhou, *The structure of Jordan $*$-derivations of prime rings*, Linear and Multilinear Algebra. **63** (2015), 411–422.
[8] C. Li, F. Lu, *Nonlinear maps preserving the Jordan triple 1-$*$-product on von Neumann algebras*, Complex Analysis and Operator Theory. **11** (2017), 109–117.
[9] C. Li, F. Lu, X. Fang, *Nonlinear mappings preserving product $XY + YX^*$ on factor von Neumann algebras*, Linear Algebra Appl. **438** (2013), 2339–2345.
[10] A. Taghavi, F. Kolivand, E. Tavakoli, *A note on strong (skew) η-Lie products preserving maps on some algebras*, Linear and Multilinear Algebra. **67** (2019), 886–895.
[11] A. Taghavi, H. Rohi, V. Darvish, *Nonlinear $*$-Jordan derivations on von Neumann algebras*, Linear and Multilinear Algebra. **64** (2016), 426–439.
[12] V. Darvish, M. Nouri, M. Razeghi, *Nonlinear bi-skew Jordan derivations on $*$-algebra*, Filomat. **36** (2022), 3231–3239.
[13] F. Zhao, C. Li, *Nonlinear $*$-Jordan triple derivations on von Neumann algebras*, Math. Slovaca. **1** (2018), 163–170.
[14] A. N. Khan, H. Alhazmi, *Multiplicative bi-skew Jordan triple derivations on prime $*$-algebra*, Georgian Math. J. **30** (2023), 389–396.
[15] C. Li, Y. Zhao, F. Zhao, *Nonlinear $*$-Jordan-type derivations on $*$-algebras*, Rocky Moutain Journal of Mathematics. **51** (2021), 601–612.
[16] M. Ashraf, M. S. Akhter, M. A. Ansari, *Nonlinear generalized bi-skew Jordan n-derivations on $*$-algebras* , Bull. Malays. Math. Sci. Soc. **47** (2024).
[17] L. Abedini, A. Taghavi, *Nonlinear maps preserving the mixed product $A \bullet B \circ C$ on von Neumann algebras*, Rocky Moutain Journal of Mathematics. **53** (2023), 671–678.
[18] C. Li, D. Zhang, *Nonlinear mixed Jordan triple $*$-derivations on factor von Neumann algebras*, Filomat. **36** (2022), 2637–2644.
[19] C. Li, Y. Zhao, F. Zhao, *Nonlinear maps preserving the mixed product $[A \bullet B, C]_*$ on von Neumann algebras*, Filomat. **35** (2021), 2775–2781.

[20] Y. Zhao, C. Li, Q. Chen, *Nonlinear maps preserving mixed product on factors*, Bulletin of the Iranian Mathematical Society. **47** (2021), 1325–1335.

[21] Y. Pang, D. Zhang, D. Ma, *The second nonlinear mixed Jordan triple derivable mapping on factor von Neumann algebras*, Bulletin of the Iranian Mathematical Society. **47** (2021), 1325–1335.

[22] N. Rehman, J. Nisar, M. Nazim, *A note on nonlinear mixed Jordan triple derivation on $*$-algebras*, Communications in Algebra. **51** (2023), 1334–1343.

[23] T. Ning, J. Zhang, *Nonlinear mixed ξ-Jordan triple derivable maps on factor von Neumann algebras*, Acta Math Sci Chinese Series. **64** (2020), 319–328.

[24] B. Ferreira, F. Wei, *Mixed $*$-Jordan-type derivations on $*$-algebras*, Journal of Algebra and Its Applications. **22** (2023).

[25] M. A. Siddeeque, A. H. Shikeh, *Nonlinear bi-skew Jordan derivations in prime $*$-rings*, Journal of Algebra and Its Applications. **53** (2023), 671–678.

[26] X. Qi, Z. Guo, T. Zhang, *Characterizing Jordan n-derivations of unital rings containing idempotents*, Bulletin of the Iranian Mathematical Society. **46** (2020), 1639–1658.