Filomat 38:22 (2024), 7743–7754
https://doi.org/10.2298/FIL2422743C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Slant helices on Riemannian manifolds

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Abstract. The notion of a slant helix in Euclidean space was defined by Izumiya and Takeuchi [5], and many authors have studied such curves in Euclidean spaces. The aim of this paper is to introduce the slant helix notion on Riemannian manifolds. The necessary conditions for a curve on a Riemannian manifold to be a slant helix are obtained in terms of differential equations. In addition, certain conditions were found for the slant helix along an immersion to be a slant helix in the ambient space. Moreover, a criterion is given for the slant helix along an immersion to be a circle in the ambient space (or vice versa).

1. Introduction

Although curves are the most basic geometrical structures of geometry, studies on curves on a Riemannian manifold are limited compared to the theory of submanifolds. In this direction, the first attempt was made by Nomizu and Yano [14]. They studied circles in Riemannian manifolds and gave a characterization for a curve on a Riemannian manifold to be a circle by differential equation

$$\nabla_X^2 X + k^2 X = 0 \tag{1}$$

where ∇ is the Levi-Civita connection of the Riemannian manifold, k is a constant and X is tangent vector field of the curve [14]. They also used this notion to characterize extrinsic spheres. Indeed, they showed that a submanifold of a Riemann manifold is an extrinsic sphere if and only if a circle in the submanifold is a circle in the ambient Riemannian manifold. T. Ikawa studied ordinary helix on Riemannian manifolds and stated that any curve is an ordinary helix on a Riemannian manifold if and only if

$$\nabla_X^3 X + F\nabla_X X = 0$$

where F is a constant and X is the tangent vector field of the curve. He also obtained certain characterizations of submanifold by using the notion of helix [3]. Ikawa also studied such curves in an indefinite-Riemannian manifold [4]. Ekmekçi generalized results of Ikawa to the case of a general helix in indefinite-Riemannian manifold [2]. Izumiya and Takeuchi have defined slant helices and conical geodesic curves in Euclidean 3-space. Those notions are generalizations of cylindrical helices. Kula et al.[9] (see also [10]) obtained characterizations of space curves to be slant helices by considering certain differential equations. The

2020 *Mathematics Subject Classification*. Primary 53B20 ; Secondary , 53A35.
Keywords. Helix, Slant helix, Circle, Riemannian manifold, Submanifold, Isotropic submanifold, Totally geodesic submanifold.
Received: 09 January 2024; Accepted: 15 January 2024
Communicated by Ljubica Velimirović
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geometry of slant helices has been also studied in semi-Riemannian geometry, [1], [12], [20]. It is seen in the literature that it is a very useful method to obtain information about the map and manifolds themselves, by examining the behavior of a curve along a map (isometric immersion, Riemannian submersion or Riemannian map), [6–8, 13, 16–19].

The main purpose of this paper is to define the concept of a slant helix on a Riemannian manifold and to examine its basic properties. To this end, in section 2, the basic notions related to the scope of this paper are presented. In the third section, a definition of the concept of slant helix on the Riemann manifold is presented. This definition agrees with the slant helix notion given in Euclidean spaces. In this section, a characterization is also given for a given curve on the manifold to be a slant helix. In addition, the characterization of the submanifold is obtained under the condition that the curve on a given submanifold is transformed to the ambient manifold as a slant helix. In section 4, the transformation of the circle and the slant helix into each other along an immersion is considered. The non-existence theorem is found if a circle is transformed into a slant helix. If a slant helix is transformed into a circle along an isometric immersion, it is shown that the immersion is totally geodesic.

2. Preliminaries

Let $(\bar{M}, \langle , \rangle)$ be a Riemannian manifold and M an $n-$ dimensional submanifold of $\bar{M}$. Assume that $\bar{\nabla}$ is the Levi-civita connection in $\bar{M}$ and ∇ is the Levi-civita connection in M. Let $\chi(\bar{M})(resp.\chi(M))$ be the Lie algebra of vector fields on $\bar{M}(resp.M)$ and $\chi^{\perp}(M)$ the set of all vector fields normal to M [21]. The Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y), \ X,Y \in \chi(M), \tag{2}$$
$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \ N \in \chi^{\perp}(M), \tag{3}$$

where $\nabla^{\perp}$ is the connection in the normal bundle and B is the second fundamental form of M[21]. A_N is called the shape operator and satisfies the relation

$$\langle A_N X, Y\rangle = \langle B(X,Y), N\rangle. \tag{4}$$

We denote the covariant derivatives for the second fundamental form B as follows:

$$(\widetilde{\nabla}_X B)(Y,Z) = \nabla^{\perp}_X B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), \tag{5}$$

$$\widetilde{\nabla}_W(\widetilde{\nabla}_Z B)(X,Y) = \nabla^{\perp}_W((\widetilde{\nabla}_Z B)(X,Y)) - (\widetilde{\nabla}_{\nabla_W Z} B)(X,Y) - (\widetilde{\nabla}_Z B)(\nabla_W X, Y) - (\widetilde{\nabla}_Z B)(X, \nabla_W Y). \tag{6}$$

The covariant differentiation of A_N is given by

$$(\nabla_X A)_N Y = \nabla_X A_N Y - A_{\nabla^{\perp}_X N} Y - A_N \nabla_X Y. \tag{7}$$

M is called a totally geodesic submanifold if its second fundamental form vanishes. The mean curvature vector field H is defined

$$H = \frac{1}{n} TrB = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

If $\nabla^{\perp}_X H = 0$, for any vector $X \in T_p(M)$, then H is called parallel. If the second fundamental form is

$$B(X,Y) = \langle X, Y\rangle H, \tag{8}$$

then M is called a totally umbilical submanifold. If the vector field $B(X,X)$ has the same length for any unit vector X in $T_p(M)$, then M is called to be isotropic at p. If M is isotropic at any point on M, then M is called isotropic. The submanifold M is isotropic at p if and only if it satisfies $\langle B(X,X), B(X,Y)\rangle = 0$ for any orthonormal vectors X and Y. Furthermore if B satisfies $B(X,Y) = 0$ for any orthonormal vectors X and Y at $p \in M$, then M is umbilical at p [3, 15].

Let c be immersed unit speed curve in a n-dimensional Riemannian manifold. We denote the unit tangent vector field, the unit normal vector field, and the binormal vector field of the curve by X, Y and Z, respectively. $\tau = \langle \nabla_X Z, Y \rangle$ is the torsion of the curve. The curve has also curvatures $k_1 > 0, k_2, k_3, k_4, ..., k_{n-1}$ and Frenet frame $N_0 = X, N_1 = Y, N_2 = Z, N_3, N_4, ..N_{n-1}$. Then, the Frenet equations are given by

$$\nabla_X N_i = -k_i N_{i-1} + k_{i+1} N_{i+1}, \ \ 0 \leq i \leq n-1.$$

In this case, c is called a Frenet curve of order n [11].

Definition 2.1. *[14] A regular Frenet curve $c = c(s)$ parameterized by arc length s with $k_1 \neq 0$ is called a circle of order 2 if there is a unit vector field Y along c and positive constant k such that*

$$\nabla_X X = kY, \ \ \nabla_X Y = -kX, \tag{9}$$

where the unit vector field X is the tangent vector field of the circle. The number $\frac{1}{k}$ is called the radius of the circle.

3. Slant helices on Riemannian Manifold

Let $c(s)$ be a regular Frenet curve on a Riemannian manifold. We denote the tangent vector field $c'(s)$ by X. Unless otherwise stated, a unit speed curve c will be considered in this paper.

Definition 3.1. *Let $c(s)$ be a Frenet curve and denote the tangent vector field of $c(s)$ by X. A regular Frenet curve $c = c(s)$ parameterized by arc length s with $k_1 \neq 0$ is called a slant helix if there are unit vector fields Y, Z along c such that*

$$\begin{aligned} \nabla_X X &= k_1 Y, \\ \nabla_X Y &= -k_1 X + k_2 Z, \\ \nabla_X Z &= -k_2 Y, \end{aligned} \tag{10}$$

and $\frac{k_1^2\left(\frac{k_2}{k_1}\right)'}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}$ is non-zero constant. The number k_1 and k_2 are called curvature and torsion of the slant helix, respectively.

We note that if $\frac{k_1^2\left(\frac{k_2}{k_1}\right)'}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} = 0$, then it follows that $\left(\frac{k_2}{k_1}\right)' = 0$, thus $\frac{k_2}{k_1}$ is constant which gives general helices in Riemannian manifold.

We first give necessary criteria for a slant helix curve on a Riemannian manifold.

Theorem 3.2. *Let $c(s)$ be a Frenet curve with curvatures $k_1, k_2 \neq 0$ on a Riemannian manifold M (dimM≥3). If $c(s)$ is a slant helix, then the unit tangent vector field X and the unit vector field Y of the curve satisfy*

$$\nabla^3{}_X X = 2k_1' \nabla_X Y + k_1 \nabla_X^2 Y + \left(\frac{k_2''}{k_2} - \frac{3}{2}\frac{k_1}{k_2}\left(\frac{k_2}{k_1}\right)' \left(\ln(k_1^2 + k_2^2)\right)' \right) \nabla_X X. \tag{11}$$

Proof. We assume that $c = c(s)$ is a slant helix with curvatures $k_1, k_2 \neq 0$. The second and third derivatives are obtained as

$$\nabla_X^2 X \ = \ \nabla_X (k_1 Y) = -k_1^2 X + k_1' Y + k_1 k_2 Z,$$

and

$$\nabla_X^3 X \ = \ 2k_1' \nabla_X Y + k_1 \nabla_X^2 Y + \frac{k_1''}{k_1} \nabla_X X \tag{12}$$

by virtue of (10). Since $c = c(s)$ is a slant helix, we get

$$\frac{k_1^2\left(\frac{k_2}{k_1}\right)'}{\left(k_1^2+k_2^2\right)^{3/2}} = \frac{k_2' k_1 - k_1' k_2}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} = constant.$$

Taking the derivative of both sides and arranging the outcome, it follows that

$$\frac{(k_2''k_1 - k_2k_1'')\left(k_1^2 + k_2^2\right)^{3/2} - (k_2'k_1 - k_2k_1')\frac{3}{2}\left(k_1^2 + k_2^2\right)^{1/2}(2k_1k_1' + 2k_2k_2')}{\left(k_1^2 + k_2^2\right)^3} = 0.$$

Hence we obtain

$$\frac{k_1''}{k_1} = \frac{k_2''}{k_2} - \frac{3}{2}\frac{k_1}{k_2}\left(\frac{k_2}{k_1}\right)'\left(\ln(k_1^2 + k_2^2)\right)'. \tag{13}$$

Substituting the last equation in the equation (12), we have (11). □

Let $\bar{M}$ be a Riemannian manifold and M a submanifold of $\bar{M}$. Then, the curvatures of a curve c on the submanifold M will be denoted by k_1, k_2, and the curvatures of the curve γ, which is the counterpart of the c on $\bar{M}$, will be denoted as $\bar{k}_1$, $\bar{k}_2$. We give the following proposition which shows that M is an isotropic submanifold under certain conditions.

Proposition 3.3. *Let M ($\dim M \geq 3$) be a connected submanifold of a Riemannian manifold $\bar{M}$ and c be Frenet curve. For each pair (u, v) of orthonormal tangent vectors, there is a slant helix c in M which is not a general helix and that is a slant helix in $\bar{M}$ satisfying the following:*

i) $c'(0) = u, (\nabla_c' c')(0) = k_1(0)v,$

ii) $6k_1(0) \neq \bar{k_1}(0), k_1, \bar{k_1} > 0, k_2, \bar{k_2} \neq 0$

where k_1, k_2 and $\bar{k_1}$, $\bar{k_2}$ are curvatures of c in M and that in $\bar{M}$, respectively. Then, submanifold M is isotropic space in $\bar{M}$.

Proof. Now, we assume that a slant helix with curvatures $k_1 > 0$ and $k_2 \neq 0$ in M is a slant helix in $\bar{M}$. From the equation (11), we have the following equation

$$\nabla_X^3 X = 2k_1'\nabla_X Y + k_1\nabla_X^2 Y + K\nabla_X X \tag{14}$$

where $K = \frac{k_2''}{k_2} - \frac{3}{2}\frac{k_1}{k_2}\left(\frac{k_2}{k_1}\right)'\left(\ln(k_1^2 + k_2^2)\right)'$. Since the curve c is a slant helix in $\bar{M}$, it follows that

$$\bar{\nabla}_X^3 X = 2\bar{k_1}'\,\bar{\nabla}_X Y + \bar{k_1}\,\bar{\nabla}_X^2 Y + \bar{K}\,\bar{\nabla}_X X$$

where $\bar{K} = \frac{\bar{k_2}''}{\bar{k_2}} - \frac{3}{2}\frac{\bar{k_1}}{\bar{k_2}}\left(\frac{\bar{k_2}}{\bar{k_1}}\right)'\left(\ln(\bar{k_1}^2 + \bar{k_2}^2)\right)'$. From (2) and (3), we obtain

$$\bar{\nabla}_X^3 X = \nabla_X^3 X + B(X, \nabla_X^2 X) + 3\bar{\nabla}_X(B(X, \nabla_X X)) - \nabla_X(A_{B(X,X)}X) - B(X, A_{B(X,X)}X) + \bar{\nabla}_X((\widetilde{\nabla}_X B)(X, X)). \tag{15}$$

Using Weingarten formula and (5), then we have

$$3\bar{\nabla}_X(B(X, \nabla_X X)) = -3A_{B(X,\nabla_X X)}X + 3(\widetilde{\nabla}_X B)(X, \nabla_X X) + 3B(\nabla_X X, \nabla_X X) + 3B(X, \nabla_X^2 X). \tag{16}$$

Also, we have

$$\begin{aligned}\bar{\nabla}_X((\widetilde{\nabla}_X B)(X, X)) = & -A_{(\widetilde{\nabla}_X B)(X,X)}X + (\widetilde{\nabla}_X^2 B)(X, X) + (\widetilde{\nabla}_{\nabla_X X} B)(X, X) \\ & +(\widetilde{\nabla}_X B)(\nabla_X X, X) + (\widetilde{\nabla}_X B)(X, \nabla_X X).\end{aligned} \tag{17}$$

Putting (16) and (17) in (15), we derive

$$\begin{aligned}\bar{\nabla}_X^3 X = & \nabla_X^3 X + 4B(X, \nabla_X^2 X) - 5A_{B(X,\nabla_X X)}X + 5(\widetilde{\nabla}_X B)(X, \nabla_X X) + 3B(\nabla_X X, \nabla_X X) - (\nabla_X A)_{B(X,X)}X \\ & -A_{B(X,X)}\nabla_X X - B(X, A_{B(X,X)}X) - 2A_{(\widetilde{\nabla}_X B)(X,X)}X + (\widetilde{\nabla}_X^2 B)(X, X) + (\widetilde{\nabla}_{\nabla_X X} B)(X, X).\end{aligned} \tag{18}$$

Substituting (14) in (18), we arrive at

$$\begin{aligned}\bar{\nabla}^3_X X &= 2k_1'\nabla_X Y + k_1\nabla^2_X Y + K\nabla_X X + 4B(X,\nabla^2_X X) - 5A_{B(X,\nabla_X X)}X\\ &+5(\widetilde{\nabla}_X B)(X,\nabla_X X) + 3B(\nabla_X X,\nabla_X X) - (\nabla_X A)_{B(X,X)}X\\ &-A_{B(X,X)}\nabla_X X - B(X,A_{B(X,X)}X) - 2A_{(\widetilde{\nabla}_X B)(X,X)}X\\ &+(\widetilde{\nabla}^2_X B)(X,X) + (\widetilde{\nabla}_{\nabla_X X}B)(X,X). \end{aligned} \tag{19}$$

Since the curve c is a slant helix in $\bar{M}$, it follows that

$$\bar{\nabla}^3_X X = 2\bar{k_1}'\ \bar{\nabla}_X Y + \bar{k_1}\ \bar{\nabla}^2_X Y + \bar{K}\ \bar{\nabla}_X X.$$

Using (2) and (3), we get

$$\begin{aligned}\bar{\nabla}^3_X X &= 2\bar{k_1}'\nabla_X Y + 2\bar{k_1}'B(X,Y) + \bar{k_1}\nabla^2_X Y + 2\bar{k_1}B(X,\nabla_X Y) - \bar{k_1}A_{B(X,Y)}X\\ &+\bar{k_1}(\widetilde{\nabla}_X B)(X,Y) + \bar{k_1}B(\nabla_X X,Y) + \bar{K}\nabla_X X + \bar{K}B(X,X).\end{aligned}$$

Substituting the last equation in (19), we obtain

$$\begin{aligned}0 &= 2k_1'\nabla_X Y + k_1\nabla^2_X Y + K\nabla_X X + 4B(X,\nabla^2_X X) - 5A_{B(X,\nabla_X X)}X\\ &+5(\widetilde{\nabla}_X B)(X,\nabla_X X) + 3B(\nabla_X X,\nabla_X X) - (\nabla_X A)_{B(X,X)}X - A_{B(X,X)}\nabla_X X\\ &-B(X,A_{B(X,X)}X) - 2A_{(\widetilde{\nabla}_X B)(X,X)}X + (\widetilde{\nabla}^2_X B)(X,X) + (\widetilde{\nabla}_{\nabla_X X}B)(X,X)\\ &-2\bar{k_1}'\nabla_X Y - 2\bar{k_1}'B(X,Y) - \bar{k_1}\nabla^2_X Y - 2\bar{k_1}B(X,\nabla_X Y) + \bar{k_1}A_{B(X,Y)}X\\ &-\bar{k_1}(\widetilde{\nabla}_X B)(X,Y) - \bar{k_1}B(\nabla_X X,Y) - \bar{K}\nabla_X X - \bar{K}B(X,X). \end{aligned} \tag{20}$$

Using (10) and taking tangential part of (20), we have

$$\begin{aligned}0 &= (-k_1(k_1^2+k_2^2) + Kk_1 - \bar{K}k_1 + \bar{k_1}(k_1^2+k_2^2))Y + (-3k_1'k_1 + 2\bar{k_1}'k_1 + \bar{k_1}k_1')X\\ &+(2k_1'k_2 + k_1k_2' - \bar{k_1}k_2' - 2\bar{k_1}'k_2)Z - 5k_1A_{B(X,Y)}X - (\nabla_X A)_{B(X,X)}X\\ &-k_1A_{B(X,X)}Y - 2A_{(\widetilde{\nabla}_X B)(X,X)}X + \bar{k_1}A_{B(X,Y)}X. \end{aligned} \tag{21}$$

Changing Y into -Y in (21) and subtracting each other, it follows that

$$(-k_1(k_1^2+k_2^2) + Kk_1 - \bar{K}k_1 + \bar{k_1}(k_1^2+k_2^2))Y \quad = \quad 5k_1A_{B(X,Y)}X + k_1A_{B(X,X)}Y - \bar{k_1}A_{B(X,Y)}X.$$

Taking inner product with the unit vector field X, we have

$$5k_1\langle A_{B(X,Y)}X, X\rangle + k_1\langle A_{B(X,X)}Y, X\rangle - \bar{k_1}\langle A_{B(X,Y)}X, X\rangle = 0.$$

Using (4) in the above equation, we obtain $(6k_1 - \bar{k_1})\langle B(X,X), B(X,Y)\rangle = 0$. For $6k_1 \neq \bar{k_1}$, we get

$$\langle B(X,X), B(X,Y)\rangle = 0. \tag{22}$$

Then [15, Lemma 1] implies that submanifold M is isotropic space. □

In the sequel, we are going to obtain a characterization of Riemannian submanifolds by imposing a geometric condition in terms of slant helices.

Theorem 3.4. *Let M (dim M ≥3) be a connected submanifold of a Riemannian manifold $\bar{M}$ and c be Frenet curve which is not a general helix. If, for $6k_1 \neq \bar{k_1}$ and $k_1'' \neq 0$, a slant helix with curvatures $k_1 > 0$ and $k_2 \neq 0$ in M is a slant helix with curvatures $\bar{k_1} > 0$ and $\bar{k_2} \neq 0$ in $\bar{M}$, then M is a totally geodesic submanifold in $\bar{M}$.*

Proof. We suppose that $c = c(s)$ is a slant helix curve with curvatures k_1 and $k_2 \neq 0$. Then, we have (20). If we take the normal part of (20), we obtain

$$\begin{aligned} 0 &= 4B(X, \nabla^2_X X) + 5(\widetilde{\nabla}_X B)(X, \nabla_X X) + 3B(\nabla_X X, \nabla_X X) - B(X, A_{B(X,X)}X) \\ &\quad + (\widetilde{\nabla}^2_X B)(X, X) + (\widetilde{\nabla}_{\nabla_X X} B)(X, X) - 2\bar{k_1}' B(X, Y) - 2\bar{k_1} B(X, \nabla_X Y) \\ &\quad - \bar{k_1}(\widetilde{\nabla}_X B)(X, Y) - \bar{k_1} B(\nabla_X X, Y) - \bar{K} B(X, X) \end{aligned}$$

and

$$\begin{aligned} 0 &= k_1^2 B(X, X) - k_1' B(X, Y) - k_1 k_2 B(X, Z) + 5k_1 \nabla^\perp_X B(X, Y) - 2k_1^2 B(Y, Y) \\ &\quad - B(X, A_{B(X,X)}X) + (\widetilde{\nabla}^2_X B)(X, X) + k_1 \nabla^\perp_Y B(X, X) - 2k_1 B(\nabla_Y X, X) \\ &\quad - 2\bar{k_1}' B(X, Y) + 2\bar{k_1} k_1 B(X, X) - 2\bar{k_1} k_2 B(X, Z) - \bar{k_1}(\widetilde{\nabla}_X B)(X, Y) \\ &\quad - \bar{k_1} k_1 B(Y, Y) - \bar{K} B(X, X). \end{aligned} \tag{23}$$

by virtue of (10). Changing Z into $-Z$ in (23) and subtracting each other, we have

$$k_2 B(X, Z)(k_1 + 2\bar{k_1}) = 0.$$

For, $k_2 \neq 0$ and $k_1, \bar{k_1} > 0$, then $B(X, Z) = 0$. Since $B(X, Z) = 0$ for orthonormal vector fields X and Z, then [14, Lemma] implies that M is umbilical in $\bar{M}$. Since M is umbilical, the equation (23) converts to

$$\begin{aligned} 0 &= k_1^2 B(X, X) - 2k_1^2 B(Y, Y) - B(X, A_{B(X,X)}X) + (\widetilde{\nabla}^2_X B)(X, X) + k_1 \nabla^\perp_Y B(X, X) + 2\bar{k_1} k_1 B(X, X) \\ &\quad - \bar{k_1}(\widetilde{\nabla}_X B)(X, Y) - \bar{k_1} k_1 B(Y, Y) - \bar{K} B(X, X). \end{aligned} \tag{24}$$

Since M is umbilical, we have

$$(\widetilde{\nabla}_X B)(X, Y) = \nabla^\perp_X B(X, Y) - B(\nabla_X X, Y) - B(X, \nabla_X Y) = 0.$$

Then, the equation (24) converts to

$$\begin{aligned} 0 &= k_1^2 B(X, X) - 2k_1^2 B(Y, Y) - B(X, A_{B(X,X)}X) + (\widetilde{\nabla}^2_X B)(X, X) \\ &\quad + k_1 \nabla^\perp_Y B(X, X) + 2\bar{k_1} k_1 B(X, X) - \bar{k_1} k_1 B(Y, Y) - \bar{K} B(X, X). \end{aligned} \tag{25}$$

Changing Y into $-Y$ in (25) and subtracting each other, we arrive at

$$\nabla^\perp_Y B(X, X) = \nabla^\perp_Y H = 0 \tag{26}$$

which means that the mean curvature vector field is parallel. Then it follows that

$$\begin{aligned} (\widetilde{\nabla}_X B)(X, X) &= \nabla^\perp_X B(X, X) - B(\nabla_X X, X) - B(X, \nabla_X X) \\ &= -k_1 B(Y, X) - k_1 B(X, Y) = 0. \end{aligned} \tag{27}$$

From (6), we get

$$\begin{aligned} \widetilde{\nabla}_X(\widetilde{\nabla}_X B)(X, X) &= \nabla^\perp_X((\widetilde{\nabla}_X B)(X, X)) - (\widetilde{\nabla}_{\nabla_X X} B)(X, X) - (\widetilde{\nabla}_X B)(\nabla_X X, X) - (\widetilde{\nabla}_X B)(X, \nabla_X X) \\ &= -\nabla^\perp_{\nabla_X X} B(X, X) + B(\nabla_{\nabla_X X} X, X) + B(X, \nabla_{\nabla_X X} X) \\ &\quad - \nabla^\perp_X B(\nabla_X X, X) + B(\nabla^2_X X, X) + B(\nabla_X X, \nabla_X X) \\ &\quad - \nabla^\perp_X B(X, \nabla_X X) + B(X, \nabla^2_X X) + B(\nabla_X X, \nabla_X X) \\ &= -\nabla^\perp_{k_1 Y} B(X, X) + B(\nabla_{k_1 Y} X, X) + B(X, \nabla_{k_1 Y} X) \\ &\quad - \nabla^\perp_X B(k_1 Y, X) + B(-k_1^2 X + k_1' Y + k_1 k_2 Z, X) + B(k_1 Y, k_1 Y) \\ &\quad - \nabla^\perp_X B(X, k_1 Y) + B(X, -k_1^2 X + k_1' Y + k_1 k_2 Z) + B(k_1 Y, k_1 Y) \\ (\widetilde{\nabla}^2_X B)(X, X) &= -k_1 \nabla^\perp_Y B(X, X) + k_1 B(\nabla_Y X, X) + k_1 B(X, \nabla_Y X) \\ &\quad - 2k_1 \nabla^\perp_X B(Y, X) - 2k_1^2 B(X, X) + 2k_1' B(X, Y) + 2k_1 k_2 B(X, Z) + 2k_1^2 B(Y, Y). \end{aligned}$$

Since M is umbilical in $\bar{M}$, we have $B(\nabla_Y X, X) = 0$. Also using (26), we get

$$(\widetilde{\nabla}^2_X B)(X, X) = -2k_1^2 B(X, X) + 2k_1^2 B(Y, Y) = 0. \tag{28}$$

Considering (26), (27) and (28), normal part becomes

$$0 = k_1^2 B(X, X) - 2k_1^2 B(Y, Y) - B(X, A_{B(X,X)}X) + 2\bar{k}_1 k_1 B(X, X) - \bar{k}_1 k_1 B(Y, Y) - \bar{K} B(X, X).$$

From (4) and (8), we conclude that

$$(-k_1^2 - \|H\|^2 + \bar{k}_1 k_1 - \bar{K})H = 0. \tag{29}$$

On the other hand, by direct computations and using (10), we have

$$-(\nabla_X A)_{B(X,X)} X = -\nabla_X (A_{B(X,X)}) X + k_1 A_{B(X,X)} Y.$$

Using this expansion in (21), we obtain

$$\begin{aligned} 0 = {} & (-k_1(k_1^2 + k_2^2) + K k_1 - \bar{K} k_1 + \bar{k}_1(k_1^2 + k_2^2))Y + (-3k_1' k_1 + 2\bar{k}_1' k_1 + \bar{k}_1 k_1')X \\ & + (2k_1' k_2 + k_1 k_2' - \bar{k}_1 k_2' - 2\bar{k}_1' k_2)Z - 5k_1 A_{B(X,Y)} X - \nabla_X (A_{B(X,X)}) X \\ & - 2A_{(\widetilde{\nabla}_X B)(X,X)} X + \bar{k}_1 A_{B(X,Y)} X. \end{aligned}$$

The umbilical M, parallel H and (27) imply that

$$\begin{aligned} 0 = {} & (-k_1(k_1^2 + k_2^2) + K k_1 - \bar{K} k_1 + \bar{k}_1(k_1^2 + k_2^2))Y + (-3k_1' k_1 + 2\bar{k}_1' k_1 + \bar{k}_1 k_1')X \\ & + (2k_1' k_2 + k_1 k_2' - \bar{k}_1 k_2' - 2\bar{k}_1' k_2)Z - \nabla_X (A_{B(X,X)}) X. \end{aligned} \tag{30}$$

Taking inner product both sides of (30) with Y, we obtain

$$(-k_1(k_1^2 + k_2^2) + K k_1 - \bar{K} k_1 + \bar{k}_1(k_1^2 + k_2^2)) - \langle \nabla_X (A_{B(X,X)} X), Y \rangle = 0. \tag{31}$$

Using (4), we can write

$$\langle A_{B(X,X)} X, Y \rangle = \langle B(X, X), B(X, Y) \rangle = 0.$$

By differentiating the last equation, we have

$$\langle \nabla_X (A_{B(X,X)} X), Y \rangle + \langle B(A_{B(X,X)} X, X), Y \rangle + \langle A_{B(X,X)} X, -k_1 X + k_2 Z \rangle = 0.$$

For $X, Y \in \chi(M)$, $B(A_{B(X,X)} X, X) \in \chi(M^\perp)$ and $\langle B(A_{B(X,X)} X, X), Y \rangle = 0$ gives

$$\langle \nabla_X (A_{B(X,X)} X), Y \rangle = k_1 \|H\|^2.$$

Then, (31) converts to

$$-k_1(k_1^2 + k_2^2) + K k_1 - \bar{K} k_1 + \bar{k}_1(k_1^2 + k_2^2) = k_1 \|H\|^2. \tag{32}$$

Taking inner product both sides of (30) with the vector X and arranging outcome, we have

$$\bar{k}_1' = \frac{3k_1'}{2} - \frac{\bar{k}_1 k_1'}{2k_1}. \tag{33}$$

Similarly taking inner product both sides of (30) with the vector Z, we arrive at

$$2k_1' k_2 + k_1 k_2' - \bar{k}_1 k_2' - 2\bar{k}_1' k_2 = 0.$$

Using (33) in the above equation, we obtain

$$\begin{aligned} & 2k_1' k_2 + k_1 k_2' - \bar{k}_1 k_2' - 2\left(\frac{3k_1'}{2} - \frac{\bar{k}_1 k_1'}{2k_1}\right) k_2 = 0 \\ & k_1(k_1 k_2' - k_1' k_2) = \bar{k}_1(k_1 k_2' - k_1' k_2). \end{aligned}$$

Since Frenet curve c is not general helix, $k_1k_2' \neq k_1'k_2$, it is seen that

$$k_1 = \bar{k_1}. \tag{34}$$

Substituting (34) in (32), we obtain $K - \bar{K} = \|H\|^2$. Substituting the last equation in (29), we get $KH = 0$. Since frenet curve c is slant helix, we conclude $\frac{k_1''}{k_1} = K$. If $k_1'' \neq 0$, we obtain $K \neq 0$. Then $H = 0$. Thus M is a totally geodesic submanifold in $\bar{M}$. □

4. Circles and Slant helices in Riemannian Manifolds

In this section, it is shown that there is no isometric immersion that carries a circle in Riemannian manifold M to the slant helix in another Riemannian manifold $\bar{M}$. But when a slant helix in Riemannian manifold M is a circle in another Riemannian manifold $\bar{M}$ along isometric immersion, the submanifold M is a totally geodesic submanifold.

Theorem 4.1. *Let M (dim M≥3) be a connected submanifold of a Riemannian manifold $\bar{M}$ and c be Frenet curve. Provided that $6k_1 \neq \bar{k_1}$ and $\bar{k_1}$ is non-constant, there is no immersion that carries a circle with curvature $k_1 > 0$ in M to the slant helix with curvatures $\bar{k_1} > 0$ and $\bar{k_2} \neq 0$ in $\bar{M}$.*

Proof. We assume that a circle with curvature $k_1 \neq 0$ in M is a slant helix in $\bar{M}$. From the equation (1), we have the following equation

$$\nabla_X^3 X = -k_1^3 Y. \tag{35}$$

Since the curve c is a slant helix in $\bar{M}$, it follows that

$$\bar{\nabla}_X^3 X = 2\bar{k_1}'\ \bar{\nabla}_X Y + \bar{k_1}\ \bar{\nabla}_X^2 Y + \bar{K}\ \bar{\nabla}_X X$$

where $\bar{K} = \frac{\bar{k_2}''}{\bar{k_2}} - \frac{3}{2}\frac{\bar{k_1}}{\bar{k_2}}\left(\frac{\bar{k_2}}{\bar{k_1}}\right)'\left(\ln(\bar{k_1}^2 + \bar{k_2}^2)\right)'$. From (2) and (3), third derivative is

$$\begin{aligned}\bar{\nabla}_X^3 X &= \nabla_X^3 X + 4B(X, \nabla_X^2 X) - 5A_{B(X,\nabla_X X)}X + 5(\widetilde{\nabla}_X B)(X, \nabla_X X)\\ &\quad +3B(\nabla_X X, \nabla_X X) - (\nabla_X A)_{B(X,X)}X - A_{B(X,X)}\nabla_X X - B(X, A_{B(X,X)}X)\\ &\quad -2A_{(\widetilde{\nabla}_X B)(X,X)}X + (\widetilde{\nabla}_X^2 B)(X,X) + (\widetilde{\nabla}_{\nabla_X X}B)(X,X).\end{aligned}$$

Substituting (35) in the above equation, we obtain

$$\begin{aligned}\bar{\nabla}_X^3 X &= -k_1^2\nabla_X X + 4B(X, \nabla_X^2 X) - 5A_{B(X,\nabla_X X)}X + 5(\widetilde{\nabla}_X B)(X, \nabla_X X) + 3B(\nabla_X X, \nabla_X X) - (\nabla_X A)_{B(X,X)}X\\ &\quad -A_{B(X,X)}\nabla_X X - B(X, A_{B(X,X)}X) - 2A_{(\widetilde{\nabla}_X B)(X,X)}X + (\widetilde{\nabla}_X^2 B)(X,X) + (\widetilde{\nabla}_{\nabla_X X}B)(X,X).\end{aligned} \tag{36}$$

Since the curve c is a slant helix in $\bar{M}$, it follows that

$$\begin{aligned}\bar{\nabla}_X^3 X &= 2\bar{k_1}'\nabla_X Y + 2\bar{k_1}'B(X,Y) + \bar{k_1}\nabla^2{}_X Y + 2\bar{k_1}B(X, \nabla_X Y) - \bar{k_1}A_{B(X,Y)}X\\ &\quad +\bar{k_1}(\widetilde{\nabla}_X B)(X,Y) + \bar{k_1}B(\nabla_X X, Y) + \bar{K}\nabla_X X + \bar{K}B(X,X)\end{aligned}$$

Substituting the last equation in (36), we obtain

$$\begin{aligned}0 &= -k_1^2\nabla_X X + 4B(X, \nabla_X^2 X) - 5A_{B(X,\nabla_X X)}X + 5(\widetilde{\nabla}_X B)(X, \nabla_X X)\\ &\quad +3B(\nabla_X X, \nabla_X X) - (\nabla_X A)_{B(X,X)}X - A_{B(X,X)}\nabla_X X - B(X, A_{B(X,X)}X)\\ &\quad -2A_{(\widetilde{\nabla}_X B)(X,X)}X + (\widetilde{\nabla}_X^2 B)(X,X) + (\widetilde{\nabla}_{\nabla_X X}B)(X,X)\\ &\quad -2\bar{k_1}'\nabla_X Y - 2\bar{k_1}'B(X,Y) - \bar{k_1}\nabla^2{}_X Y - 2\bar{k_1}B(X, \nabla_X Y) + \bar{k_1}A_{B(X,Y)}X\\ &\quad -\bar{k_1}(\widetilde{\nabla}_X B)(X,Y) - \bar{k_1}B(\nabla_X X, Y) - \bar{K}\nabla_X X - \bar{K}B(X,X).\end{aligned} \tag{37}$$

Using (10) and taking the tangent part of (37), we have

$$(-k_1^3-\bar{K}k_1+\bar{k_1}k_1^2)Y+2\bar{k_1}'k_1X-5k_1A_{B(X,Y)}X-(\nabla_XA)_{B(X,X)}X-k_1A_{B(X,X)}Y-2A_{(\widetilde{\nabla}_XB)(X,X)}X+\bar{k_1}A_{B(X,Y)}X=0. \quad (38)$$

Changing Y into -Y in (38) and subtracting each other, it follows that

$$(-k_1^3-\bar{K}k_1+\bar{k_1}k_1^2)Y = 5k_1A_{B(X,Y)}X+k_1A_{B(X,X)}Y-\bar{k_1}A_{B(X,Y)}X.$$

Taking the inner product with the unit vector field X, we have

$$5k_1\langle A_{B(X,Y)}X,X\rangle+k_1\langle A_{B(X,X)}Y,X\rangle-\bar{k_1}\langle A_{B(X,Y)}X,X\rangle=0.$$

Using (4) in the above equation, we obtain $(6k_1-\bar{k_1})\langle B(X,X),B(X,Y)\rangle=0$. For $6k_1\neq\bar{k_1}$, we get

$$\langle B(X,X),B(X,Y)\rangle=0. \quad (39)$$

Then submanifold M is isotropic space. Taking the normal part of (37), we obtain

$$\begin{aligned} 0 = {} & k_1^2B(X,X)+5k_1\nabla^{\perp}{}_XB(X,Y)-2k_1^2B(Y,Y)-B(X,A_{B(X,X)}X)\\ & +(\widetilde{\nabla}_X^2B)(X,X)+k_1\nabla_Y^{\perp}B(X,X)-2k_1B(\nabla_YX,X)\\ & -2\bar{k_1}'B(X,Y)+2\bar{k_1}k_1B(X,X)-\bar{k_1}(\widetilde{\nabla}_XB)(X,Y)-\bar{k_1}k_1B(Y,Y)-\bar{K}B(X,X) \end{aligned} \quad (40)$$

by virtue of (9). Considering

$$(\widetilde{\nabla}_X^2B)(X,X) = \nabla_X^{\perp}((\widetilde{\nabla}_XB)(X,X))-(\widetilde{\nabla}_{\nabla_XX}B)(X,X)-(\widetilde{\nabla}_XB)(\nabla_XX,X)-(\widetilde{\nabla}_XB)(X,\nabla_XX),$$

(40) convert to

$$\begin{aligned} 0 = {} & k_1^2B(X,X)+5k_1\nabla^{\perp}{}_XB(X,Y)-2k_1^2B(Y,Y)-B(X,A_{B(X,X)}X)\\ & +\nabla_X^{\perp}((\widetilde{\nabla}_XB)(X,X))-(\widetilde{\nabla}_{\nabla_XX}B)(X,X)-2(\widetilde{\nabla}_XB)(\nabla_XX,X)\\ & +k_1\nabla_Y^{\perp}B(X,X)-2k_1B(\nabla_YX,X)-2\bar{k_1}'B(X,Y)+2\bar{k_1}k_1B(X,X)\\ & -\bar{k_1}(\widetilde{\nabla}_XB)(X,Y)-\bar{k_1}k_1B(Y,Y)-\bar{K}B(X,X). \end{aligned} \quad (41)$$

Changing X into -X in (41) and subtracting each other, we have

$$-4\bar{k_1}'B(X,Y)=0\Rightarrow\bar{k_1}'B(X,Y)=0.$$

Provided that $\bar{k_1}$ is non-constant, $B(X,Y)=0$. Taking into account (39) and $B(X,Y)=0$, then M is umbilical in $\bar{M}$. Since M is umbilical, the equation (40) converts to

$$\begin{aligned} 0 = {} & k_1^2B(X,X)-2k_1^2B(Y,Y)-B(X,A_{B(X,X)}X)+(\widetilde{\nabla}_X^2B)(X,X)\\ & +k_1\nabla_Y^{\perp}B(X,X)+2\bar{k_1}k_1B(X,X)-\bar{k_1}k_1B(Y,Y)-\bar{K}B(X,X). \end{aligned} \quad (42)$$

Changing Y into $-Y$ in (42) and subtracting each other, we arrive at

$$\nabla_Y^{\perp}B(X,X)=\nabla_Y^{\perp}H=0$$

which means that the mean curvature vector field is parallel. Then it follows that $(\widetilde{\nabla}_XB)(X,X)=0$ and $(\widetilde{\nabla}_X^2B)(X,X)=0$. Then, the normal part becomes

$$k_1^2B(X,X)-2k_1^2B(Y,Y)-B(X,A_{B(X,X)}X)+2\bar{k_1}k_1B(X,X)-\bar{k_1}k_1B(Y,Y)-\bar{K}B(X,X)=0.$$

From (4) and (8), we conclude that

$$(-k_1^2-\|H\|^2+\bar{k_1}k_1-\bar{K})H=0.$$

The umbilical M, parallel H and (7) imply that, (38) arrive at

$$(-k_1^3 - \bar{K}k_1 + \bar{k_1}k_1^2)Y + 2\bar{k_1}'k_1X - \nabla_X(A_{B(X,X)})X = 0. \tag{43}$$

Taking inner product both sides of (43) with X, we obtain

$$2\bar{k_1}'k_1 - \langle\nabla_X(A_{B(X,X)}X), X\rangle = 0. \tag{44}$$

Using (4), we can write

$$\langle A_{B(X,X)}X, X\rangle = \langle B(X,X), B(X,X)\rangle$$

and by differentiating the last equation, we have $\langle\nabla_X(A_{B(X,X)}X), X\rangle = 0$. Then, (44) converts to $2\bar{k_1}'k_1 = 0$. In the last equation, it is either $k_1 = 0$ or $\bar{k_1}' = 0$. If $k_1 = 0$, the curve in the manifold cannot be a circle. If $\bar{k_1}' = 0$, M cannot be umbilical. Thus, there is a contradiction and the proof is completed. □

Theorem 4.2. *Let M ($\dim M \geq 3$) be a connected submanifold of a Riemannian manifold $\bar{M}$ and c be Frenet curve. If a slant helix for $k_1 \neq 0$ and $k_2 \neq 0$ in M is a circle with curvatures $\bar{k_1} > 0$ in $\bar{M}$, then k_1 and k_2 are constants. As a result, the submanifold M is a totally geodesic in $\bar{M}$.*

Proof. We assume that a slant helix in M is a circle in $\bar{M}$. From the equation (11), we have the following equation

$$\nabla_X^3 X \;=\; 2k_1'\nabla_X Y + k_1\nabla_X^2 Y + K\nabla_X X \tag{45}$$

where $K = \frac{k_2''}{k_2} - \frac{3}{2}\frac{k_1}{k_2}\left(\frac{k_2}{k_1}\right)'\left(\ln(k_1^2 + k_2^2)\right)'$. Since the curve c is a circle in $\bar{M}$, it follows that

$$\bar{\nabla}_X^3 X = -\bar{k_1}^2\bar{\nabla}_X X. \tag{46}$$

Using (2), (3), (45) and (46), we obtain

$$\begin{aligned} 0 \;=\;& 2k_1'\nabla_X Y + k_1\nabla_X^2 Y + K\nabla_X X + 4B(X, \nabla_X^2 X) - 5A_{B(X,\nabla_X X)}X \\ &+5(\widetilde{\nabla}_X B)(X, \nabla_X X) + 3B(\nabla_X X, \nabla_X X) - (\nabla_X A)_{B(X,X)}X \\ &-A_{B(X,X)}\nabla_X X - B(X, A_{B(X,X)}X) - 2A_{(\widetilde{\nabla}_X B)(X,X)}X + (\widetilde{\nabla}_X^2 B)(X,X) \\ &+(\widetilde{\nabla}_{\nabla_X X}B)(X,X) + \bar{k_1}^2\nabla_X X + \bar{k_1}^2 B(X,X). \end{aligned} \tag{47}$$

Using (10) and taking the tangent part of (47), we have

$$\begin{aligned} 0 \;=\;& -3k_1'k_1X + (-k_1(k_1^2 + k_2^2) + Kk_1 + \bar{k_1}^2k_1)Y + (2k_1'k_2 + k_1k_2')Z - 5k_1A_{B(X,Y)}X \\ &-(\nabla_X A)_{B(X,X)}X - k_1A_{B(X,X)}Y - 2A_{(\widetilde{\nabla}_X B)(X,X)}X. \end{aligned} \tag{48}$$

Changing Y into $-Y$ in the last equation and subtracting each other, it follows that

$$(-k_1(k_1^2 + k_2^2) + Kk_1 + \bar{k_1}^2k_1)Y = 5k_1A_{B(X,Y)}X + k_1A_{B(X,X)}Y.$$

Taking the inner product with the unit vector field X, we have

$$5k_1\langle A_{B(X,Y)}X, X\rangle + k_1\langle A_{B(X,X)}Y, X\rangle = 0.$$

Using (4) in the above equation, we obtain $6k_1\langle B(X,X), B(X,Y)\rangle = 0$. For $k_1 \neq 0$, we get

$$\langle B(X,X), B(X,Y)\rangle = 0. \tag{49}$$

Then submanifold M is isotropic space. Taking the normal part of (47), we obtain

$$\begin{aligned} 0 \;=\;& k_1^2B(X,X) - k_1'B(X,Y) - k_1k_2B(X,Z) + 5k_1\nabla_X^{\perp}B(X,Y) - 2k_1^2B(Y,Y) \\ &-B(X, A_{B(X,X)}X) + (\widetilde{\nabla}_X^2 B)(X,X) + k_1\nabla_Y^{\perp}B(X,X) - 2k_1B(\nabla_Y X, X) + \bar{k_1}^2B(X,X). \end{aligned} \tag{50}$$

Changing Z into $-Z$ in (50) and subtracting each other, we have $-2k_1k_2B(X,Z) = 0$. For, $k_1 \neq 0$ and $k_2 \neq 0$, then $B(X,Z) = 0$. Taking into account (49) and $B(X,Z) = 0$, then M is umbilical in $\bar{M}$. Since M is umbilical, the equation (50) converts to

$$k_1^2B(X,X) - 2k_1^2B(Y,Y) - B(X, A_{B(X,X)}X) + (\widetilde{\nabla}_X^2B)(X,X) + k_1\nabla_Y^{\perp}B(X,X) + \bar{k_1}^2B(X,X) = 0. \quad (51)$$

Changing Y into $-Y$ in (51) andsubtracting each other, we arrive at

$$\nabla_Y^{\perp}B(X,X) = \nabla_Y^{\perp}H = 0$$

which means that the mean curvature vector field is parallel. Then it follows that $(\widetilde{\nabla}_XB)(X,X) = 0$ and $(\widetilde{\nabla}_X^2B)(X,X) = 0$. Then, the normal part becomes

$$k_1^2B(X,X) - 2k_1^2B(Y,Y) - B(X, A_{B(X,X)}X) + \bar{k_1}^2B(X,X) = 0.$$

From (4) and (8), we conclude that

$$(-k_1^2 - \|H\|^2 + \bar{k_1}^2)H = 0.$$

The umbilical M, parallel H and (7) imply that

$$-3k_1'k_1X + (-k_1(k_1^2 + k_2^2) + Kk_1 + \bar{k_1}^2k_1)Y + (2k_1'k_2 + k_1k_2')Z - \nabla_X(A_{B(X,X)})X = 0. \quad (52)$$

Taking inner product both sides of (52) with X, we obtain

$$-3k_1'k_1 - \langle\nabla_X(A_{B(X,X)})X, X\rangle = 0. \quad (53)$$

Using (4), we can write $\langle A_{B(X,X)}X, X\rangle = \langle B(X,X), B(X,X)\rangle$ and by differentiating the last equation, we have $\langle\nabla_X(A_{B(X,X)}X), X\rangle = 0$. Then, (53) converts to $-3k_1'k_1 = 0$. Thus, k_1 is constant. Taking inner product both sides of (52) with the vector Y and arranging outcome, we have

$$-k_1^2 - k_2^2 + K + \bar{k_1}^2 = \|H\|^2. \quad (54)$$

Similarly taking inner product both sides of (52) with the vector Z, we arrive at

$$2k_1'k_2 + k_1k_2' = 0.$$

Since k_1 is constant, k_2 is constant. Thus, K is zero. Using (54) in (52), we obtain $k_2^2H = 0$. Since $k_2 \neq 0$, we have $H = 0$. Thus M is a totally geodesic submanifold in $\bar{M}$. □

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