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Characterizations for the (b, c)-core inverse in rings

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Abstract. It is known that group inverse, the Moore–Penrose inverse and the inverse along an element have strongly connections with the classical inverse. The (b, c)-inverse and the (b, c)-core inverse are two new types of outer generalized inverses, extending several generalized inverses. In this paper, we mainly establish the criteria for the (b, c)-inverse and the (b, c)-core inverse by units in rings.

1. Introduction

Let S be a semigroup. An element $a \in S$ is regular in the sense of von Neumann if there exists some $x \in S$ such that $a = axa$. Such an x is called an inner inverse or $\{1\}$-inverse of a, and is denoted by a^-. By $a\{1\}$ we denote the set of all inner inverses of a.

For any $a, b, c \in S$, the element a is called (b, c)-invertible if there exists some $y \in S$ such that $y \in bSy \cap ySc$, $yab = b$ and $cay = c$. Such an y is called a (b, c)-inverse of a. It is unique if it exists, and is denoted by $a^{(b,c)}$. We denote by $S^{(b,c)}$ the set of all (b, c)-invertible elements in S. In particular, a is called invertible along b [5] if it is (b, b)-invertible. The (b, c)-inverse encompasses the Moore–Penrose inverse [8], the Drazin inverse [3], the group inverse and the inverse along an element.

Let $*$ be an involution on a semigroup S, that is the involution $*$ satisfies $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for any $x, y \in S$. A semigroup S is called a $*$-semigroup if there exists an involution on S. In what follows, we assume that S is a $*$-semigroup.

We follow [8]. An element $a \in S$ is Moore–Penrose invertible if there exists an $x \in S$ satisfying the following four equations (1) $axa = a$, (2) $xax = x$, (3) $(ax)^* = ax$, (4) $(xa)^* = xa$. Such an x is called a Moore–Penrose inverse of a. It is unique if it exists, and is usually denoted by $a^\dagger$. If $a, x \in S$ satisfy the equations $\{i_1, \dots, i_k\} \subseteq \{1, 2, 3, 4\}$, then x is called a $\{i_1, \dots, i_k\}$-inverse of a, and is denoted by $a^{(i_1,\dots,i_k)}$. As usual, by $S^\dagger$, $S^{(1,3)}$ and $S^{(1,4)}$ we denote the sets of all Moore–Penrose invertible, $\{1, 3\}$-invertible and $\{1, 4\}$-invertible elements in S, respectively.

The present author Zhu [11] introduced the (b, c)-core inverse of a in S. Let $a, b, c \in S$. We call that a is (b, c)-core invertible if there exists some $x \in S$ such that $caxc = c$, $xS = bS$ and $Sx = Sc^*$. Such an x is called a

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(b, c)-core inverse of a. It is unique if it exists, and is denoted by $a^{\oplus}_{(b,c)}$. It was also shown that a is (b, c)-core invertible if and only if a is (b, c)-invertible and c is {1,3}-invertible if and only if ca is (b, c^*)-invertible. The new introduced (b, c)-core inverse extends the core inverse [1, 10], the w-core inverse [16] and the Moore–Penrose inverse. We denote by $S^{\oplus}_{(b,c)}$ the set of all (b, c)-core invertible elements in S. More details on the w-core inverse can be found in [15, 17].

Recall that an element $a \in R$ is left invertible if there exists some $x \in R$ such that $xa = 1$, and a is right invertible if $ay = 1$ for some $y \in R$. An element a is invertible if it is both left and right invertible. As usual, by R_l^{-1}, R_r^{-1} and R^{-1} we denote the sets of all left invertible, right invertible and invertible elements in R, respectively.

A map $* : R \to R$ is an involution of R if it satisfies $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x + y)^* = x^* + y^*$ for all $x, y \in R$. Throughout this paper, any ring R is assumed to be a unital $*$-ring, that is a ring R with unity 1 and an involution $*$.

Several articles derived the the criteria of group inverses, Drazin inverses and Moore–Penrose inverses of a regular element $a \in R$ by using units. Such as, Puystjens and Hartwig [9] proved that $a \in R^\#$ if and only if $a + 1 - aa^- \in R^{-1}$ if and only if $a + 1 - a^-a \in R^{-1}$, where $R^\#$ denotes the set of all group invertible elements in R. Patrício and Araújo [7] derived that $a \in R^\dagger$ if and only if $aa^* + 1 - aa^- \in R^{-1}$ if and only if $a^*a + 1 - a^-a \in R^{-1}$. In [12], the authors obtained the fact that $a \in R^\dagger$ if and only if $a \in aa^*aR$ if and only if $a \in Raa^*a$. Based on this, the author derived in [14] that $a \in R^\dagger$ if and only if $aa^* + 1 - aa^- \in R_l^{-1}$ if and only if $aa^* + 1 - aa^- \in R_r^{-1}$.

In 2011, Mary [5] introduced the inverse along an element, which recovers the classical group inverses, Drazin inverses and Moore–Penrose inverses. Furthermore, it is shown that $a^{(d,d)}$ exists if and only if $da + 1 - dd^- \in R^{-1}$ if and only if $ad + 1 - d^-d \in R^{-1}$, provided that d is regular. This recovers the classical criterion for group inverses and Moore–Penrose inverses by picking $d = a$ and $d = a^*$, respectively.

As far as we know presently, there is no such a characterization for the general (b, c)-inverse for the case $b \neq c$.

Motivated by [7, 9, 11], it is of interest to establish the characterization for the (b, c)-inverse and the (b, c)-core inverse by using units in a $*$-ring R.

2. Characterizations for (b, c)-core inverses by units

In this section, we aim to derive the characterization for the (b, c)-inverse by units, provided that b and c are relevant.

Let us now recall Green's relations [4] in a ring R: (i) $a\mathcal{L}b \Leftrightarrow Ra = Rb \Leftrightarrow$ there exist some $x, y \in R$ such that $a = xb$ and $b = ya$. (ii) $a\mathcal{R}b \Leftrightarrow aR = bR \Leftrightarrow$ there exist some $s, t \in R$ such that $a = bs$ and $b = at$. (iii) $a\mathcal{H}b \Leftrightarrow a\mathcal{L}b$ and $a\mathcal{R}b$.

Under the relation $b\mathcal{H}c$, we observe that b and c have almost the same property. For instance, if b is regular ({1,3}-invertible or {1,4}-invertible), then so is c. The notation for the commutator of b and c is $[b, c] = bc - cb$.

Lemma 2.1. *Let $a, b, c \in R$ with $b\mathcal{H}c$. Then we have*

(i) *b is regular if and only if c is regular. Moreover, $b = cc^-b = bc^-c$ and $c = cb^-b = bb^-c$.*

(ii) *$b \in R^{(1,3)}$ if and only if $c \in R^{(1,3)}$. Moreover, $[bb^{(1,3)}, cc^{(1,3)}] = 0$.*

(iii) *$b \in R^{(1,4)}$ if and only if $c \in R^{(1,4)}$. Moreover, $[b^{(1,4)}b, c^{(1,4)}c] = 0$.*

Proof. Given $b\mathcal{H}c$, i.e., $bR = cR$ and $Rb = Rc$, then there is some $t \in R$ such that $c = bt = bb^-bt = bb^-c$, Also, $b\mathcal{H}c$ implies $b = cs$ for some $s \in R$.

(i) Suppose b is regular. Then $c = bb^-c = csb^-c$, i.e., c is regular. So, $b = cc^-cs = cc^-b$, and dually $b = bc^-c$. Conversely, if c is regular then so is b, and we have $c = cb^-b = bb^-c$.

(ii) Suppose $b \in R^{(1,3)}$, i.e., $b \in Rb^*b$. Then $c \in Rb^*c$ by $c = bt$. As $b = cs$, then $c \in R(cs)^*c = Rs^*c^*c \subseteq Rc^*c$ and $c \in R^{(1,3)}$. For the converse part, if $c \in Rc^*c$, we get $b = cs \in Rc^*cs = Rc^*b = R(bt)^*b \subseteq Rb^*b$, and $b \in R^{(1,3)}$. Applying (i), it follows that $b = cc^{(1,3)}b$ and $bb^{(1,3)} = cc^{(1,3)}bb^{(1,3)} = (cc^{(1,3)}bb^{(1,3)})^* = bb^{(1,3)}cc^{(1,3)}$. Hence, $[bb^{(1,3)}, cc^{(1,3)}] = 0$.

(iii) By a similar proof of (ii). □

It is well known that $b \in R^{\dagger}$ if and only if $b \in R^{(1,3)} \cap R^{(1,4)}$. From Lemma 2.1, one can get that $b \in R^{\dagger}$ if and only if $c \in R^{\dagger}$, under the hypothesis $b\mathcal{H}c$.

Lemma 2.2. *Let $a, b \in R$. Then we have*
(i) $1 + ab$ *is left invertible if and only if* $1 + ba$ *is left invertible.*
(ii) $1 + ab$ *is right invertible if and only if* $1 + ba$ *is right invertible.*

The lemma above is well known as Jacobson' Lemma. In particular, if $1+ab$ is invertible then so is $1+ba$. Moreover, $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.

Lemma 2.3. [2, Theorem 2.2] *Let $a, b, c \in R$. Then a is (b, c)-invertible if and only if $b \in Rcab$ and $c \in cabR$. In particular, if $b = vcab$ and $c = cabw$ for some $v, w \in R$, then $a^{(b,c)} = bw = vc$.*

Proposition 2.4. *Let $a, b, c \in R$ with b regular. If $b\mathcal{H}c$, then the following statements are equivalent*:
(i) $a \in R^{(b,c)}$.
(ii) $u = 1 + ab - b^-b \in R^{-1}$ *and* $v = 1 + ca - cc^- \in R^{-1}$.
(iii) $u' = 1 + ba - bb^- \in R^{-1}$ *and* $v' = 1 + ac - c^-c \in R^{-1}$.
In this case, $a^{(b,c)} = (u')^{-1}b = c(v')^{-1}$.

Proof. (i) ⇒ (ii) Given (i), then, by Lemma 2.3, we have $b = vcab$ and $c = cabw$ for some $v, w \in R$. Note that $vcb^-bab = vcab = b$ by Lemma 2.1(i). Then $(b^-vcb^-b + 1 - b^-b)(b^-bab + 1 - b^-b) = 1$, i.e., $b^-bab + 1 - b^-b \in R_l^{-1}$. Also, it follows from $bc^-c = b$ that $(b^-bab + 1 - b^-b)(b^-bw + 1 - b^-b) = 1$. Indeed, we have

$$\begin{aligned} & (b^-bab + 1 - b^-b)(b^-bw + 1 - b^-b) \\ = \; & b^-babw + 1 - b^-b \\ = \; & b^-(bc^-c)abw + 1 - b^-b \\ = \; & b^-bc^-(cabw) + 1 - b^-b \\ = \; & b^-(bc^-c) + 1 - b^-b \\ = \; & b^-b + 1 - b^-b \\ = \; & 1. \end{aligned}$$

Consequently, $b^-bab + 1 - b^-b \in R_r^{-1}$. So, $b^-bab + 1 - b^-b = 1 + b^-b(ab - 1) \in R^{-1}$. Again, Lemma 2.2 ensures that $u = 1 + ab - b^-b = 1 + (ab - 1)b^-b \in R^{-1}$. Similarly, we get $1 + ca - cc^- \in R^{-1}$.

(ii) ⇔ (iii) follows from Lemma 2.2. Indeed, $u = 1 + ab - b^-b = 1 + (a - b^-)b \in R^{-1}$ if and only if $u' = 1 + b(a - b^-) = 1 + ba - bb^- \in R^{-1}$. A similar argument for v and v'.

(iii) ⇒ (i) If $u' = 1 + ba - bb^- \in R^{-1}$ then $u'b = bab$ and $b = (u')^{-1}bab = (u')^{-1}(bc^-c)ab \in Rcab$. Also, $c = cac(v')^{-1} = cabb^-c(v')^{-1} \in cabR$. So, $a \in R^{(b,c)}$ and $a^{(b,c)} = (u')^{-1}b = c(v')^{-1}$ by Lemma 2.3. □

If $b\mathcal{H}c$ and b is regular, then c is regular, $b = bc^-c$ and $c = cb^-b$ by Lemma 2.1. Hence, $u = 1 + ab - b^-b \in R^{-1}$ in Proposition 2.4(ii) can be reduced to $u = 1 + abc^-c - b^-bc^-c \in R^{-1}$, and in terms of Lemma 2.2, $1 + c^-cab - c^-cb^-b = 1 + c^-cab - c^-c = 1 + c^-c(ab - 1) \in R^{-1}$, whence $1 + abc^-c - c^-c = 1 + ab - c^-c \in R^{-1}$.

Analogously, by $b = cc^-b$ and $c = bb^-c$, then $1+ba-bb^- \in R^{-1}$ can be reduced to $1+ba-cc^- \in R^{-1}$. Similar arguments show that $v = 1 + ca - cc^- \in R^{-1}$ if and only if $t = 1 + ca - bb^- \in R^{-1}$, and $v' = 1 + ac - c^-c \in R^{-1}$ if and only if $t' = 1 + ac - b^-b \in R^{-1}$.

We hence have the following characterization for the (b, c)-inverse.

Corollary 2.5. *Let $a, b, c \in R$ with b regular. If $b\mathcal{H}c$, then the following statements are equivalent*:
(i) $a \in R^{(b,c)}$.
(ii) $s = 1 + ab - c^-c \in R^{-1}$ *and* $v = 1 + ca - cc^- \in R^{-1}$.
(iii) $s' = 1 + ba - cc^- \in R^{-1}$ *and* $v' = 1 + ac - c^-c \in R^{-1}$.
(iii) $u = 1 + ab - b^-b \in R^{-1}$ *and* $t = 1 + ca - bb^- \in R^{-1}$.
(iv) $u' = 1 + ba - bb^- \in R^{-1}$ *and* $t' = 1 + ac - b^-b \in R^{-1}$.

We next come to our another main result of this section, under Green's relations $b\mathcal{H}c$.

Theorem 2.6. *Let $a, b, c \in R$ with b regular. If $b\mathcal{H}c$, then the following statements are equivalent*:
(i) $a \in R^{(b,c)}$.
(ii) $u = caba + 1 - cc^- \in R^{-1}$.
(iii) $v = abac + 1 - c^-c \in R^{-1}$.
(iv) $s = baca + 1 - bb^- \in R^{-1}$.
(v) $t = acab + 1 - b^-b \in R^{-1}$.
In this case, $a^{(b,c)} = u^{-1}cab = bacv^{-1} = s^{-1}bac = babt^{-1}$.

Proof. It only need prove (i) ⇔ (ii) ⇔ (iii), as (i) ⇔ (iv) ⇔ (v) can be probed similarly.
(i) ⇒ (ii) As $a \in R^{(b,c)}$, then, by Lemma 2.1 and Corollary 2.5, $ba + 1 - cc^- = cc^-ba + 1 - cc^- \in R^{-1}$ and $cacc^- + 1 - cc^- \in R^{-1}$. So, $caba + 1 - cc^- = (cacc^- + 1 - cc^-)(cc^-ba + 1 - cc^-) \in R^{-1}$.
(ii) ⇔ (iii) by Lemma 2.2.
(iii) ⇒ (i) If $v = abac + 1 - c^-c$ then $cv = cabac$ and $c = cabacv^{-1} \in cabR$. As (iii) ⇔ (ii), then we have $uc = cabac$ and $c = u^{-1}cabac$. So, $b = cc^-b = u^{-1}cabacc^-b = u^{-1}cabab = u^{-1}ca(bc^-c)ab \in Rcab$. By Lemma 2.3, $a \in R^{(b,c)}$ and $a^{(b,c)} = bacv^{-1} = u^{-1}cab$.
We next give another expression of $a^{(b,c)}$. As $s = baca + 1 - bb^- \in R^{-1}$, then $sb = bacab$ and $b = s^{-1}bacab$. Also, $b = bacabt^{-1}$ since $t = acab + 1 - b^-b \in R^{-1}$. So, $c = cb^-b = cb^-(bacabt^{-1}) = (cb^-b)acabt^{-1} = cacabt^{-1} = ca(bb^-c)abt^{-1} = cab(b^-cabt^{-1})$. Moreover, $a^{(b,c)} = s^{-1}bac = cabt^{-1}$. □

3. Criteria for the (b, c)-core inverse

The following result presents the characterization for the (b, c)-core inverse by units. Several auxiliary lemmas are given, which play important roles in the proof of the sequel results.

Lemma 3.1. [12, Theorem 2.16] and [13, Theorem 3.12] *Let $a \in R$. Then the following statements are equivalent*:
(i) $a \in R^{\dagger}$.
(ii) $a \in aa^*aR$.
(iii) $a \in Raa^*a$.
In this case, $a^{\dagger} = (ax)^* = (ya)^*$, *where $x, y \in R$ satisfy* $a = aa^*ax = yaa^*a$.

Lemma 3.2. [11, Theorem 2.7] *Let $a, b, c \in R$. Then $a \in R^{\textcircled{\#}}_{(b,c)}$ if and only if $a \in R^{(b,c)}$ and $c \in R^{(1,3)}$. In this case,* $a^{\textcircled{\#}}_{(b,c)} = a^{(b,c)}c^{(1,3)}$.

Theorem 3.3. *Let $a, b, c \in R$ with $b\mathcal{H}c$. Then the following statements are equivalent*:
(i) $a \in R^{\textcircled{\#}}_{(b,c)}$.
(ii) $c \in R^{(1,3)}$ *and* $u = caba + 1 - cc^{(1,3)} \in R^{-1}$.
(iii) $c \in R^{(1,3)}$ *and* $v = abac + 1 - c^{(1,3)}c \in R^{-1}$.
In this case, $a^{\textcircled{\#}}_{(b,c)} = bau^{-1}$.

Proof. To begin with, (ii) ⇔ (iii) follows from Lemma 2.2.
(i) ⇒ (ii) As $a \in R^{\textcircled{\#}}_{(b,c)}$, then, by Lemma 3.2, $a \in R^{(b,c)}$ and $c \in R^{(1,3)}$. It follows from Theorem 2.6 that $caba + 1 - cc^{(1,3)} \in R^{-1}$ since $c^{(1,3)} \in c\{1\}$.
(iii) ⇒ (i) Since $v = abac + 1 - c^{(1,3)}c \in R^{-1}$, it follows from Theorem 2.6 that $a \in R^{(b,c)}$, which together with $c \in R^{(1,3)}$ imply $a \in R^{\textcircled{\#}}_{(b,c)}$ by Theorem 3.2.
We next give the formula of the $a^{\textcircled{\#}}_{(b,c)}$. Since $u^*c = ((caba)^* + 1 - cc^{(1,3)})c = (caba)^*c$, we have $c = (u^*)^{-1}(caba)^*c = (u^*)^{-1}(aba)^*c^*c$ and $abau^{-1} \in c\{1, 3\}$. So, $a^{\textcircled{\#}}_{(b,c)} = a^{(b,c)}c^{(1,3)} = a^{(b,c)}abau^{-1} = (a^{(b,c)}ab)au^{-1} = bau^{-1}$. □

Recall from [11] that an element $a \in R$ is called dual (b, c)-core invertible if there is some $y \in S$ such that $byab = b$, $yR = b^*R$ and $Ry = Rc$. Such an element y is called a dual (b, c)-core inverse of a. The dual (b, c)-core inverse of a is denoted by $a_{(b,c)\textcircled{\#}}$. By $R_{(b,c)\textcircled{\#}}$ we denote the set of all dual (b, c)-core invertible elements in R.
Characterizations for the dual (b, c)-core inverse can be given as follows.

Theorem 3.4. *Let $a, b, c \in R$ with $b\mathcal{H}c$. Then the following statements are equivalent:*
(i) $a \in R_{(b,c)\#}$.
(ii) $b \in R^{(1,4)}$ *and* $s = baca + 1 - bb^{(1,4)} \in R^{-1}$.
(iii) $b \in R^{(1,4)}$ *and* $t = acab + 1 - b^{(1,4)}b \in R^{-1}$.
In this case, $a_{(b,c)\#} = t^{-1}ac$.

As was shown in [11], $a \in R^{\#}_{(b,c)}$ if and only if $a \in R^{(b,c)}$ and $c \in R^{(1,3)}$. Dually, $a \in R_{(b,c)\#}$ if and only if $a \in R^{(b,c)}$ and $b \in R^{(1,4)}$. In particular, $a \in R^{\#}_{(b,c)} \cap R_{(b,c)\#}$ if and only if $a \in R^{(b,c)}$, $b \in R^{(1,4)}$ and $c \in R^{(1,3)}$. It is concluded that $a \in R^{\#}_{(b,c)} \cap R_{(b,c)\#}$ if and only if $a \in R^{(b,c)}$ and $b \in R^{\dagger}$ (or $c \in R^{\dagger}$), provided that $b\mathcal{H}c$.

We next present the criterion for both (b, c)-core and dual (b, c)-core invertible elements by units, under the Green's relations $b\mathcal{H}c$.

Lemma 3.5. [7, Theorem 1.2] and [13, Corollary 3.17] *Let $a \in R$ be regular. Then the following statements are equivalent:*
(i) $a \in R^{\dagger}$.
(ii) $u = aa^* + 1 - aa^- \in R^{-1}$.
(iii) $v = a^*a + 1 - a^-a \in R^{-1}$.
In this case, $a^{\dagger} = (u^{-1}a)^* = (av^{-1})^*$.

Theorem 3.6. *Let $a, b, c \in R$ with b regular. If $b\mathcal{H}c$, then the following statements are equivalent:*
(i) $a \in R^{\#}_{(b,c)} \cap R_{(b,c)\#}$.
(ii) $u = bb^*baca + 1 - bb^- \in R^{-1}$.
(iii) $v = b^*bacab + 1 - b^-b \in R^{-1}$.
(iv) $s = cc^*caba + 1 - cc^- \in R^{-1}$.
(v) $t = c^*cabac + 1 - c^-c \in R^{-1}$.

Proof. (i) ⇒ (ii) Given $a \in R^{\#}_{(b,c)} \cap R_{(b,c)\#}$, then $a \in R^{(b,c)}$, $b \in R^{(1,4)}$ and $c \in R^{(1,3)}$. So, $b \in R^{\dagger}$ since $b\mathcal{H}c$. It follows from Lemma 3.5 that $bb^* + 1 - bb^- \in R^{-1}$, and consequently $bb^*bb^- + 1 - bb^- \in R^{-1}$ from Lemma 2.2. Note that $a \in R^{(b,c)}$. Then $baca + 1 - bb^- \in R^{-1}$ in terms of Theorem 2.6. Therefore, $u = bb^*baca + 1 - bb^- = (bb^*bb^- + 1 - bb^-)(baca + 1 - bb^-) \in R^{-1}$.

(ii) ⇔ (iii) follows from Jacobson's Lemma.

(iii) ⇒ (i) As $v = b^*bacab + 1 - b^-b \in R^{-1}$, then $bv = bb^*bacab$ and $b = bb^*bacabv^{-1} \in bb^*bR$. So, by Lemma 3.1, $b \in R^{\dagger}$, and hence $c \in R^{\dagger}$ since $b\mathcal{H}c$. Given $b \in R^{\dagger}$, then $bb^*bb^- + 1 - bb^- \in R^{-1}$ by Lemmas 2.2 and 3.5. So, $baca + 1 - bb^- = (bb^*bb^- + 1 - bb^-)^{-1}(bb^*baca + 1 - bb^-) \in R^{-1}$, which together with Theorem 2.6 imply $a \in R^{(b,c)}$. So, $a \in R^{\#}_{(b,c)} \cap R_{(b,c)\#}$.

(i) ⇔ (iv) ⇔ (v) can be proved similarly. □

References

[1] O.M. Baksalary, G. Trenkler, Core inverse of matrices, Linear Multilinear Algebra 58 (2010) 681-697.
[2] M.P. Drazin, A class of outer generalized inverses, Linear Algebra Appl. 436 (2012) 1909-1923.
[3] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Amer. Math. Monthly 65 (1958) 506-514.
[4] J.A. Green, On the structure of semigroups, Ann. Math. 54 (1951) 163-172.
[5] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl. 434 (2011) 1836-1844.
[6] X. Mary, P. Patrício, Generalized inverses modulo $\mathcal{H}$ in semigroups and rings, Linear Multilinear Algebra 61 (2013) 886-891.
[7] P. Patrício, C.M. Araújo, Moore-Penrose invertibility in involutory rings: $aa^{\dagger} = bb^{\dagger}$, Linear Multilinear Algebra 58 (2010) 445-452.
[8] R. Penrose, A generalized inverse for matrices, Proc. Camb. Phil. Soc. 51 (1955) 406-413.
[9] R. Puystjens,R.E. Hartwig, The group inverse of a companion matrix, Linear Multilinear Algebra 43 (1997) 137-150.
[10] D.S. Rakić, N.C. Dinčić, D.S. Djordjević, Group, Moore-Penrose, core and dual core inverse in rings with involution, Linear Algebra Appl. 463 (2014) 115-133.
[11] H.H. Zhu, The (b, c)-core inverse and its dual in semigroups with involution, J. Pure Appl. Algebra. 228 (2024) 107526.
[12] H.H. Zhu, J.L. Chen, P. Patrício, Further results on the inverse along an element in semigroups and rings, Linear Multilinear Algebra 64 (2016) 393-403.
[13] H.H. Zhu, J.L. Chen, P. Patrício, X. Mary, Centralizer's applications to the inverse along an element, Appl. Math. Comput. 315 (2017) 27-33.

[14] H.H. Zhu, J.L. Chen, P. Patrício, Reverse order law for the inverse along an element, Linear Multilinear Algebra 65 (2017) 166-177.
[15] H.H. Zhu, L.Y. Wu, A new class of partial orders, Algebra Colloq, 30 (2023) 585-598.
[16] H.H. Zhu, L.Y. Wu, J.L. Chen, A new class of generalized inverses in semigroups and rings with involution, Comm. Algebra 51 (2023) 2098-2113.
[17] H.H. Zhu, L.Y. Wu, D. Mosić, One-sided w-core inverses in rings with involution, Linear Multilinear Algebra 71 (2023) 528-544.
[18] H.H. Zhu, X.X. Zhang, J.L. Chen, Generalized inverses of a factorization in a ring with involution, Linear Algebra Appl. 472 (2015) 142-150.