Filomat 38:22 (2024), 7729–7734
https://doi.org/10.2298/FIL2422729K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On interpolative metric spaces

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Abstract. The purpose of this article is to expand the "open discussion" on the definition and necessity of the interpolation metric space and keep it on the agenda of researchers in nonlinear functional analysis. The secondary aim of this article is to indicate that the outcomes of this "open discussion" have the potential to stop the recent recession in the metric fixed point theory.

1. Introduction

The metric fixed point theory has a background of around a century. An initial result in the metric fixed point theory, which has a very productive history, was reported by Banach in 1922. In this century, the theory and methods of the metric fixed point theory have been developed extensively. In connection with this progress, metric fixed point theory applications have been established in distinct and various quantitative sciences.

In the last two decades, some fixed point theory publications have overlapped their results. In addition, we note that some recently published results are equivalent to those of previously published ones. In other words, the fixed point theory is in recession.

In order to overcome the recession, several attempts have been made by proposing new types of contractions (Kannan type, Reich-Ciric-Rus type, Hardy-Rogers type, Meir-Keeler type, Wardowski type contraction, etc.) or new abstract spaces (such as 2-metric, D-metric, G-metric, A metric, S-metric, b-metric, quasi-metric, semi-metric, partial-metric, metric-like, cone metric, etc.) and so on. In time, it was proved that some of these contractions are equivalent to existing contractions, see, e.g., [5] and related references therein. In the same way, it was demonstrated that some of the abstract space notions are equivalent to the existing notions. For instance, cone metric (also known as Banach valued metric) can be considered a transformed form of the standard metric space. In particular, Du [3, 1] proved that by scalarization function ζ, cone metric d_c is equivalent to $d = \zeta \circ d_c$.

In this paper, we shall discuss the notion of interpolative metric [4], which is a natural generalization of a standard metric space.

In what follows, we shall state the definition of (α, c)-interpolative metric.

2020 *Mathematics Subject Classification*. 47H07, 47H08, 47H10
Keywords. metric spaces, interpolative metric space, fixed point, Reich-Rus-Ciric type contraction, Kannan type contraction, Bianchini type contraction
Received: 14 January 2024; Accepted: 12 March 2024
Communicated by Dragan S. Djordjević
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Definition 1.1. *[4] Let X be a nonempty set. We say that $d : X \times X \to [0, +\infty)$ is (α, c)-interpolative metric if*

$(m1)$ *$d(x, y) = 0$, if and only if, $x = y$ for all $x, y \in X$*

$(m2)$ *$d(x, y) = d(y, x)$, for all $x, y \in X$,*

$(m3)$ *there exist an $\alpha \in (0, 1)$ and $c \geq 0$ such that*

$$d(x, y) \leq d(x, z) + d(z, y) + c\left[(d(x, z))^{\alpha} (d(z, y))^{1-\alpha}\right],$$

for all $(x, y, z) \in X \times X \times X$.

Then, we call (X, d) an (α, c)-interpolative metric space.

Note that each metric space can be considered an (α, c)-interpolative metric space with $c = 0$. In the following example, we shall clarify that the converse is invalid.

Example 1.2. *Let (X, ρ) be a standard metric space. Define a function $d : X \times X \to [0, \infty)$ as follow*

$$d(x, y) := \rho(x, y)(\rho(x, y) + A),$$

where $A > 0$. Since ρ is a metric on X, the conditions $(m1)$ and $(m2)$ are straightforward. For $(m3)$, it is enough to consider $c \geq 2$ for any $\alpha \in (0, 1)$. Thus, (X, d) is $(\frac{1}{2}, 2)$-interpolative metric space.

Indeed, we have

$$\begin{aligned} d(x, y) &= \rho(x, y)(\rho(x, y) + A) \\ &\leq (\rho(x, z) + \rho(z, y))(\rho(x, z) + \rho(z, y) + A) \\ &= [\rho(x, z)(\rho(x, z) + A) + \rho(x, z)\rho(z, y)] + [\rho(z, y)(\rho(z, y) + A) + \rho(z, y)\rho(x, z)] \\ &= [\rho(x, z)(\rho(x, z) + A)] + [\rho(z, y)(\rho(z, y) + A)] + 2\rho(x, z)\rho(z, y) \\ &= d(x, z) + d(z, y) + 2\,(\rho(x, z))^{\frac{1}{2}}\,(\rho(x, z))^{\frac{1}{2}}\,(\rho(z, y))^{\frac{1}{2}}\,(\rho(z, y))^{\frac{1}{2}} \\ &\leq d(x, z) + d(z, y) + 2\,(\rho(x, z))^{\frac{1}{2}}\,[\rho(x, z) + A]^{\frac{1}{2}}\,(\rho(z, y))^{\frac{1}{2}}\,[\rho(z, y) + A]^{\frac{1}{2}} \\ &= d(x, z) + d(z, y) + 2(d(x, z))^{\frac{1}{2}}(d(z, y))^{\frac{1}{2}} \end{aligned}$$

It is clear that the function $d(x, y)$ does not form metric. Note also that the above estimation for the pair $(\frac{1}{2}, 2)$ is very rough, and it can be improved in several ways.

Example 1.3. *Let X be a non-empty set and define a function $d : X \times X \to [0, \infty)$ as follow*

$$d(x, y) := |x - y|^3, \text{ for all } x, y \in X.$$

Regarding the notion of the absolute value function, we conclude that the conditions $(m1)$ and $(m2)$ are satisfied trivially. For $(m3)$, it is enough to consider $c \geq 6$ for any $\alpha = \frac{1}{3} \in (0, 1)$. Then, (X, d) is $(\frac{1}{3}, 6)$-interpolative metric space.

More precisely, by a simple calculation and manipulation, we derive that

$$\begin{aligned} d(x, y) = |x - y|^3 &= |x - z + z - y|^3 \\ &= |x - z|^3 + |z - y|^3 + 3|x - z|^2|z - y| + 3|x - z||z - y|^2 \\ &\leq d(x, z) + d(z, y) + 3\left[(d(x, z))^{\frac{2}{3}}\,(d(z, y))^{\frac{1}{3}}\right] + 3\left[(d(x, z))^{\frac{1}{3}}\,(d(z, y))^{\frac{2}{3}}\right] \\ &\text{without loss of generality, we assume } d(x, z) \geq d(z, y) \\ &\leq d(x, z) + d(z, y) + 6\left[(d(x, z))^{\frac{2}{3}}\,(d(z, y))^{\frac{1}{3}}\right] \end{aligned} \tag{1}$$

Consequently, we conclude that $(m3)$ is fulfilled. Hence, (X, d) is $(\frac{1}{3}, 6)$-interpolative metric space.

Suppose that $r > 0$ and $x \in X$. Denote

$$\mathfrak{B}(x, r) = \{y \in X : d(x, y) < r\},$$

as an open ball in (α, c)-interpolative metric space (X, d)

Definition 1.4. *Let (X, d) be a (α, c)-interpolative metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ converges to x in X, if and only if, $d(x_n, x) \to 0$, as $n \to \infty$.*

Definition 1.5. *Let (X, d) be a (α, c)-interpolative metric space and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence in X, if and only if,* $\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0$.

Definition 1.6. *Let (X, d) be a (α, c)-interpolative metric space. We say that (X, d) is a complete (α, c)-interpolative metric space if every Cauchy sequence converges in X.*

In this paper, we shall consider the notion of (α, c)-interpolative metric spaces [4] in the framework of fixed point theory to derive the analog of some certain fixed theorems (such as, Ćirić type, Bianchini type, Kannan type fixed point theorems) in this new structure.

2. Main Result

In this section, we shall consider Ćirić type fixed point theorem in the context of an (α, c)-interpolative metric spaces. First, we state the renowned Ćirić fixed point theorem in the setting of an (α, c)-interpolative metric spaces.

Theorem 2.1. *Let (X, d) be a (α, c)-interpolative metric space and let $T : X \to X$ be a mapping. Suppose that there exists q with* $0 < q < 1$ *such that*

$$d(Tx, Ty) \leq qM_d(x, y), \tag{2}$$

for all $x, y \in X$, in which,

$$M_d(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Then, T possesses a unique fixed point in X.

Proof. Take an arbitrary point $x \in X$ and rename it as $x_0 := x$. By using this initial point, let us construct a sequence $\{x_n\}$ as follows: $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for any $n_0 \in \mathbb{N}_0$ then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$; that is, x_{n_0} becomes the fixed point and the proof is completed. Based on this discussion, throughout the proof, we shall presume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Consequently, we derive that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}_0$.

Due to the assumption (2) of the theorem, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq qM_d(x_n, x_{n-1}) \\ &= q\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{3}$$

Since $0 < q < 1$, the case $M_d(x_n, x_{n-1}) = d(x_n, x_{n+1})$ is impossible. Hence, $M_d(x_n, x_{n-1}) = d(x_{n-1}, x_n)$ for all $n \in \mathbf{N}_0$. In brief, the equation (3) turns into

$$d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}_0.$$

Recursively, one can derive from the above inequality that

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1), \text{ for all } n \in \mathbb{N}_0. \tag{4}$$

Taking limit from both side of 3 implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{5}$$

Note that the limit (5) yields also that there exists $k \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq 1 \text{ for all } n \geq k. \tag{6}$$

Now suppose that, $m, n \in \mathbb{N}$ and $m > n > k$. If $x_n = x_m$, we have $T^m(x_0) = T^n(x_0)$. Thus we have, $T^{m-n}(T^n(x_0)) = T^n(x_0)$. Thus, we have $T^n(x_0)$ is the fixed point of T^{m-n}. Also,

$$T(T^{m-n}(T^n(x_0))) = T^{m-n}(T(T^n(x_0))) = T(T^n(x_0)).$$

It means that, $T(T^n(x_0))$ is the fixed point of T^{m-n}. Thus, $T(T^n(x_0)) = T^n(x_0)$. So $T^n(x_0)$ is the fixed point of T. So without loss of generality, we can suppose that $x_n \neq x_m$.

In what follows, we prove that the constructive iterative sequence $\{x_n\}$ is Cauchy. For this aim, we, first, assert that

$$\lim_{n\to\infty} d(x_n, x_{n+r+1}) = 0, \tag{7}$$

for $r \in \mathbb{N}$. For $r = 1$,

$$d(x_n, x_{n+2}) \quad \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + c\left[(d(x_n, x_{n+1}))^{\alpha} (d(x_{n+1}, x_{n+2}))^{1-\alpha}\right]. \tag{8}$$

By letting $n \to \infty$ and keeping (5) in mind, we derive that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{9}$$

Further, we have

$$d(x_n, x_{n+3}) \quad \leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + c\left[(d(x_n, x_{n+2}))^{\alpha} (d(x_{n+1}, x_{n+3}))^{1-\alpha}\right]. \tag{10}$$

On account of (5) and (10), by taking $n \to \infty$ we derive that

$$\lim_{n\to\infty} d(x_n, x_{n+3}) = 0. \tag{11}$$

Suppose that we have the following limit

$$\lim_{n\to\infty} d(x_n, x_{n+r}) = 0, \text{ for some } r \in \mathbb{N}. \tag{12}$$

Then, by the statement of the Theorem, we have

$$d(x_n, x_{n+r+1}) \quad \leq d(x_n, x_{n+r}) + d(x_{n+r}, x_{n+r+1}) + c\left[(d(x_n, x_{n+r}))^{\alpha} (d(x_{n+r}, x_{n+r+1}))^{1-\alpha}\right]. \tag{13}$$

By taking the limits (5) and (12) into account, by letting $n \to \infty$ we observe that

$$\lim_{n\to\infty} d(x_n, x_{n+r+1}) = 0. \tag{14}$$

Consequently, one can deduce that

$$d(x_{n+1}, x_m) < 1, \tag{15}$$

for $m > n > k$. As a next step, for $m > n > k$, we shall consider

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m) + c[(d(x_n, x_{n+1}))^{\alpha} (d(x_{n+1}, x_m))^{1-\alpha}] \\ &\leq q^{n-k} d(x_k, x_{k+1}) + d(x_{n+1}, x_m) + c[\left(q^{n-k} d(x_k, x_{k+1})\right)^{\alpha} (d(x_{n+1}, x_m))^{1-\alpha}] \end{aligned} \tag{16}$$

On account of (15), we find that $(d(x_{n+1}, x_m))^{1-\alpha} < 1$. Hence, the right-hand side of the inequality (16) becomes

$$\begin{aligned} &\leq q^{n-k} d(x_k, x_{k+1}) + d(x_{n+1}, x_m)[c\left(q^{n-k}\right)^{\alpha} (d(x_{n+1}, x_m))^{\alpha}] d(x_k, x_{k+1}) \\ &\leq q^{n-k} d(x_k, x_{k+1}) + [1 + c\left(q^{n-k}\right)^{\alpha}] d(x_{n+1}, x_m) d(x_k, x_{k+1}). \end{aligned} \tag{17}$$

It is easy to observe the following estimation:

$$d(x_n, x_m) \leq q^{n-k} + [1 + c(q^{n-k})^{\alpha}] d(x_{n+1}, x_m). \tag{18}$$

Notice also that

$$\begin{aligned} d(x_{n+1}, x_m) &\leq d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m) \\ &+c[(q^{n-k+1} d(x_k, x_{k+1}))^{alpha} (d(x_{n+2}, x_m))^{1-alpha}] \\ &\leq q^{n-k+1} d(x_k, x_{k+1}) + [(d(x_{n+2}, x_m))^{\alpha} + c(q^{n-k+1})^{\alpha}] d(x_{n+2}, x_m) d(x_k, x_{k+1}) \\ &\leq q^{n-k+1} + [1 + c(q^{n-k+1})^{\alpha}] d(x_{n+2}, x_m). \end{aligned} \tag{19}$$

Emerging the inequalities (18) and (19), we find that

$$\begin{aligned} d(x_n, x_m) \quad & q^{n-k} + q^{n-k+1}[1 + c(q^{n-k})^{\alpha}] \\ & +[1 + c(q^{n-k})^{\alpha}][1 + c(q^{n-k+1})^{\alpha}] d(x_{n+2}, x_m) \end{aligned} \tag{20}$$

Keeping all these observations in mind, we determined that

$$\begin{aligned} d(x_n, x_m) \quad & q^{n-k} \sum_{i=0}^{m-n-1} q^i \prod_{j=0}^{i-1} (1 + cq^{n-k+j})^{\alpha} \\ &\leq q^{n-k} \sum_{i=0}^{m-n-1} q^i \prod_{j=0}^{i-1} (1 + cq^j)^{\alpha} \quad \text{since } q < 1. \end{aligned} \tag{21}$$

The right-hand side of the above inequality is dominated by the sequence $\sum_{i=0}^{\infty} S_i$, which is convergent where,

$$S_i = \prod_{j=0}^{i-1} (1 + cq^j)^{\alpha}.$$

In brief, the sequence $\{x_n\}$ is Cauchy.

Thus, we conclude that the constructed iterative sequence $\{x_n\}$ is a Cauchy. Since (X, d) is a complete (α, c)-interpolative metric space, the sequence $\{x_n\}$ converges to $z \in X$. We claim that z is the fixed point of T. On the contrary, assume $d(z, Tz) > 0$. Note that

$$d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq q \max\{d(x_n, z), d(x_n, x_{n+1}), d(Tz, z)\}. \tag{22}$$

Thus, taking the limit of both sides of (22), we have $q < 1$, which is a contradiction. Therefore, $Tz = z$, and z is the fixed point of T in X. Also, the uniqueness of the fixed point is straightforward from (2). □

The following can be considered as the analog of Bianchini [2] theorems in this new setting.

Corollary 2.2. *Let (X, d) be a (α, c)-interpolative metric space and let $T : X \to X$ be a mapping. Suppose that there* *exists q with* $0 < q < 1$ *such that*

$$d(Tx, Ty) \leq qN_d(x, y), \tag{23}$$

for all $x, y \in X$*, where,*

$$N_d(x, y) = \max \{d(x, Tx), d(y, Ty)\}.$$

Then, T has a unique fixed point in X.

The proof of this corollary is the mimic of the proof of Theorem 2.1, so we skip it.

Corollary 2.3. *Let* (X, d) *be a* (α, c)*-interpolative metric space and let* $T : X \to X$ *be a mapping. Suppose that there exists q with* $0 < q < 1$ *such that*

$$d(Tx, Ty) \leq qK_d(x, y), \tag{24}$$

for all $x, y \in X$*, where,*

$$K_d(x, y) = \frac{d(x, Tx) + d(y, Ty)}{2}.$$

Then, T has a unique fixed point in X.

The proof is straightforward due to the fact that $\frac{a+b}{2} \leq \max\{a, b\}$ where $a, b \geq 0$.

The upcoming corollary is the analog of Banach's Contraction Principle in the context of (α, c)-interpolative metric space.

Corollary 2.4. *Let* (X, d) *be a* (α, c)*-interpolative metric space and let* $T : X \to X$ *be a mapping. Suppose that there exists q with* $0 < q < 1$ *such that*

$$d(Tx, Ty) \leq qd(x, y), \tag{25}$$

for all $x, y \in X$*. Then, T has a unique fixed point in X.*

When we verbatim the steps of the proof Theorem 2.1, we derive the result without encountering any difficulties. With this in mind, we skip the proof of this corollary.

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