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Solution of Dirac dynamic system by Laplace transform on time scales in quantum physics

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Abstract. In this study, we solve a Dirac dynamic system on time scale with constant graininess by using Laplace transform which converts differential equations into algebraic equations and convolution into multiplication. It has many applications in science and engineering. Furthermore, we give some numerical examples for special time scales related to the Laplace transform.

1. Introduction

The theory of time scale calculus was first introduced by Stephan Hilger under the consultancy of Aulbach [22, 23]. It offers a formalism to study hybrid discrete dynamical systems. It has many important applications in many different areas. This theory is a unification of differential calculus with calculus of finite differences.

The studies about spectral theory on time scales have focused on Sturm-Liouville equation. Sturm-Liouville theory on time scales was firstly studied by Erbe and Hilger in 1993,[17] . Some important results on the properties of eigenvalues and eigenfunctions for this problem on time scales were given in various publications,[1, 2, 4, 5, 16, 19, 20, 24, 25, 33]. Although there are a few numbers studies related to the spectral theory of Dirac dynamic system, very important studies have been carried out on the time scale about it [18, 28]. Due to the difficulties on the time scale, it is very tiring to study spectral theory for different equations and systems on time scale.

In quantum physics, the Dirac equation is a relativistic wave equation derived by Paul Dirac in 1928. In its free form, or including electromagnetic interactions, it describes all spin-1/2 massive particles such as electrons and quarks for which parity is a symmetry. It is consistent with both the principles of quantum mechanics and the theory of special relativity, and was the first theory to account fully for special relativity in the context of quantum mechanics. It was validated by accounting for the fine details of the hydrogen spectrum in a completely rigorous way. In the classical case, there are many studies on spectral theory of the Dirac system [26, 29–31, 35, 36].

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In this study, our aim is to solve below Dirac eigenvalue problems on a time scale $\mathbb{T}$ by using Laplace transform:

$$By^{\Delta}(t) + Q(t)y(t) = \lambda y(t), \tag{1}$$

$$By^{\Delta}(t) + Q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \tag{2}$$

with initial conditions

$$y_1(0,\lambda) = a, \quad y_2(0,\lambda) = b, \tag{3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & r(t) \end{pmatrix},$$

$t \in (0,+\infty)_{\mathbb{T}}$; $q(t) = c$, $r(t) = d$; a, b are real constants; $(a^2 + b^2)(c^2 + d^2) \neq 0$; λ is a spectral parameter; $y^{\sigma}(t) = y(\sigma(t))$ and $y(t) = \begin{pmatrix} y_1(t,\lambda) \\ y_2(t,\lambda) \end{pmatrix}$ is the vector-valued eigenfunction of this problem, where $y_i : \mathbb{T} \to \mathbb{R}$, $y_i^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$, $i = 1, 2$. Throughout this study, we assume that the graininess function of the considered time scale is $\mu(t) \equiv h \geq 0$.

Actually, the system (1) is derived from the system (2) when $\sigma(t) = t$. Therefore, we will only deal with the problem (2)-(3). The desired results for the problem (1)-(3) are valid for $h = 0$ on the results obtained for the problem (2)-(3).

This study includes five sections. In section 1, we give historical development and structure of the problem. Then, we express some concepts and basic features of Laplace transform on $\mathbb{T}$ in section 2. We solve a Dirac dynamic system by Laplace transform on $\mathbb{T}$ in section 3. Next section, we give two numerical examples on different time scales to embody the solution technique of the problem. Finally, we complete the paper with conclusion.

2. Preliminaries

The fundamental terminology of time scale calculus such as Hilger derivative, $\mathbb{T}^{\kappa}$ region, delta integration, σ and ρ operators, μ graininess function, *rd*-continuity, regulated function can be reached in [6, 7] . In addition, we have to express some very important notions related to time scale theory.

Definition 2.1. *[6] $p : \mathbb{T} \to \mathbb{R}$ is a regressive function if $1 + \mu(t)p(t) \neq 0$ holds for all $t \in \mathbb{T}^{\kappa}$.*

$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$ indicates the set of all regressive and *rd*-continuous functions on $\mathbb{T}$. $\mathcal{R}$ forms an Abelian group with the addition operation $\oplus$ defined by $(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t)$ for all $t \in \mathbb{T}^{\kappa}$, $p, q \in \mathcal{R}$. In addition, additive inverse of p for this group is denoted by

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)},$$

for all $t \in \mathbb{T}^{\kappa}$, $p \in \mathcal{R}$.

Definition 2.2. *[6] Exponential function on $\mathbb{T}$ is defined by*

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right),$$

for $s, t \in \mathbb{T}$ and $p \in \mathcal{R}$. Here, $\xi_h(z)$ is cylinder transformation, where $\xi_h(z) = \frac{1}{h}Log(1 + hz)$ for $h > 0$. For details on exponential function, we refer to [6, 7].

Definition 2.3. *[6] Let $p \in C_{rd}$. If $-\mu p^2 \in \mathcal{R}$, the hyperbolic functions* $\cosh_p$ *and* $\sinh_p$ *are defined by*

$$\cosh_p = \frac{e_p + e_{-p}}{2} \quad \textit{and} \quad \sinh_p = \frac{e_p - e_{-p}}{2}.$$

If $\mu p^2 \in \mathcal{R}$, the trigonometric functions $\cos_p$ *and* $\sin_p$ *are defined by*

$$\cos_p = \frac{e_{ip} + e_{-ip}}{2} \quad \textit{and} \quad \sin_p = \frac{e_{ip} - e_{-ip}}{2i}.$$

More detailed information on these functions can be found in [6, 7]. For a constant $\alpha \in \mathcal{R}$, the functions $e_\alpha(t,0)$, $\sin_\alpha(t,0)$ and $\sinh_\alpha(t,0)$ have the below forms for common time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$, respectively.

$\mathbb{T}$	$e_\alpha(t,0)$	$\sin_\alpha(t,0)$	$\sinh_\alpha(t,0)$
$\mathbb{R}$	$e^{\alpha t}$	$\sin(\alpha t)$	$\sinh(\alpha t)$
$\mathbb{Z}$	$(1+\alpha)^t$	$\frac{(1+i\alpha)^t-(1-i\alpha)^t}{2i}$	$\frac{(1+\alpha)^t-(1-\alpha)^t}{2}$
$h\mathbb{Z}$	$(1+\alpha h)^{\frac{t}{h}}$	$\frac{(1+i\alpha h)^{\frac{t}{h}}-(1-i\alpha h)^{\frac{t}{h}}}{2i})$	$\frac{(1+\alpha h)^{\frac{t}{h}}-(1-\alpha h)^{\frac{t}{h}}}{2}$

Table 1. Representations of $e_\alpha(t,0)$, $\sin_\alpha(t,0)$ and $\sinh_\alpha(t,0)$ on $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = h\mathbb{Z}$

Usual Laplace transformation can be defined for functions on time scale which uses the same table of transforms for any arbitrary time scale. It can be used to solve dynamical equations. Here, we remind some principle notions and theorems related to Laplace transform on $\mathbb{T}$ [3, 8–15, 21, 27, 32, 34].

Definition 2.4. *[6] Suppose that $y : \mathbb{T}_0 \to \mathbb{R}$ is regulated. Then, Laplace transform of y is defined by*

$$\mathcal{L}\{y\}(z) := \int_0^\infty y(t) e^\sigma_{\ominus z}(t,0)\Delta t, \tag{4}$$

for $z \in \mathcal{D}\{y\}$, where $\mathbb{T}_0$ is a time scale, $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = \infty$; $\mathcal{D}\{y\}$ consists of all complex numbers $z \in \mathcal{R}$ when the improper integral exists.

Remark 2.5. *In classical case, Laplace transform is also well suited to solving systems of differential equations. Laplace transform is applied to both sides of each equation converts a system of differential equations into a system of linear algebraic equations. These algebraic equations are solved using various techniques. Then, by applying inverse Laplace transform to this solution system, the solution of systems of differential equations is obtained.*

Before application of Laplace transformation to dynamical systems, some of its basic properties must be given. It was easily seen from the Definition 2.4 that $\mathcal{L}$ is linear as follows:

Theorem 2.6. *[6] Let x and y be regulated on $\mathbb{T}_0$ and α, β be constants. Then*

$$\mathcal{L}\{\alpha x + \beta y\}(z) = \alpha\mathcal{L}\{x\}(z) + \beta\mathcal{L}\{y\}(z),$$

for $z \in \mathcal{D}\{x\} \cap \mathcal{D}\{y\}$.

Theorem 2.7. *[6] If $y : \mathbb{T}_0 \to \mathbb{C}$ is a function whose first order delta derivative is regulated, then*

$$\mathcal{L}\left\{y^\Delta\right\}(z) = z\mathcal{L}\{y\}(z) - y(0), \tag{5}$$

for all regressive $z \in \mathbb{C}$ when $\lim_{t\to\infty}\{y(t)e_{\ominus z}(t,0)\} = 0$.

Lemma 2.8. *[6] If $\mathbb{T}_0$ has constant forward-step function $\mu(t) \equiv h \geq 0$, then*

$$\mathcal{L}\{y^\sigma\}(z) = (1 + hz)\mathcal{L}\{y\}(z) - hy(0). \tag{6}$$

The following table gives Laplace transforms of some basic functions for usage in section 3.

$y(t)$	1	t	$e_\alpha(t,0)$
$\mathcal{L}\{y\}(z)$	$\frac{1}{z}$	$\frac{1}{z^2}$	$\frac{1}{z-\alpha}$

$y(t)$	$\sin_\alpha(t,0)$	$\sinh_\alpha(t,0)$	$e_\alpha(t,0)\sin_{\frac{\beta}{1+\mu\alpha}}(t,0)$	$e_\alpha(t,0)\sinh_{\frac{\beta}{1+\mu\alpha}}(t,0)$
$\mathcal{L}\{y\}(z)$	$\frac{\alpha}{z^2+\alpha^2}$	$\frac{\alpha}{z^2-\alpha^2}$	$\frac{\beta}{(z-\alpha)^2+\beta^2}$	$\frac{\beta}{(z-\alpha)^2-\beta^2}$

$y(t)$	$\cos_\alpha(t,0)$	$\cosh_\alpha(t,0)$	$e_\alpha(t,0)\cos_{\frac{\beta}{1+\mu\alpha}}(t,0)$	$e_\alpha(t,0)\cosh_{\frac{\beta}{1+\mu\alpha}}(t,0)$
$\mathcal{L}\{y\}(z)$	$\frac{z}{z^2+\alpha^2}$	$\frac{z}{z^2-\alpha^2}$	$\frac{z-\alpha}{(z-\alpha)^2+\beta^2}$	$\frac{z-\alpha}{(z-\alpha)^2-\beta^2}$

Table 2. Laplace transforms of Some Common Functions on $\mathbb{T}$

3. Main Results

Here, we obtain eigenfunction expansion for Dirac problem (2)-(3) on $\mathbb{T}$ with constant graininess by Laplace transform.

We will solve firstly below classical Dirac problem:

$$By'(t) + Q(t)y(t) = \lambda y(t), \qquad t \in (0,+\infty)_{\mathbb{R}} \tag{7}$$

$$y_1(0,\lambda) = a, \quad y_2(0,\lambda) = b, \tag{8}$$

by Laplace transform for $\mathbb{T} = \mathbb{R}$, where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; Q(t) = \begin{pmatrix} q(t) & 0 \\ 0 & r(t) \end{pmatrix},$$

$q(t) = c$, $r(t) = d$ are real constants; λ is a spectral parameter and $y = \begin{pmatrix} y_1(t,\lambda) \\ y_2(t,\lambda) \end{pmatrix}$ is the vector-valued function.

For sake of shortness we assume that $\delta_{k,m}(\lambda) = (\lambda - c)^k(\lambda - d)^m$, $k, m = -1, 0, 1$ throughout the study.

Theorem 3.1. *The eigenfunction of the problem (7)-(8) has following form*

$$y_1(t) = \begin{cases} a\cosh\sqrt{-\delta_{1,1}(\lambda)}\,t - b\sqrt{-\delta_{-1,1}(\lambda)}\sinh\sqrt{-\delta_{1,1}(\lambda)}\,t, & \text{if } \delta_{1,1}(\lambda) < 0, \\ a - b\,\delta_{0,1}(\lambda)\,t, & \text{if } \delta_{1,1}(\lambda) = 0, \\ a\cos\sqrt{\delta_{1,1}(\lambda)}\,t - b\sqrt{\delta_{-1,1}(\lambda)}\sin\sqrt{\delta_{1,1}(\lambda)}\,t, & \text{if } \delta_{1,1}(\lambda) > 0, \end{cases} \tag{9}$$

$$y_2(t) = \begin{cases} b\cosh\sqrt{-\delta_{1,1}(\lambda)}\,t + a\sqrt{-\delta_{1,-1}(\lambda)}\sinh\sqrt{-\delta_{1,1}(\lambda)}\,t, & \text{if } \delta_{1,1}(\lambda) < 0, \\ b + a\,\delta_{1,0}(\lambda)\,t, & \text{if } \delta_{1,1}(\lambda) = 0, \\ b\cos\sqrt{\delta_{1,1}(\lambda)}\,t + a\sqrt{\delta_{1,-1}(\lambda)}\sin\sqrt{\delta_{1,1}(\lambda)}\,t, & \text{if } \delta_{1,1}(\lambda) > 0. \end{cases} \tag{10}$$

Proof. Let us reorganize the system (7) as

$$\begin{cases} y_1' + (\lambda - d)y_2 = 0, \\ y_2' + (c-\lambda)y_1 = 0. \end{cases} \tag{11}$$

We begin by applying the "usual"Laplace transform known as

$$\mathcal{L}\{y\}(s) := \int_0^\infty y(t)e^{-st}dt,$$

whenever the right side integral is convergent, to both sides of each equation of (11):

$$\begin{cases} \mathcal{L}\{y_1'\}(s) + (\lambda - d)\mathcal{L}\{y_2\}(s) = 0, \\ \mathcal{L}\{y_2'\}(s) + (c-\lambda)\mathcal{L}\{y_1\}(s) = 0. \end{cases}$$

Then, from the initial conditions (8) and the formula

$$\mathcal{L}\{y'\}(s) = s\mathcal{L}\{y\}(s) - y(0),$$

we get a system of differential equations which is transformed into a pair of simultaneous linear algebraic equations for the functions Y_1 and Y_2 :

$$\begin{cases} sY_1(s) + (\lambda - d)Y_2(s) = a, \\ sY_2(s) + (c-\lambda)Y_1(s) = b, \end{cases}$$

which has following solutions

$$Y_1(s) = \frac{as - b(\lambda - d)}{s^2 + (\lambda - c)(\lambda - d)}, \qquad Y_2(s) = \frac{bs + a(\lambda - c)}{s^2 + (\lambda - c)(\lambda - d)},$$

where $Y_1(s) = \mathcal{L}\{y_1\}(s)$ and $Y_2(s) = \mathcal{L}\{y_2\}(s)$.

After decomposing the expressions into sums of simple fractions on Y_1 and Y_2, by using the table of "usual"Laplace transform, the solutions (9) and (10) are obtained. □

Now, using Laplace transform on time scales, we will solve Dirac dynamic system.

Theorem 3.2. *The eigenfunction of the problem (2)-(3) has the below forms:*

(i) *If* $h\,\delta_{1,1}(\lambda) = 0$, *then*

$$y_1(t) = \begin{cases} a\cosh_{\sqrt{-\delta_{1,1}(\lambda)}}(t,0) - b\sqrt{-\delta_{-1,1}(\lambda)}\sinh_{\sqrt{-\delta_{1,1}(\lambda)}}(t,0), & \text{if } \delta_{1,1}(\lambda) < 0, \\ a - b\,\delta_{0,1}(\lambda)\,t, & \text{if } \delta_{1,1}(\lambda) = 0, \\ a\cos_{\sqrt{\delta_{1,1}(\lambda)}}(t,0) - b\sqrt{\delta_{-1,1}(\lambda)}\sin_{\sqrt{\delta_{1,1}(\lambda)}}(t,0), & \text{if } \delta_{1,1}(\lambda) > 0, \end{cases}$$

$$y_2(t) = \begin{cases} b\cosh_{\sqrt{-\delta_{1,1}(\lambda)}}(t,0) + a\sqrt{-\delta_{1,-1}(\lambda)}\sinh_{\sqrt{-\delta_{1,1}(\lambda)}}(t,0), & \text{if } \delta_{1,1}(\lambda) < 0, \\ b + a\,\delta_{1,0}(\lambda)\,t, & \text{if } \delta_{1,1}(\lambda) = 0, \\ b\cos_{\sqrt{\delta_{1,1}(\lambda)}}(t,0) + a\sqrt{\delta_{1,-1}(\lambda)}\sin_{\sqrt{\delta_{1,1}(\lambda)}}(t,0), & \text{if } \delta_{1,1}(\lambda) > 0. \end{cases}$$

(ii) *If* $h>0$ *and* $\delta_{1,1}(\lambda)=-\frac{1}{h^2}$, *then*

$$y_1(t)=\frac{a+bh\delta_{0,1}(\lambda)}{2}e_{-\frac{1}{2h}}(t,0),\qquad y_2(t)=\frac{b-ah\delta_{1,0}(\lambda)}{2}e_{-\frac{1}{2h}}(t,0).$$

(iii) *If* $h>0$ *and* $\delta_{1,1}(\lambda)\neq-\frac{1}{h^2}$, *then*

$$y_1(t)=e_{-\alpha}(t,0)\left\{a\cosh_{\frac{\sqrt{-\beta}}{1-h\alpha}}(t,0)-b\sqrt{\delta_{-1,1}(\lambda)}\,\frac{|1+\delta_{1,1}h^2|}{1+\delta_{1,1}h^2}\,\sinh_{\frac{\sqrt{-\beta}}{1-h\alpha}}(t,0)\right\},$$

$$y_2(t)=e_{-\alpha}(t,0)\left\{b\cosh_{\frac{\sqrt{-\beta}}{1-h\alpha}}(t,0)+a\sqrt{\delta_{1,-1}(\lambda)}\,\frac{|1+\delta_{1,1}h^2|}{1+\delta_{1,1}h^2}\,\sinh_{\frac{\sqrt{-\beta}}{1-h\alpha}}(t,0)\right\},$$

for $\delta_{1,1}(\lambda)<0$, *and*

$$y_1(t)=e_{-\alpha}(t,0)\left\{a\cos_{\sqrt{\delta_{1,1}}}(t,0)-b\sqrt{\delta_{-1,1}(\lambda)}\sin_{\sqrt{\delta_{1,1}}}(t,0)\right\},$$

$$y_2(t)=e_{-\alpha}(t,0)\left\{b\cos_{\sqrt{\delta_{1,1}}}(t,0)+a\sqrt{\delta_{1,-1}(\lambda)}\sin_{\sqrt{\delta_{1,1}}}(t,0)\right\},$$

for $\delta_{1,1}(\lambda)>0$, *where* $\alpha=\frac{h\delta_{1,1}(\lambda)}{1+\delta_{1,1}(\lambda)h^2}$ *and* $\beta=\frac{\delta_{1,1}(\lambda)}{(1+\delta_{1,1}(\lambda)h^2)^2}$.

Proof. Let us reorganize the system (2) as

$$\begin{cases} y_1^\Delta+(\lambda-d)y_2^\sigma=0,\\ y_2^\Delta+(c-\lambda)y_1^\sigma=0.\end{cases}\tag{12}$$

By applying the Laplace transform (4) defined on $\mathbb{T}$ to both sides of each equation of (12), we get

$$\begin{cases} \mathcal{L}\left\{y_1^\Delta\right\}(z)+(\lambda-d)\mathcal{L}\left\{y_2^\sigma\right\}(z)=0,\\ \mathcal{L}\left\{y_2^\Delta\right\}(z)+(c-\lambda)\mathcal{L}\left\{y_1^\sigma\right\}(z)=0.\end{cases}$$

From the initial conditions (3) and the formulas (5), (6), we get a system of dynamic equations is transformed into a pair of simultaneous linear algebraic equations for the functions Y_1 and Y_2 :

$$\begin{cases} zY_1(z)+(\lambda-d)(1+hz)Y_2(z)=a+bh(\lambda-d),\\ zY_2(z)+(c-\lambda)(1+hz)Y_1(z)=b+ah(c-\lambda),\end{cases}$$

which has following solutions

$$Y_1(z)=\frac{(ah^2\delta_{1,1}(\lambda)+a)z-b\delta_{0,1}(\lambda)+ah\delta_{1,1}(\lambda)}{(1+\delta_{1,1}(\lambda)h^2)z^2+2h\delta_{1,1}(\lambda)z+\delta_{1,1}(\lambda)},\tag{13}$$

$$Y_2(z)=\frac{(bh^2\delta_{1,1}(\lambda)+b)z+a\delta_{1,0}(\lambda)+bh\delta_{1,1}(\lambda)}{(1+\delta_{1,1}(\lambda)h^2)z^2+2h\delta_{1,1}(\lambda)z+\delta_{1,1}(\lambda)},\tag{14}$$

where $Y_1(z)=\mathcal{L}\{y_1\}(z)$ and $Y_2(z)=\mathcal{L}\{y_2\}(z)$.

(i) Let $h\,\delta_{1,1}(\lambda)=0$.

If $h\geq0$ and $\delta_{1,1}(\lambda)=0$, then the formulas (13) and (14) turn into the forms

$$Y_1(z)=\frac{a}{z}-\frac{b\delta_{0,1}(\lambda)}{z^2},\qquad Y_2(z)=\frac{b}{z}+\frac{a\delta_{1,0}(\lambda)}{z^2},$$

respectively.

If $h = 0$, then the formulas (13) and (14) turn into the forms

$$Y_1(z) = \frac{az}{z^2 - \sqrt{-\delta_{1,1}(\lambda)}^2} - \frac{b\delta_{0,1}(\lambda)}{z^2 - \sqrt{-\delta_{1,1}(\lambda)}^2},$$

$$Y_2(z) = \frac{bz}{z^2 - \sqrt{-\delta_{1,1}(\lambda)}^2} + \frac{a\delta_{1,0}(\lambda)}{z^2 - \sqrt{-\delta_{1,1}(\lambda)}^2},$$

for $\delta_{1,1}(\lambda) < 0$, and

$$Y_1(z) = \frac{az}{z^2 + \sqrt{\delta_{1,1}(\lambda)}^2} - \frac{b\delta_{0,1}(\lambda)}{z^2 + \sqrt{\delta_{1,1}(\lambda)}^2},$$

$$Y_2(z) = \frac{bz}{z^2 + \sqrt{\delta_{1,1}(\lambda)}^2} + \frac{a\delta_{1,0}(\lambda)}{z^2 + \sqrt{\delta_{1,1}(\lambda)}^2},$$

for $\delta_{1,1}(\lambda) > 0$, respectively.

The proof of case (i) of the theorem is completed by Table 2.

(ii) Let $h > 0$ and $\delta_{1,1}(\lambda) = -\frac{1}{h^2}$.

It is easy to see that the formulas (13) and (14) are as follows:

$$Y_1(z) = \frac{\frac{a+bh\delta_{0,1}(\lambda)}{2}}{z + \frac{1}{2h}}, \qquad Y_2(z) = \frac{\frac{b-ah\delta_{1,0}(\lambda)}{2}}{z + \frac{1}{2h}},$$

respectively. Therefore, from Table 2, it yields that the proof of the case (ii) of the theorem.

(iii) Let $h > 0$ and $\delta_{1,1}(\lambda) \neq -\frac{1}{h^2}$. Then, the formulas (13) and (14) can be rewritten as

$$Y_1(z) = \frac{\frac{-b\delta_{0,1}(\lambda)}{1+\delta_{1,1}(\lambda)h^2} + a\left(z + \frac{h\delta_{1,1}(\lambda)}{1+\delta_{1,1}(\lambda)h^2}\right)}{\left(z + \frac{h\delta_{1,1}(\lambda)}{1+\delta_{1,1}(\lambda)h^2}\right)^2 + \frac{\delta_{1,1}(\lambda)}{(1+\delta_{1,1}(\lambda)h^2)^2}},$$

$$Y_2(z) = \frac{\frac{a\delta_{1,0}(\lambda)}{1+\delta_{1,1}(\lambda)h^2} + b\left(z + \frac{h\delta_{1,1}(\lambda)}{1+\delta_{1,1}(\lambda)h^2}\right)}{\left(z + \frac{h\delta_{1,1}(\lambda)}{1+\delta_{1,1}(\lambda)h^2}\right)^2 + \frac{\delta_{1,1}(\lambda)}{(1+\delta_{1,1}(\lambda)h^2)^2}},$$

respectively. Consequently, if the last two formulas are arranged for the states $\frac{\delta_{1,1}(\lambda)}{(1+\delta_{1,1}(\lambda)h^2)^2} < 0$ (or $\delta_{1,1}(\lambda) < 0$) and $\frac{\delta_{1,1}(\lambda)}{(1+\delta_{1,1}(\lambda)h^2)^2} > 0$ (or $\delta_{1,1}(\lambda) > 0$) according to the Table 2, the proof is completed. □

The relationship of the obtained results with classical case solutions is highlighted by the following corollary.

Corollary 3.3. *If $\mathbb{T} = \mathbb{R}$, then $h = 0$ for all $t \in \mathbb{T}$. Therefore, the eigenfunctions of the problem (2)-(3) are as in the case (i) of the theorem 3.2.*

Corollary 3.4. *If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, then $\mu(t) = h$, $h > 0$ for all $t \in \mathbb{T}$. Therefore, the eigenfunctions of the problem (2)-(3) are as in the cases (ii) and (iii) of the theorem 3.2.*

4. Numerical Examples

In this section, we examine the structure of Dirac dynamic system for various time scales. The fact that the solutions obtained on different time scales are different shows that this study will make an important contribution to the application areas of the Laplace transformation.

Example 4.1. *Let us consider the following Dirac dynamic system:*

$$By^{\Delta}(t) + Q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \tag{15}$$

$$y_1(0,\lambda) = 0, \quad y_2(0,\lambda) = 1, \tag{16}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; Q(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$t \in \mathbb{T}$, *λ is a spectral parameter; $y^{\sigma}(t) = y(\sigma(t))$ and $y(t) = \begin{pmatrix} y_1(t,\lambda) \\ y_2(t,\lambda) \end{pmatrix}$ is the vector-valued eigenfunction.*

Due to the structure of the matrix Q, we get $\delta_{1,1}(\lambda) = (\lambda - 1)^2 \geq 0$. We will examine the solutions on two distinct time scales.

i) Let $\mathbb{T} = \left\{\frac{n+1}{2} : n \in \mathbb{N}\right\}$, then $\sigma(t) = t + \frac{1}{2}$ for any $t \in \mathbb{T}$. $\mu(t) = h = \frac{1}{2}$ holds for all $t \in \mathbb{T}$ on account of this. Hence, the eigenfunctions of the problem (15)-(16) are as follows:

$$y_1(t) = 0, \qquad y_2(t) = 1, \tag{17}$$

for $\lambda = 1$ (or $\delta_{1,1}(\lambda) = 0$),

$$y_1(t) = -e_{-\frac{2(\lambda-1)^2}{4+(\lambda-1)^2}}(t,0)\sin_{|\lambda-1|}(t,0),$$

$$y_2(t) = -e_{-\frac{2(\lambda-1)^2}{4+(\lambda-1)^2}}(t,0)\cos_{|\lambda-1|}(t,0),$$

for $\delta_{1,1}(\lambda) > 0$.

ii) Let $\mathbb{T} = \{-1, 0\} \cup [1, 2]$, then $\sigma(t) = t + 1$ for $t \in \{-1, 0\}$ and $\sigma(t) = t$ for $t \in [1, 2]$. Therefore, $h = 1$ and $h = 0$ hold for these cases, respectively. Hence, the eigenfunctions of the problem (15)-(16) are as follows:

If $h = 1$, then the eigenfunctions are in the form (17) for $\lambda = 1$ (or $\delta_{1,1}(\lambda) = 0$),

$$y_1(t) = -e_{-\frac{(\lambda-1)^2}{1+(\lambda-1)^2}}(t,0)\sin_{|\lambda-1|}(t,0),$$

$$y_2(t) = -e_{-\frac{(\lambda-1)^2}{1+(\lambda-1)^2}}(t,0)\cos_{|\lambda-1|}(t,0),$$

for $\delta_{1,1}(\lambda) > 0$.

If $h = 0$, then the eigenfunctions are in the form (17) for $\lambda = 1$ (or $\delta_{1,1}(\lambda) = 0$),

$$y_1(t) = -\sin_{|\lambda-1|}(t,0),$$

$$y_2(t) = \cos_{|\lambda-1|}(t,0),$$

for $\delta_{1,1}(\lambda) > 0$.

Example 4.2. *Let us consider the following Dirac dynamic system:*

$$By^{\Delta}(t) + Q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \tag{18}$$

$$y_1(0,\lambda) = 1, \quad y_2(0,\lambda) = 1, \tag{19}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; Q(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$t \in \mathbb{T}$, *λ is a spectral parameter; $y^{\sigma}(t) = y(\sigma(t))$ and* $y(t) = \begin{pmatrix} y_1(t,\lambda) \\ y_2(t,\lambda) \end{pmatrix}$ *is the vector-valued eigenfunction.*

Due to the structure of the matrix Q, we get $\delta_{1,1}(\lambda) = \lambda^2 - 1$.

Let $\mathbb{T} = \left\{\frac{n+1}{2} : n \in \mathbb{N}\right\}$, then the eigenfunctions of the problem (15)-(16) are as follows:

$$y(t) = \begin{pmatrix} 1 + 2t \\ 1 \end{pmatrix} \qquad \text{for} \quad \lambda = -1,$$

$$y(t) = \begin{pmatrix} 1 \\ 1 + 2t \end{pmatrix} \qquad \text{for} \quad \lambda = 1,$$

$$y(t) = \begin{pmatrix} e_{-\frac{2(\lambda^2-1)}{4+(\lambda^2-1)}}(t,0)\left\{\cos_{\sqrt{\lambda^2-1}}(t,0) - \sqrt{\frac{\lambda-1}{\lambda+1}}\sin_{\sqrt{\lambda^2-1}}(t,0)\right\} \\ e_{-\frac{2(\lambda^2-1)}{4+(\lambda^2-1)}}(t,0)\left\{\cos_{\sqrt{\lambda^2-1}}(t,0) + \sqrt{\frac{\lambda+1}{\lambda-1}}\sin_{\sqrt{\lambda^2-1}}(t,0)\right\} \end{pmatrix}$$

for $\lambda \in \mathbb{R}/(-1,1)$ (or $\delta_{1,1}(\lambda) > 0$),

$$y(t) = \begin{pmatrix} e_{-\frac{2(\lambda^2-1)}{4+(\lambda^2-1)}}(t,0)\left\{\cosh_{\sqrt{1-\lambda^2}}(t,0) - \sqrt{\frac{\lambda-1}{\lambda+1}}\sinh_{\sqrt{1-\lambda^2}}(t,0)\right\} \\ e_{-\frac{2(\lambda^2-1)}{4+(\lambda^2-1)}}(t,0)\left\{\cosh_{\sqrt{1-\lambda^2}}(t,0) + \sqrt{\frac{\lambda+1}{\lambda-1}}\sinh_{\sqrt{1-\lambda^2}}(t,0)\right\} \end{pmatrix}$$

for $\lambda \in (-1,1)$ (or $\delta_{1,1}(\lambda) < 0$).

5. Conclusion

In this study, Dirac system, which has an important place in particle physics, has been handled and solved on time scale by using Laplace transform. The obtained results are very important because they are the general state of the classical results. These results can be applied to different problems in particle physics and valuable studies can be done. The given numerical examples will be useful to understand the method of obtaining solutions.

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