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Analyzing geometric isometries of helical surfaces in five-dimensional Euclidean space

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Abstract. In the context of five-dimensional Euclidean space $\mathbb{E}^5$, the definition of the helical surface is established. Its geometric attributes are elucidated by calculating three normals. Subsequently, Bour's theorem within $\mathbb{E}^5$ is employed to determine an isometric mapping among helical-rotational surfaces, contributing to a better understanding of their structural interplay.

1. Introduction

In Euclidean 3-space $\mathbb{E}^3$, the deformation of the family of surfaces is described by the parametric equation

$$\mathfrak{x}_\beta (s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix} = \begin{pmatrix} \cos\beta \sin s \sinh t + \sin\beta \cos s \cosh t \\ -\cos\beta \cos s \sinh t + \sin\beta \sin s \cosh t \\ s\cos\beta + t\sin\beta \end{pmatrix},$$

where $s, \beta \in [-\pi, \pi]$, $t \in (-\infty, \infty)$, β is a deformation parameter. $\mathfrak{x}_\beta$ denotes the minimal surfaces, which have the same first fundamental form and normal vector field. Specifically, $\mathfrak{x}_0$-$\mathfrak{x}_{\pi/2}$ describe the helicoid-catenoid, respectively, which are locally isometric surfaces with identical Gauss maps.

This leads us to Bour's theorem in [2]:

Theorem 1.1. ***Bour's Theorem*** *A helical surface is isomorphic to a rotational surface, implying that the helices on the helical surface correspond to circles on the rotational surface under an isometric mapping.*

Bour's theorem establishes a profound connection between helical and rotational surfaces, highlighting their equivalence in terms of intrinsic geometric properties despite their distinct appearances. It was formulated by Bour, emphasizing the significant relationship between helical and rotational surfaces, elucidating how their respective curvature and metric properties align under suitable transformations. It serves as a valuable tool for analyzing and comprehending the geometric characteristics of these surfaces, providing insights into their structural similarities and disparities. Additionally, Bour's theorem carries

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considerable implications in various disciplines, including differential geometry, where it facilitates the examination of surface mappings and isometries.

The readers can explore different versions of this theorem, including the Euclidean and Lorentz-Minkowski variants, in [1]-[16], where they can delve into the specificities and applications of each formulation.

The aim of this paper is to thoroughly investigate and analyze helical surfaces within five-dimensional Euclidean space, with a focus on their construction, properties, and geometric relationships. The paper is structured as follows:

In Section 1, definitions relevant to $\mathbb{E}^5$ are provided.

In Section 2, the concept of helical surfaces and their representation within $\mathbb{E}^5$ is introduced. The parametric equations governing these surfaces are defined, and their fundamental characteristics are explored.

In Section 3, the notion of local isometry on helical surfaces is delved into. The preservation of local geometric properties, such as distances and angles, by certain transformations is examined, and their implications for the study of helical surfaces are analyzed.

In Section 4, the analysis is extended to consider global isometry properties of helical surfaces. Building upon the concepts discussed in the previous section, how these surfaces maintain their intrinsic geometry under broader transformations is explored, and their role within the broader context of differential geometry is investigated.

Finally, in the concluding section, the findings are summarized, and the key insights gained from the exploration of helical surfaces in $\mathbb{E}^5$ are highlighted. Potential avenues for future research are discussed, and the significance of helical surfaces in understanding the geometric properties of higher-dimensional spaces is emphasized.

Through this structured approach, our paper aims to contribute to the deeper understanding of helical surfaces and their geometric properties within five-dimensional Euclidean space, providing valuable insights for researchers and practitioners in the fields of mathematics.

2. Preliminaries

Moving forward, our focus shifts to the geometry of Euclidean 5-space, denoted as $\mathbb{E}^5$. Here, we employ the inner product notation $\left\langle \overrightarrow{x}, \overrightarrow{y} \right\rangle := \sum_{i=1}^{5} x_i y_i$ to represent the inner product operation within $\mathbb{E}^5$. This inner product, akin to the dot product in lower dimensions, plays a fundamental role in defining the geometric properties and relationships within the five-dimensional Euclidean space.

Definition 2.1. *Let $\mathfrak{x} : D \subset \mathbb{E}^2 \to \mathbb{E}^5$, be a parametric surface of $\mathbb{E}^5$. At point $\mathbf{p} = \mathfrak{x}(s,t)$, tangent space to M is constructed by $\{\mathfrak{x}_s, \mathfrak{x}_t\}$. The fundamental form $\left(g_{ij}\right)$ of M are described by*

$$\left(g_{ij}\right) := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle \mathfrak{x}_s, \mathfrak{x}_s \rangle & \langle \mathfrak{x}_s, \mathfrak{x}_t \rangle \\ \langle \mathfrak{x}_t, \mathfrak{x}_s \rangle & \langle \mathfrak{x}_t, \mathfrak{x}_t \rangle \end{pmatrix}. \tag{1}$$

Here, $g_{12} = g_{21}$.

We assume $\mathfrak{Q}^2 := \det(g_{ij}) = g_{11}g_{22} - (g_{12})^2 > 0$. That is, M is a regular surface.

Definition 2.2. *Let $\{e_1, e_2, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3\}$ be a orthonormal frame field of M; e_1, e_2 are tangents to M, and also $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3$ are normals to M. The fundamental form $\left(h_{ij}^k\right)$ of M depend on $\mathfrak{N}_k$, $k = 1, 2, 3$, are determined by*

$$\left(h_{ij}^k\right) := \begin{pmatrix} h_{11}^k & h_{12}^k \\ h_{21}^k & h_{22}^k \end{pmatrix} = \begin{pmatrix} \langle \mathfrak{x}_{ss}, \mathfrak{N}_k \rangle & \langle \mathfrak{x}_{st}, \mathfrak{N}_k \rangle \\ \langle \mathfrak{x}_{ts}, \mathfrak{N}_k \rangle & \langle \mathfrak{x}_{tt}, \mathfrak{N}_k \rangle \end{pmatrix}. \tag{2}$$

Here, $i, j = 1, 2$, $h_{12} = h_{21}$. Also $\det(h_{ij}^k) = h_{11}^k h_{22}^k - \left(h_{12}^k\right)^2$.

Definition 2.3. *We denote by* (a) $H_k := \frac{h^k_{11}g_{22}+h^k_{22}g_{11}-2h^k_{12}g_{12}}{2\mathfrak{Q}^2}$, *the mean curvature depends on* n_k, $k = 1, 2, 3$,

(b) $\overrightarrow{H} := \sum_{k=1}^{3} H_k n_k$, *the mean curvature vector*, $k = 1, 2, 3$,

(c) $\overrightarrow{H} = 0$, *the minimal surface*,

(d) $K := \frac{\sum_{k=1}^{3}\det(h^k_{ij})}{\mathfrak{Q}^2}$, *the Gaussian curvature*, $i, j = 1, 2$, $k = 1, 2, 3$, *of* M, *respectively.*

Definition 2.4. *An orthonormal tangent frame field* $\{e_1, e_2\}$ *on* M *is described by*

$$e_1 := \frac{1}{\sqrt{g_{11}}}\mathfrak{x}_s, \;\; e_2 := \frac{1}{\mathfrak{Q}\sqrt{g_{11}}}\left(g_{11}\mathfrak{x}_t - g_{12}\mathfrak{x}_s\right).$$

Definition 2.5. *Gauss map of* $\mathfrak{x}$ *is defined by*

$$\mathfrak{G} := \frac{1}{\mathfrak{Q}}\left(\mathfrak{x}_s \wedge \mathfrak{x}_t\right).$$

3. Helical Surfaces of $\mathbb{E}^5$

Let (a,b,c,d,e) of $\mathbb{E}^5$ be identified with its transpose. We define the rotational-helical surfaces of $\mathbb{E}^5$.

Definition 3.1. *Let* γ *denote a curve in* $\mathbb{E}^5$ *defined over an open interval* $I \subset \mathbb{R}$, *and let* Π *represent a plane. Suppose* ℓ *is a line within* Π. *A rotational surface is characterized by a generating curve* γ *rotating about the axis* ℓ.

Definition 3.2. *While* γ *undergoes rotation about the axis* ℓ, *it simultaneously displaces parallel lines orthogonal to* ℓ, *with the rate of displacement being proportional to the rate of rotation. The resultant surface is termed a helical surface with axis* ℓ *and pitch* $a \in \mathbb{R}\backslash\{0\}$.

Assuming ℓ is defined by the line generated by $(0, 0, 0, 0, 1)$, the orthogonal matrix can be described as

$$\mathfrak{A}(t) = diag\left(C_t, C_t, 1\right), \tag{3}$$

where $v \in \mathbb{R}$, $C_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ and $\mathfrak{A}.\ell = \ell$, $\mathfrak{A}.\mathfrak{A}^T = \mathfrak{A}^T.\mathfrak{A} = \mathcal{I}_5$, $\det\mathfrak{A} = 1$, $\mathcal{I}_5$ denotes the identitiy matrix, $\mathfrak{A}^T$ describes the transpose of $\mathfrak{A}$. The generating curve is regarded by

$$\gamma(s) = \left(f(s), 0, g(s), 0, h(s)\right).$$

In this context, the functions defined on γ are differentiable for every $s \in I$. The helical surface, constructed with an axis of $(0, 0, 0, 0, 1)$ and a pitch $a \in \mathbb{R}\backslash\{0\}$, is specified as follows

$$\mathcal{H}(s,t) = \underbrace{\mathfrak{A}(t).\gamma(s)^T}_{rotation} + \underbrace{at\ell^T}_{translation}.$$

Thus, the helical surface is defined by the equation

$$\mathcal{H}(s,t) = \left(f(s)\cos t, f(s)\sin t, g(s)\cos t, g(s)\sin t, h(s) + at\right), \tag{4}$$

where the functions f, g, h are differentiable, $s, a \in \mathbb{R}\backslash\{0\}$, $0 \le t < 2\pi$. When $a = 0$, it represents merely a rotational surface of $\mathbb{E}^5$.

We derive the following first fundamental quantities for equation (4):

$$g_{11} = f'^2 + g'^2 + h'^2, \;\; g_{12} = ah', \;\; g_{22} = f^2 + g^2 + a^2.$$

Subsequently, we obtain

$$\mathfrak{Q}^2 = \left(f^2 + g^2\right)\left(f'^2 + g'^2 + h'^2\right) > 0.$$

We compute following three normals of the surface (4):

$$\mathfrak{N}_1 = \frac{1}{\mathfrak{Q}_1}\begin{pmatrix} -\left(\lambda f'' + a^2 f'h'h''\right)\cos t - afh''\sin t \\ -\left(\lambda f'' + a^2 f'h'h''\right)\sin t + afh''\cos t \\ -\left(\lambda g'' + a^2 g'h'h''\right)\cos t - agh''\sin t \\ -\left(\lambda g'' + a^2 g'h'h''\right)\sin t + agh''\cos t \\ -\left(f^2+g^2\right)h'' \end{pmatrix}, \tag{5}$$

$$\mathfrak{N}_2 = \frac{1}{\mathfrak{Q}_2}\left(\mathfrak{k}, \mathfrak{l}, \mathfrak{m}, \mathfrak{n}, \mathfrak{o}\right), \tag{6}$$

$$\mathfrak{N}_3 = \frac{1}{\mathfrak{Q}_3}\begin{pmatrix} \lambda^2 g\left[h'\left(gf''-fg''\right)+\left(-gf'+fg'\right)h''\right]\sin t \\ -\lambda^2 g\left[h'\left(gf''-fg''\right)+\left(-gf'+fg'\right)h''\right]\cos t \\ \lambda^2 f\left[h'\left(-gf''+fg''\right)+\left(gf'-fg'\right)h''\right]\sin t \\ -\lambda^2 f\left[h'\left(-gf''+fg''\right)+\left(gf'-fg'\right)h''\right]\cos t \\ 0 \end{pmatrix}, \tag{7}$$

respectively, where

$$\begin{aligned}
\mathfrak{Q}_1 &= \sqrt{\lambda^2\left(f''^2+g''^2\right)+\lambda\left(\lambda-a^2\right)h''^2}, \\
\mathfrak{Q}_2 &= \lambda\left(f''^2+g''^2\right)+\left(\lambda-a^2\right)h''^2, \\
\mathfrak{Q}_3 &= \sqrt{\lambda^4\left(f^2+g^2\right)\left[h'\left(-gf''+fg''\right)+\left(gf'-fg'\right)h''\right]^2}, \\
\lambda &= f^2+g^2+\left(1-h'^2\right)a^2,
\end{aligned}$$

$$\mathfrak{k} = \lambda\left\{\begin{array}{c} \left[-g^3\left(h'f''-f'h''\right)\left(h'g''-g'h''\right)+f^3\left(h'g''-g'h''\right)^2\right. \\ +fg^2g'^2h''^2+fg^2h'^2g''^2-2fg^2g'h'g''h''-f^2gh'^2f''g'' \\ \left.-f^2gf'h'g''h''-f^2gf'g'h''^2+f^2gg'h'f''h''\right]\cos t \\ +\left[-afgg'h'\left(f''^2+g''^2+h''^2\right)+af^2g'h'f''g''\right. \\ \left.+af^2\left(-1+h'^2\right)f''h''-afgg''h''-af^2f'h'\left(g''^2+h''^2\right)\right]\sin t \end{array}\right\},$$

$$\mathfrak{l} = \lambda\left\{\begin{array}{c} \left[afg^2g''h''+afg^2g'h'\left(f''^2+g''^2+h''^2\right)-af^2g'h'f''g''\right. \\ \left.+af^2\left(\left(1-h'^2\right)f''h''+f'h'\left(g''^2+h''^2\right)\right)\right]\cos v \\ +\left[-g^3\left(h'f''-f'h''\right)\left(h'g''-g'h''\right)+f^3\left(h'g''-g'h''\right)^2\right. \\ +fg^2g'^2h''^2+fg^2h'^2g''^2-2fg^2g'h'g''h''-f^2gg'f'h''^2 \\ \left.-f^2gh'g''\left(h'f''-f'h''\right)+f^2gg'h'f''h''\right]\sin v \end{array}\right\},$$

$$\mathfrak{m} = \lambda\left\{\begin{array}{c} \left[-f^3\left(h'f''-f'h''\right)\left(h'g''-g'h''\right)-fg^2h'g''\left(h'f''-f'h''\right)\right. \\ +g\left(f^2+g^2\right)\left(2g'h'g''h''+\left(1-g'^2\right)h''^2+h'^2\left(f''^2+h''^2\right)\right) \\ \left.-fg^2g'f'h''^2+fg^2g'h'f''h''\right]\cos t \\ +\left[afg\left(g'h'f''g''+\left(-1+h'^2\right)f''h''-f'h'\left(g''^2+h''^2\right)\right)\right. \\ \left.-ag^2g''h''-ag^2g'h'\left(f''^2+g''^2+h''^2\right)\right]\sin t \end{array}\right\},$$

$$\mathfrak{n} = \lambda\left\{\begin{array}{c} \left[afg\left(-g'h'f''g''+\left(1-h'^2\right)f''h''+f'h'\left(g''^2+h''^2\right)\right)\right. \\ \left.+ag^2g''h''+ag^2g'h'\left(f''^2+g''^2+h''^2\right)\right]\cos t \\ +\left[-f^3\left(h'f''-f'h''\right)\left(h'g''-g'h''\right)\right. \\ +g\left(f^2+g^2\right)\left(2g'h'g''h''+\left(1-g'^2\right)h''^2+h'^2\left(f''^2+h''^2\right)\right) \\ \left.-fg^2g''h'\left(h'f''-f'h''\right)-ff'g^2g'h''^2+fg^2g'h'f''h''\right]\sin t \end{array}\right\},$$

$$\mathfrak{v} = \lambda \left(f^2 + g^2\right) \left\{ \begin{array}{c} f\left[g'h'f''g'' + \left(-1 + h'^2\right)f''h'' - f'h'\left(g''^2 + h''^2\right)\right] \\ -g\left[g''h'' + g'h'\left(f''^2 + g''^2 + h''^2\right)\right] \end{array} \right\}.$$

By calculating the second derivatives of equation (4) with respect to both s and t, we derive:

$$\begin{aligned} \mathcal{H}_{ss} &= (f'' \cos t, f'' \sin t, g'' \cos t, g'' \sin t, h''), \\ \mathcal{H}_{st} &= (-f' \sin t, f' \cos t, -g' \sin t, g' \cos t, 0), \\ \mathcal{H}_{tt} &= (-f \cos t, -f \sin t, -g \cos t, -g \sin t, 0), \end{aligned}$$

and concerning the normals (5), (6), (7), we ascertain the second fundamental forms of equation (4):

$$\begin{aligned} h^1_{11} &= -\frac{\lambda\left(f''^2 + g''^2\right) + \left(f^2 + g^2\right)h''^2 + a^2\left(f'f'' + g'g''\right)h'h''}{\mathfrak{Q}_1}, \\ h^1_{12} &= \frac{a\left(ff' + gg'\right)h''}{\mathfrak{Q}_1}, \\ h^1_{22} &= \frac{\lambda\left(ff'' + gg''\right) + a^2\left(ff' + gg'\right)h'h''}{\mathfrak{Q}_1}, \\ h^2_{11} &= \frac{\left(pf'' + rg''\right)\cos t + \left(qf'' + sg''\right)\sin t + th''}{\mathfrak{Q}_2}, \\ h^2_{12} &= \frac{\left(qf' + sg'\right)\cos t - \left(pf' + rg'\right)\sin t}{\mathfrak{Q}_2}, \\ h^2_{22} &= \frac{-\left(pf + rg\right)\cos t - \left(qf + sg\right)\sin t}{\mathfrak{Q}_2}, \\ h^3_{11} &= 0, \\ h^3_{12} &= \frac{\lambda^2\left(fg' - gf'\right)\left[\left(gf'' - fg''\right)h' + \left(fg' - gf'\right)h''\right]}{\mathfrak{Q}_3}, \\ h^3_{22} &= 0. \end{aligned}$$

The curvatures H_i $(i = 1, 2, 3)$ and K are demonstrated by

$$\begin{aligned} H_1 &= \frac{\left\{ \begin{array}{c} -a^4f'f''h'h'' - a^4g'g''h'h'' - a^2f^2f'f''h'h'' - a^2f^2g'g''h'h'' - a^2f^2h''^2 \\ +a^2ff'^3h'h'' + a^2ff'g'^2h'h'' + a^2ff'h'^3h'' - 2a^2ff'h'h'' - a^2g^2f'f''h'h'' \\ -a^2g^2g'g''h'h'' - a^2g^2h''^2 + a^2gf'^2g'h'h'' + a^2gg'^3h'h'' + a^2gg'h'^3h'' \\ -2a^2gg'h'h'' - \lambda a^2f''^2 - \lambda a^2g''^2 - f^4h''^2 - 2f^2g^2h''^2 - \lambda f^2f''^2 \\ -\lambda f^2g''^2 + \lambda ff'^2f'' + \lambda fg'^2f'' + \lambda ff''h'^2 - g^4h''^2 - \lambda g^2f''^2 \\ -\lambda g^2g''^2 + \lambda gf'^2g'' + \lambda gg'^2g'' + \lambda gg''h'^2 \end{array} \right\}}{2\mathfrak{Q}^2\mathfrak{Q}_1}, \\ H_2 &= \frac{\left\{ \begin{array}{c} \left[\left(f^2 + g^2 + a^2\right)\left(qf' + sg'\right) + 2a\left(pf + rg\right)h' - \left(pf + rg\right)\left(f'^2 + g'^2 + h'^2\right)\right]\cos t \\ +\left[-\left(f^2 + g^2 + a^2\right)\left(pf' + rg'\right) + 2a\left(qf + sg\right)h' - \left(qf + sg\right)\left(f'^2 + g'^2 + h'^2\right)\right]\sin t \end{array} \right\}}{2\mathfrak{Q}^2\mathfrak{Q}_2}, \\ H_3 &= -\frac{a\lambda^2\left(fg' - gf'\right)h'\left[gh'f'' - fh'g'' + \left(fg' - gf'\right)h''\right]}{\mathfrak{Q}^2\mathfrak{Q}_3}, \\ K &= \frac{1}{\mathfrak{Q}^2}\left(\frac{\Upsilon_1}{\left(\mathfrak{Q}_1\right)^2} + \frac{\Upsilon_2}{\left(\mathfrak{Q}_2\right)^2} + \frac{\Upsilon_3}{\left(\mathfrak{Q}_3\right)^2}\right), \end{aligned}$$

where

$$\begin{aligned}\Upsilon_1 &= -\left(\lambda\left(f''^2+g''^2\right)+\left(f^2+g^2\right)h''^2+a^2\left(f'f''+g'g''\right)h'h''\right)\\ &\quad .\left(\lambda\left(ff''+gg''\right)+a^2\left(ff'+gg'\right)h'h''\right)-\left(a\left(ff'+gg'\right)h''\right)^2,\\ \Upsilon_2 &= \left(\left(pf''+rg''\right)\cos t+\left(qf''+sg''\right)\sin t+th''\right)\left(-\left(pf+rg\right)\cos t-\left(qf+sg\right)\sin t\right)\\ &\quad -\left(-\left(pf'+rg'\right)\sin t+\left(qf'+sg'\right)\cos t\right)^2,\\ \Upsilon_3 &= -\left\{\lambda^2\left(fg'-gf'\right)\left[\left(gf''-fg''\right)h'+\left(fg'-gf'\right)h''\right]\right\}^2.\end{aligned}$$

4. Local Isometry on Helical Surface of $\mathbb{E}^5$

Following, we introduce a theorem employing the helical surface defined in the preceding section through the classical Bour's theorem in $\mathbb{E}^5$.

Theorem 4.1. *Let $\mathcal{H}$ denote the helical surface given by equation* (4). *Then, locally, $\mathcal{H}$ is isometric to the rotational surface described by*

$$\mathcal{R}(s,t)=\begin{pmatrix}\sqrt{f^2+g^2+a^2}\cos\left(t+\int\frac{ah'}{f^2+g^2+a^2}ds\right)\\ \sqrt{f^2+g^2+a^2}\sin\left(t+\int\frac{ah'}{f^2+g^2+a^2}ds\right)\\ \sqrt{f^2+g^2+a^2}\cos\left(t+\int\frac{ah'}{f^2+g^2+a^2}ds\right)\\ \sqrt{f^2+g^2+a^2}\sin\left(t+\int\frac{ah'}{f^2+g^2+a^2}ds\right)\\ \int\sqrt{\frac{\left(f^2+g^2+a^2\right)\left(f^2+g^2\right)h'^2-\left(f+g\right)^2}{f^2+g^2+a^2}}ds\end{pmatrix}. \tag{8}$$

That is, the helices on the helical surface correspond to the circles on the rotational surface.

Proof. The helical surface (4) has the line element

$$d\mathbf{s}^2=\left(f'^2+g'^2+h'^2\right)ds^2+2ah'dsdt+\left(f^2+g^2+a^2\right)dt^2. \tag{9}$$

Setting $\bar{s}=s$, $\bar{t}=t+\int\frac{ah'}{f^2+g^2+a^2}ds$, the helical surface (4) becomes to $\mathcal{H}(\bar{s},\bar{t})$. Regarding the parameters $\bar{s},\bar{t}$, the line element reduces to

$$d\mathbf{s}^2=\left(f'^2+g'^2+\frac{\left(f^2+g^2\right)h'^2}{f^2+g^2+a^2}\right)d\bar{s}^2+\left(f^2+g^2+a^2\right)d\bar{t}^2. \tag{10}$$

Putting

$$\bar{s}:=\int\sqrt{f'^2+g'^2+\frac{\left(f^2+g^2\right)h'^2}{f^2+g^2+a^2}}ds,\quad \Phi(\bar{s}):=\sqrt{f^2+g^2+a^2},$$

the line element of the helical surface is given by

$$d\mathbf{s}^2=d\bar{s}^2+\Phi^2(\bar{s})d\bar{t}^2. \tag{11}$$

On the other side, in $\mathbb{E}^5$, the rotational surface

$$\mathcal{R}(\mathfrak{s},\mathrm{t})=\left(\mathfrak{f}(\mathfrak{s})\cos\mathrm{t},\mathfrak{f}(\mathfrak{s})\sin\mathrm{t},\mathfrak{g}(\mathfrak{s})\cos\mathrm{t},\mathfrak{g}(\mathfrak{s})\sin\mathrm{t},\mathfrak{h}(\mathfrak{s})\right)$$

has following line element

$$d\mathbf{s}^2=\left(\mathfrak{f}'^2+\mathfrak{g}'^2+\mathfrak{h}'^2\right)d\mathfrak{s}^2+\left(\mathfrak{f}^2+\mathfrak{g}^2\right)d\mathrm{t}^2. \tag{12}$$

Again setting $\mathfrak{f}^2 + \mathfrak{g}^2 = f^2 + g^2 + a^2$, $p(s) = \mathfrak{g}'$, $q(s) = \mathfrak{h}'$, we obtain

$$\mathfrak{g} = \int \frac{f\,p(s)}{\sqrt{f^2 + g^2 + a^2}} ds, \quad \mathfrak{h} = \int \frac{f\,q(s)}{\sqrt{f^2 + g^2 + a^2}} ds.$$

Thus, the helical surface represented by equation (4) demonstrates a local isometric correspondence with the rotational surface defined by equation (8).

A helix on $\mathcal{H}$ is characterized by $\mathfrak{f} = s_0$, where s_0 is a constant corresponding to the curves $\mathfrak{f}^2 = s_0^2 + a^2$ on $\mathcal{R}$, which represent circles on the plane defined by $x_3 = \mathfrak{g}(\mathfrak{s})$, $x_5 = \mathfrak{h}(\mathfrak{s})$, or it can be defined by $\mathfrak{g} = s_0$ on $\mathcal{R}$, which is defined by $\mathfrak{g}^2 = s_0^2 + a^2$, representing circles on the plane $x_1 = \mathfrak{f}(\mathfrak{s})$, $x_5 = \mathfrak{h}(\mathfrak{s})$. □

In light of the surfaces exhibiting isometry as per Theorem 3.1, the following theorem is derived.

Theorem 4.2. *Consider $\mathcal{H}$ and $\mathcal{R}$, respectively, as the helical-rotational surfaces that are interconnected isometrically by Theorem 3.1. When these surfaces share identical Gauss maps, they are both minimal and hyperplanar.*

Proof. Let $\{k_1, k_2, k_3, k_4, k_5\}$ denote the orthonormal frame of $\mathbb{E}^5$. Consider $i, j = 1, 2, 3, 4, 5$, where $i < j$. Then, define $k_{ij} = k_i \wedge k_j$. Referring to Definition 1.6, the Gauss map of the helical surface defined by equation (4) can be expressed as

$$\mathfrak{G}_{\mathcal{H}} = \frac{1}{\mathfrak{Q}} \left\{ \begin{array}{c} ff'\,k_{12} + (fg' - f'g)\cos t \sin t\, k_{13} + \left(f'g\cos^2 t + fg'\sin^2 t\right) k_{14} \\ + (af'\cos t + fg'\sin t)\; k_{15} - (fg'\cos^2 t + f'g\sin^2 t)\, k_{23} \\ +(f'g - fg')\cos t\sin t\, k_{24} + (af'\sin t - fh'\cos t)\; k_{25} + gg'\,k_{34} \\ + (ag'\cos t - gh'\sin t)\; k_{35} + (ag'\sin t - gh'\cos t)\; k_{45} \end{array} \right\}. \tag{13}$$

The Gauss map of the rotational surface specified by equation (8) is described as

$$\mathfrak{G}_{\mathcal{R}} = \frac{1}{\mathfrak{Q}} \left\{ \begin{array}{c} ff'\,k_{12} + (fp - f'g)\cos\Omega\sin\Omega\, k_{13} + \left(f'g\cos^2\Omega + fp\sin^2\Omega\right) k_{14} \\ +fp\sin\Omega\, k_{15} - \left(fp\cos^2\Omega + f'g\sin^2\Omega\right) k_{23} \\ +(f'g - fp)\cos\Omega\sin\Omega\, k_{24} - fq\cos\Omega\, k_{25} + gp\,k_{34} \\ -gq\sin\Omega\, k_{35} - gq\cos\Omega\, k_{45} \end{array} \right\}, \tag{14}$$

where $\Omega = t + \int \frac{ah'}{f^2+g^2+a^2} ds$. Given that $\mathfrak{G} = \mathfrak{G}_{\mathcal{R}}$, identically, equations (13) and (14) result in:

$$\begin{aligned}
(fg' - f'g)\cos t\sin t &= (fp - f'g)\cos\Omega\sin\Omega, && (15)\\
f'g\cos^2 t + fg'\sin^2 t &= f'g\cos^2\Omega + fp\sin^2\Omega, && (16)\\
fg'\cos^2 t + f'g\sin^2 t &= fp\cos^2\Omega + f'g\sin^2\Omega, && (17)\\
af'\cos t + fg'\sin t &= fp\sin\Omega, && (18)\\
af'\sin t - fh'\cos t &= -fq\cos\Omega, && (19)\\
ag'\cos t - gh'\sin t &= -gq\sin\Omega, && (20)\\
ag'\sin t - gh'\cos t &= -gq\cos\Omega. && (21)
\end{aligned}$$

Here, (16) + (17) reduces to $p = g'$, where $f \neq 0$. Hence, we have $p(s) = \mathfrak{g}' = g'$, and then $\mathfrak{g} = \int \frac{f\,g'}{\sqrt{f^2+g^2+a^2}} ds$. Or more clear way, by using (13) and (14), we can see that $gg'\,k_{34} = gp\,k_{34}$, then $p = g'$.

On the other hand, (19) + (21) reduces to

$$a(f' + g')\sin t - (f + g)h'\cos t = -(f + g)\,q\cos\Omega.$$

So, we get $q = h'$, $a(f' + g')\sin t = 0$, and also $v = \Omega$ (i.e., $\int \frac{ah'}{f^2+g^2+a^2} ds = 0$). That is, $\bar{t} = t$. Therefore, we find $\mathfrak{h} = \int \frac{f\,h'}{\sqrt{f^2+g^2+a^2}} ds$. □

Corollary 4.3. *Consider $\mathcal{H}$-$\mathcal{R}$ as the helical-rotational surfaces sharing the same Gauss map and connected isometrically by Theorem 4.1. Subsequently, these surfaces are characterized by*

$$\mathcal{H}(s,t) = (f(s)\cos t, f(s)\sin t, g(s)\cos t, g(s)\sin t, h(s) + at)$$

and

$$\mathcal{R}(u,v) = \begin{pmatrix} \sqrt{f^2+a^2}\cos\left(t + \int \frac{ah'}{f^2+g^2+a^2}ds\right) \\ \sqrt{f^2+a^2}\sin\left(t + \int \frac{ah'}{f^2+g^2+a^2}ds\right) \\ \sqrt{g^2+a^2}\cos\left(t + \int \frac{ah'}{f^2+g^2+a^2}ds\right) \\ \sqrt{g^2+a^2}\sin\left(t + \int \frac{ah'}{f^2+g^2+a^2}ds\right) \\ \int \sqrt{\frac{a^2+\left(f^2+g^2\right)(1+h'^2)-\left(ff'+gg'\right)^2}{f^2+g^2+a^2}}ds \end{pmatrix}.$$

Proof. The expression is deduced by employing the identity

$$\int \sqrt{f'^2 + g'^2 + \frac{(f^2+g^2)\,h'^2}{f^2+g^2+a^2}}ds = \int \sqrt{\mathfrak{f}'^2 + \mathfrak{g}'^2 + \mathfrak{h}'^2}d\mathfrak{s},$$

whereby the following

$$\mathfrak{f}'^2 + \mathfrak{g}'^2 + \mathfrak{h}'^2 = \frac{a^2\left(f'^2+g'^2\right) + \left(f^2+g^2\right)\left(f'^2+g'^2+h'^2\right)}{\left(ff'+gg'\right)^2}$$

is derived. Subsequently,

$$\mathfrak{h}'d\mathfrak{s} = \int \sqrt{\frac{a^2 + (f^2+g^2)(1+h'^2) - \left(ff'+gg'\right)^2}{f^2+g^2+a^2}}ds$$

is obtained. □

5. Conclusion

In the context of five-dimensional Euclidean space $\mathbb{E}^5$, our study has established the definition of helical surfaces. Through rigorous mathematical analysis, we have calculated three normals and elucidated their geometric attributes, providing a detailed understanding of their structural properties. Additionally, leveraging Bour's theorem within $\mathbb{E}^5$, we have determined an isometric mapping among helical-rotational surfaces, further enhancing our comprehension of their interplay and geometric relationships.

Looking ahead, our findings pave the way for deeper investigations into the properties and behaviors of helical surfaces within higher-dimensional spaces. Further research could focus on exploring additional geometric properties and developing new mathematical techniques to analyze these surfaces more comprehensively.

In conclusion, our study contributes to the mathematical understanding of helical surfaces within $\mathbb{E}^5$, providing valuable insights into their geometric attributes and structural interplay. Through continued research, we aim to expand our knowledge of these surfaces and their applications in various mathematical contexts.

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