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A solution of a nonstandard Dirichlet Finsler (p, q)-Laplacian

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Abstract. In this article, we study the existence of generalized solution for the nonstandard Finsler (p, q)-Laplacian. The method is based on a nice application of Brouwer's fixed point theorem, Nemytskij operator and Galerkin basis of the Sobolve space.

1. Introduction

The study of the Finsler p-Laplacian, is a PDE which generalizes the Finsler Laplacian in an analogous manner to the way the p-Laplacian generalizes the Laplacian. The Finsler Laplacian is indeed a linear elliptic operator with constant coefficients. Due to do this, $\mathbb{R}^N$ endowed with the (possibly asymmetric) norm F can be viewed as a Finsler manifold, which reflects into a rich geometric structure to explore since it breaks the usual symmetry properties coming from the peculiarities of the Euclidean norm, like the directional independence.

Such a generalization to anisotropic (or Finsler) PDEs is also of interest from two points of views

(1) In its own right: For example Eigenfunctions of the p-Laplacian have weaker regularities in the Finslerian setting than the Riemannian one, due to the non-linearity of the Finslerian p-Laplacian, or in the case of Liouville equation, it is interesting to consider operators appearing in the Euler-Lagrange equations for Wulff-type functionals $\int F(\nabla u)^N dx$.

(2) From the applied point of view, as it is motivated by concrete applications in many fields, such as in the study of sharp geometric inequalities and capacity, blowup analysis, digital image processing, crystalline mean curvature flow and crystalline fracture theory (see [2, 3, 5–7, 22] and the references therein).

However, this generalization is sometimes challenging, since one has to abandon typical arguments which apply only in the Euclidean setting and find new ways to handle.

Here, we improve [11, Theorem 3.4] and with the inspiration of [11, 16], the existence of a generalized solution for the nonstandard Dirichlet problem with Finsler competing (p, q)-Laplacian

$$\begin{cases} -\mathcal{Q}_p u + \mu \mathcal{Q}_q u = f(x, \phi \star u, \nabla(\phi \star u)) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

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is proved, where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, with a Lipschitz boundary $\partial\Omega$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, $\phi \in L^1(\mathbb{R}^N)$, the convolution $\phi \star u$ is given by Definition 2.8 and finally, $-Q_p u = -div\left(F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)\right)$ denotes Finsler p-Laplacian operator where F is a sufficiently smooth norm on $\mathbb{R}^N$ (see Section 2). Also we assume

(H_1) $|f(x,t,\xi)| \leq \sigma(x) + c_1|t|^{p-1} + c_2|\xi|^{p-1}$ for a.e. $x \in \Omega$ and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $\xi = (\xi_1, \cdots, \xi_N)$, $\sigma \in L^{\gamma'}(\Omega)$ for $\gamma \in (1, p^*)$, $\gamma' = \frac{\gamma}{\gamma-1}$.

(H_2) There are two constants $c_1 \geq 0, c_2 \geq 0$, satisfying

$$\|\phi\|^{p-1}_{L^1(\mathbb{R}^N)}\left(c_1 S_p^p + c_2 N^{p-1} S_p\right) < a^p, \tag{2}$$

where $\phi \in L^1(\mathbb{R}^N)$ and S_p is given by Remark (2.6) and $a > 0$ is given by condition (5).

The differential operator in (1), i.e.

$$u \to Q_p u - \mu Q_q u$$

is the difference of the Finsler degenerated p-Laplacian and q-Laplacian. In fact, the negative anisotropic φ-Laplacian (for $\varphi = p, q$)

$$-Q_\varphi u : W_0^{1,\varphi}(\Omega) \to W^{-1,\varphi'}(\Omega)$$

expressed as

$$\langle -Q_\varphi u, v\rangle_{W_0^{1,p}(\Omega)} = \int_\Omega F^{\varphi-1}(\nabla u)F_\xi(\nabla u) \cdot \nabla v dx$$

for all $u, v \in W_0^{1,\varphi}(\Omega)$, where $\varphi' := \frac{\varphi}{\varphi-1}$.

Since $1 < q < p < \infty$, then the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ is hold and the operator $-Q_p + \mu Q_q$ is well defined on $W_0^{1,p}(\Omega)$.

The sign of $-Q_p + \mu Q_q$ for $\mu > 0$ and sufficiently large is different from $\mu > 0$ and sufficiently small. This makes some difficulty for studying (1). We owe essential ideas to [11, 16] to overcome the lack of ellipticity, monotonicity and variational structure in problem (1). Therefore we can prove the existence of a generalized solution of Finsler (p,q)-Laplacian (1) by Theorem 1.1. The positive point is that our result improve [11, Theorem 3.4]. In [11, Theorem 3.4] Motreanu $\|\rho\|^{p-1}_{L^1(\mathbb{R}^N)}\left(c_1 S_p^p + c_2 N^{p-1} S_p\right) < 1$, but we apply $\|\rho\|^{p-1}_{L^1(\mathbb{R}^N)}\left(c_1 S_p^p + c_2 N^{p-1} S_p\right) < a^p$, where $a > 0$ and is given by Finsler norm F.

Theorem 1.1. *Suppose that (H_1) and (H_2) are hold. Then there exists a generalized solution to Finsler (p,q)-Laplacian (1). In particular, if $\mu \leq 0$, there exists a weak solution to Finsler (p,q)-Laplacian (1).*

The rest of the paper is organized as follows: In Section 2, Finsler Laplacian operator and some related facts are recalled. In Section 3, the associated Nemytskij operator is introduced and then we show the Finsler competing (p,q)-Laplacian (1) has a solution, i.e. the proof of Theorem 1.1 is presented.

2. Finsler Laplacian operator

In this section, first we recall Finsler p-Laplacian operator and study some its properties (see [4, 6, 10, 21] and references therein for more details).

The Finsler p-Laplacian operator or anisotropic p-Laplacian operator is introduced by

$$-Q_p u := -div\left(F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)\right) = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(F^{p-1}(\nabla u)F_{\xi_i}(\nabla u)\right), \tag{3}$$

where $1 < p < +\infty$, $\nabla_\xi F$ is the gradient of F, $F_{\xi_i} = \frac{\partial F}{\partial \xi_i}$ and F is a sufficiently smooth norm on $\mathbb{R}^N$ (see [14, 21, 23] for more details).

Definition 2.1. *We say that $F : \mathbb{R}^N \to \mathbb{R}$ is a norm*

1) $F : \mathbb{R}^N \to [0, +\infty)$ *is a convex function of class* $C^2(\mathbb{R}^N \backslash \{0\})$,

2) *F is even and positively homogeneous of degree 1, i.e.* $F(t\xi) = |t|F(\xi)$ *for any* $t \in \mathbb{R}, \xi \in \mathbb{R}^N$.

3) $F(\xi) > 0$ *for any* $\xi \neq 0$.

Note that we do not require F to be symmetric, so it may happen that $F(\xi) \neq F(-\xi)$.

The following properties are easy consequences of 1-homogeneity and convexity of F.

Proposition 2.2. *Let F be a norm in* $\mathbb{R}^N$, *then the following holds:*

(i) *If* $F \in C^1(\mathbb{R}^N \backslash \{0\})$, *then for* $\xi \in \mathbb{R}^N \backslash \{0\}, t \neq 0$

$$F_{\xi_i}(\xi)\xi_i = F(\xi), \; F_{\xi_i}(t\xi) = sign(t) F_{\xi_i}(\xi).$$

(ii) *If* $F \in C^2(\mathbb{R}^N \backslash \{0\})$, *then for* $\xi \in \mathbb{R}^N \backslash \{0\}, t \neq 0$

$$\sum_{j=1}^{N} F_{\xi_i \xi_j}(\xi)\xi_j = 0 \text{ for any } i = 1, \cdots, n, \; F_{\xi_i \xi_j}(t\xi) = \frac{1}{|t|} F_{\xi_i \xi_j}(\xi).$$

(iii) $|F(x) - F(y)| \leq F(x + y) \leq F(x) + F(y)$
$|F_{\xi}(x)| \leq C$ *for any* $x \neq 0$ *if* $F \in C^1(\mathbb{R}^N \backslash \{0\})$.

We assume also

4) Notice that

$$\nabla_{\xi}^2 \left(F^p\right)(\xi) \text{ is positive definite in } \mathbb{R}^N \backslash \{0\}, \text{ with } 1 < p < +\infty. \tag{4}$$

5) There exists two constants $0 < a \leq b < \infty$ such that

$$a|\xi| \leq F(\xi) \leq b|\xi|, \text{ for any } \xi \in \mathbb{R}^N. \tag{5}$$

Remark 2.3. *The above conditions guarantees that the operator* $\mathcal{Q}_p(u)$ *is elliptic, hence there exists a positive constant* γ *such that*

$$\frac{1}{p} \sum_{i,j=1}^{N} \nabla_{\xi_i \xi_j}^2 \left(F^p\right)(\eta)\xi_i \xi_j \geq \gamma |\eta|^{p-2} |\xi|^2$$

for some positive constant γ, *for any* $\eta \in \mathbb{R}^N \backslash \{0\}$ *and for any* $\xi \in \mathbb{R}^N$.

Example 2.4. *The most well known example is*

$$F(\xi) = |\xi|_q := \left(\sum_{i=1}^{N} |\xi_i|^q \right)^{\frac{1}{q}} \text{ in } \mathbb{R}^N$$

for $1 < q < \infty$. *In this case,* $\mathcal{Q}_{q,p}(u) := div(\|u\|_q^{p-q} \nabla^q u)$, *where* $\nabla^q u = (|u_{x_1}|^{q-2} u_{x_1}, \cdots, |u_{x_n}|^{q-2} u_{x_n})$, *i.e.*

$$\mathcal{Q}_{q,p}(u) := \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left(\sum_{j=1}^{N} |\frac{\partial u}{\partial x_j}|^q \right)^{\frac{p-q}{q}} |\frac{\partial u}{\partial x_i}|^{q-2} \frac{\partial u}{\partial x_i} \right).$$

Remark 2.5. *When $q = 2$, i.e. $F(\xi) = |\xi|$ is the Euclidean norm, $\mathcal{Q}$ is just the classical isotropic p-Laplacian. This operator is closely related to a smooth, convex hypersurface in $\mathbb{R}^N$, which is called the Wulff's shape (or equilibrium crystal shape) of F. The study of the Wulff shape was initiated in Wulff's work [22] on crystal shapes.*
In addition, $\mathcal{Q}$ is a generalization of $\overrightarrow{p}$-Laplacian (see [1, 9, 13, 15–19] and references therein).

For a general norm F, $\mathcal{Q}_p$ is anisotropic and can be highly nonlinear. In literature, several papers (see [14] and references there in) are devoted to the study of the smallest eigenvalue of

$$\begin{cases} -\mathcal{Q}_p u = \lambda |u|^{p-2}p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

denoted by $\lambda_{1,p}$, in bounded domains, which has the variational characterization

$$\lambda_{1,p} = \min_{\varphi \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega F^p(\nabla\varphi)dx}{\int_\Omega |\varphi|^p dx}. \tag{7}$$

We denote by p^* the critical Sobolev exponent, i.e., $p^* = \frac{Np}{N-p}$ if $N > p$ and $p^* = +\infty$ if $N \le p$.

Remark 2.6. *The embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for $1 \le r < p^*$ and is continuous for $r = p^*$. Thus for $1 \le r \le p^*$ there exists a positive constant S_r such that*

$$\|u\|_r \le S_r \|\nabla u\|_p \text{ for all } W_0^{1,p}(\Omega).$$

Remark 2.7. *By Remark 2.6, (5) and (7) we can conclude*

$$\left(\int_\Omega F^r(\nabla\varphi)dx\right)^{\frac{1}{r}} \le b\|\varphi\|_r \le bS_r\|\nabla\varphi\|_p,$$

for $1 \le r \le p^*$

Definition 2.8. *Assume $\phi \in L^1(\mathbb{R}^N)$, $u \in W_0^{1,p}(\Omega)$ and the convolution $\phi \star u(x)$ is defined by*

$$\phi \star u(x) := \int_{\mathbb{R}^N} \phi(x-y)u(y)dy \text{ for a.e. } x \in \mathbb{R}^N.$$

Remark 2.9. *With respect to the Definition 2.8*

(i)

$$\frac{\partial}{\partial x_i}(\phi \star u) = \phi \star \frac{\partial u}{\partial x_i} \in L^{p_i}(\mathbb{R}^N), \text{ for all } i = 1, 2, \cdots, N.$$

(ii)

$$\|\phi \star u\|_{L^r(\mathbb{R}^N)} \le \|\phi\|_{L^1(\mathbb{R}^N)}\|u\|_{L^r(\Omega)} \tag{8}$$

whenever $r \in [1, p^\star]$.

(iii)

$$\|\phi \star \frac{\partial u}{\partial x_i}\|_{L^{p_i}(\mathbb{R}^N)} \le \|\phi\|_{L^1(\mathbb{R}^N)}\|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)} \tag{9}$$

for all $i = 1, \cdots, N$.

(iv) *By* (9), *one can have*

$$\|\phi \star u\|_{W_0^{1,p}(\mathbb{R}^N)} \leq N\|\phi\|_{L^1(\mathbb{R}^N)}\|u\|_{W_0^{1,p}(\mathbb{R}^N)}. \tag{10}$$

See [11, 16] for more details.

Before ending this section we mean a generalized solution for (1).

Definition 2.10. *A function $u \in W_0^{1,p}(\Omega)$ is called a generalized solution to problem* (1) *if there exists a sequence $\{u_n\}_{n\geq 1}$ in $W_0^{1,p}(\Omega)$ such that*

(I) $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *as* $n \to \infty$.

(II) $-\mathcal{Q}_p u_n + \mu \mathcal{Q}_q u_n - f(\cdot, \phi \star u_n(\cdot), \nabla(\phi \star \nabla u)(\cdot)) \rightharpoonup 0$ *in* $W^{-1,p'}(\Omega)$ *as* $n \to \infty$.

(III) $\lim_{n\to\infty} \langle -\mathcal{Q}_p u_n + \mu \mathcal{Q}_q u_n, u_n - u\rangle_{W_0^{1,p}(\Omega)} = 0$, *where*

$$\langle \left(-\mathcal{Q}_p + \mu \mathcal{Q}_q\right)(u), v\rangle_{W_0^{1,p}(\Omega)} = \int_\Omega \left(F^{p-1}(\nabla u) - \mu F^{q-1}(\nabla u)\right) F_\xi(\nabla u) \cdot \nabla v \, dx.$$

Definition 2.11. *The function $u \in W_0^{1,p}(\Omega)$ is called a weak solution of* (1) *if*

$$\int_\Omega \left(F^{p-1}(\nabla u) - \mu F^{q-1}(\nabla u)\right) F_\xi(\nabla u) \cdot \nabla v \, dx = \int_\Omega f(x, \phi \star u(x), \nabla(\phi \star u(x)))v(x)dx$$

for all $v \in W_0^{1,p}(\Omega)$.

Remark 2.12. *In Definition 2.10, set $u_n = u$ for all n , then any weak solution is a generalized solution to Finsler (p,q)-Laplacian* (1).

3. Weak and generalized solutions

Here, we study the behavior of the Nemytskij operator and construct a sequence (by the Galerkin basis of the space) which converges strongly to the generalized (weak) solution of (1) when $\mu \geq 0$ ($\mu < 0$). First we recall an embedding result.
Since $q < p$ and Ω is bounded then

$$\begin{aligned} &W_0^{1,p}(\Omega) \text{ is continuously embedded in } W_0^{1,q}(\Omega) \text{ and} \\ &W^{-1,q'}(\Omega) \text{ is continuously embedded in } W^{-1,p'}(\Omega). \end{aligned} \tag{11}$$

Assume the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ (see (1)) is defined by

$$\begin{aligned} \langle A(u), v\rangle_{W_0^{1,p}(\Omega)} = &\int_\Omega \left(F^{p-1}(\nabla u) - \mu F^{q-1}(\nabla u)\right) F_\xi(\nabla u) \cdot \nabla v \, dx \\ &- \int_\Omega f(x, \phi \star u(x), \nabla(\phi \star u)(x))v(x)dx. \end{aligned} \tag{12}$$

Lemma 3.1. *The operator A defined by* (12) *is continuous, when (H_1) and (H_2) are hold.*

Proof. Define the operator

$$T : W_0^{1,p}(\Omega) \to L^p(\Omega) \times L^p(\Omega) \times \cdots \times L^p(\Omega)$$

by $T(u) = (\phi \star u\Big|_{\Omega}, \nabla(\phi \star u)\Big|_{\Omega})$. Relations (8) and (10) imply that T is linear and continuous. By (H_1) and $(H_2)$and the Krasnoselskii's theorem [8], the Nemytskii operator

$$\begin{aligned} \mathcal{N} :& L^p(\Omega) \times (L^p(\Omega) \times \cdots \times L^p(\Omega)) \to L^{p'}(\Omega) \\ & (v, w_1, \cdots, w_N) \mapsto f(\cdot, v(\cdot), w_1(\cdot), \cdots, w_N(\cdot)) \end{aligned}$$

is well defined and continuous and so the composition operator

$$W_0^{1,p}(\Omega) \to L^{p'}(\Omega), \ \ u \mapsto f(\cdot, \phi \star u(\cdot), \nabla(\phi \star u)(\cdot)) \tag{13}$$

is continuous. Notice that $L^{p'}(\Omega)$ is continuously embedded in $W^{-1,p'}(\Omega)$.

The operator $-\mathcal{Q}_\varrho : W_0^{1,\varrho}(\Omega) \to W^{-1,\varrho'}(\Omega)$ (for $\varrho = p, q$) is continuous. Therefore embedding (11) implies $-\mathcal{Q}_p + \mu\mathcal{Q}_q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous and finally the operator A is continuous. □

Definition 3.2. *A Galerkin basis of the space $W_0^{1,p}(\Omega)$ is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of vector subspaces of $W_0^{1,p}(\Omega)$ satisfying*

1) *$\dim(X_n) < \infty$ for all n.*
2) *$X_n \subset X_{n+1}$ for all n.*
3) *$\overline{\bigcup_{n=1}^{\infty} X_n} = W_0^{1,p}(\Omega)$.*

Remark 3.3. *There exists a Galerkin basis because the Banach space $W_0^{1,p}(\Omega)$ with $1 < p < +\infty$ is separable.*

A consequence of Brouwer's fixed point theorem will resolve each approximate problem on X_n. Due to do this, we construct a sequence $\{u_n\}$ by the next Proposition.

Proposition 3.4. *Assume (H_1) and (H_2) are hold. Then for each $n \geq 1$ there exists $u_n \in X_n$ such that*

$$\begin{aligned} &\int_\Omega \Big(F^{p-1}(\nabla u_n) - \mu F^{q-1}(\nabla u_n)\Big) F_\xi(\nabla u_n) \cdot \nabla v \, dx \\ &\qquad = \int_\Omega f(x, \phi \star u_n(x), \nabla(\phi \star u_n(x)))v(x)dx \end{aligned} \tag{14}$$

for all $v \in X_n$. In addition, $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$.

Proof. We define $A_n : X_n \to X_n^\star$ by

$$\begin{aligned} \langle A_n(u), v\rangle_{X_n} = &\int_\Omega \Big(F^{p-1}(\nabla u) - \mu F^{q-1}(\nabla u)\Big) F_\xi(\nabla u) \cdot \nabla v \, dx \\ &- \int_\Omega f(x, \phi \star u(x), \nabla(\phi \star u(x)))v(x)dx \end{aligned}$$

for all $u, v \in X_n$ and all $n \in \mathbb{N}$. The operator A_n is continuous (by Lemma 3.1) and condition (5) we get

$$\begin{aligned} &\langle A_n(v), v\rangle_{X_n} \\ &= \int_\Omega \Big(F^p(\nabla v) - \mu F^q(\nabla v)\Big)dx - \int_\Omega f(x, \phi \star v(x), \nabla(\phi \star v(x)))v(x)dx \\ &\geq \int_\Omega \Big(a^p|\nabla v|^p - \mu b^q|\nabla v|^q\Big)dx - \int_\Omega f(x, \phi \star v(x), \nabla(\phi \star v(x)))v(x)dx \\ &\geq a^p\|v\|^p_{W_0^{1,p}(\Omega)} - \mu b^q|\Omega|^{\frac{p-q}{p}}\|v\|^q_{W_0^{1,p}(\Omega)} - \|\sigma\|_{L^{\gamma'}(\Omega)}\|v\|_{L^\gamma(\Omega)} \\ &\quad - c_1\|\phi \star v\|^{p-1}_{L^p(\Omega)}\|v\|_{L^p(\Omega)} - c_2\|\nabla(\phi \star v)\|^{p-1}_{L^p(\mathbb{R}^N,\mathbb{R}^N)}\|v\|_{L^p(\Omega)} \end{aligned}$$

for all $v \in X_n$, by (H_1), (H_2) and Hölder inequality. Now (8), (10) and Sobolev embedding show

$$\begin{aligned}
&\langle A_n(v), v\rangle_{X_n}\\
&\quad= \int_\Omega \Big(F^p(\nabla v) - \mu F^q(\nabla v)\Big)dx - \int_\Omega f(x, \phi \star v(x), \nabla(\phi \star v(x)))v(x)dx\\
&\quad\geq a^p \|v\|^p_{W_0^{1,p}(\Omega)} - \mu b^q |\Omega|^{\frac{p-q}{p}} \|\nabla v\|^q_{W_0^{1,p}(\Omega)} - \|\sigma\|_{L^{\gamma'}(\Omega)} \|v\|_{L^r(\Omega)}\\
&\qquad - c_1 \|\phi\|^{p-1}_{L^1(\mathbb{R}^N)} \|v\|^p_{L^p(\Omega)} - c_2 N^{p-1} \|\phi\|^{p-1}_{L^1(\mathbb{R}^N)} \|\nabla v\|^{p-1}_{L^p(\Omega)} \|v\|_{L^p(\Omega)}\\
&\quad\geq a^p \|v\|^p_{W_0^{1,p}(\Omega)} - \mu b^q |\Omega|^{\frac{p-q}{p}} \|v\|^q_{W_0^{1,p}(\Omega)} - S_\gamma \|\sigma\|_{L^{\gamma'}(\Omega)} \|v\|_{W_0^{1,p}(\Omega)}\\
&\qquad - c_1 S_p^p \|\phi\|^{p-1}_{L^1(\mathbb{R}^N)} \|v\|^p_{W_0^{1,p}(\Omega)} - c_2 S_p N^{p-1} \|\phi\|^{p-1}_{L^1(\mathbb{R}^N)} \|v\|^p_{W_0^{1,p}(\Omega)}
\end{aligned} \tag{15}$$

for all $v \in X_n$, where $|\Omega|$ is the Lebesgue measure of Ω, S_p is given by (2.6), $p > q > 1$.
Assume $\lambda_{1,p} > 0$ is given by (7) and for $R = R(n) > 0$ sufficiently large we get

$$\langle A_n(v), v\rangle_{X_n} \geq 0 \text{ whenever } v \in X_n \text{ with } \|v\|_{W_0^{1,p}(\Omega)} = R,$$

By a consequence of Brouwer's fixed point theorem (see, e.g., [20, p. 37]) (since X_n is a finite dimensional space) there exists $u_n \in X_n$ solving the equation $A_n(u_n) = 0$ and this shows $u_n \in X_n$ is a solution for problem (14).

$\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. To show this, let $v = u_n \in X_n$ in (15), then

$$\begin{aligned}
&\left(a^p - \|\phi\|^{p-1}_{L^1(\mathbb{R}^N)} \left(c_1 S_p^p + c_2 S_p N^{p-1}\right)\right) \|u_n\|^p_{W_0^{1,p}(\Omega)}\\
&\leq \mu b^q |\Omega|^{\frac{p-q}{p}} \|u_n\|^q_{W_0^{1,p}(\Omega)} + S_\gamma \|\sigma\|_{L^{\gamma'}(\Omega)} \|v\|_{W_0^{1,p}(\Omega)}.
\end{aligned}$$

Since $p > q > 1$, then (2) shows that $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p}(\Omega)$. □

Now we can prove the existence of solution of the problem (1), i.e. we present the proof of Theorem 1.1.

Proof. Assume $\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ is given by Proposition 3.4 which is bounded in $W_0^{1,p}(\Omega)$ and the reflexively, there exists a subsequence still denoted by $\{u_n\}_{n\geq 1}$ which is bounded and

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \tag{16}$$

with some $u \in W_0^{1,p}(\Omega)$. The continuity of the operator in (13), shows the sequence $\{f(\cdot, \phi \star u_n, \nabla(\phi \star u_n))\}_{n\geq 1}$ is bounded in $L^{p'}$. Suppose

$$-Q_p u_n + \mu Q_q u_n - f(\cdot, \phi \star u_n, \nabla(\phi \star u_n)) \rightharpoonup \eta \text{ in } W^{-1,p'}(\Omega) \tag{17}$$

with some $\eta \in W^{-1,p'}(\Omega)$, by the reflexivity of $W^{-1,p'}(\Omega)$.
Assume $v \in \cup_{n\geq 1} X_n$. Fix an integer $m \geq 1$ such that $v \in X_m$. Proposition 3.4 provides that (14) holds for all $n \geq m$. Letting $n \to \infty$ in (14), by means of (17) we get

$$\langle \eta, v\rangle_{W_0^{1,p}(\Omega)} = 0 \text{ for all } v \in \cup_{n\geq 1} X_n.$$

by the density of $\cup_{n\geq 1} X_n$ in $W_0^{1,p}(\Omega)$ (see (*iii*) in the definition of the Galerkin basis), it turns out that $\eta = 0$ and so in $W^{-1,p'}(\Omega)$ we have

$$-Q_p u_n + \mu Q_q u_n - f(\cdot, \phi \star u_n, \nabla(\phi \star u_n)) \rightharpoonup 0. \tag{18}$$

Let $v = u_n$ in (14), we obtain

$$\int_\Omega \Big(F^{p-1}(\nabla u_n) - \mu F^{q-1}(\nabla u_n)\Big)F_\xi(\nabla u_n)\cdot\nabla u_n\,dx - \int_\Omega f(\cdot,\phi\star u_n,\nabla(\phi\star u_n))u_n\,dx = 0 \tag{19}$$

for all $n \geq 1$, while (18) gives

$$\int_\Omega \Big(F^{p-1}(\nabla u_n) - \mu F^{q-1}(\nabla u_n)\Big)F_\xi(\nabla u_n)\cdot\nabla u\,dx - \int_\Omega f(\cdot,\phi\star u_n,\nabla(\phi\star u_n))u\,dx \to 0 \tag{20}$$

as $n \to \infty$. Altogether, (19) and (20) yield

$$\int_\Omega \Big(F^{p-1}(\nabla u_n) - \mu F^{q-1}(\nabla u_n)\Big)F_\xi(\nabla u_n)\cdot\nabla(u_n - u)\,dx - \int_\Omega f(\cdot,\phi\star u_n,\nabla(\phi\star u_n))(u_n - u)dx \to 0 \tag{21}$$

as $n \to \infty$. Remark 2.6 and (16), imply that $u_n \to u$ strongly in $L^p(\Omega)$, and since $\{f(\cdot,\phi\star u_n,\nabla(\phi\star u_n))\}$ is bounded, then

$$\lim_{n\to\infty}\int_\Omega f(\cdot,\phi\star u_n,\nabla(\phi\star u_n))(u_n - u)dx = 0. \tag{22}$$

By inserting (22) into (21) we get

$$\lim_{n\to\infty}\int_\Omega \Big(F^{p-1}(\nabla u_n) - \mu F^{q-1}(\nabla u_n)\Big)F_\xi(\nabla u_n)\cdot\nabla(u_n - u)\,dx = 0. \tag{23}$$

Thus the condition of Definition 2.10 are satisfied and this implies that $u \in W_0^{1,p}(\Omega)$ is a generalized solution to problem (1).

Now, we prove the existence of weak solution in the case $\mu \leq 0$. Assume u is a generalized solution to problem (1) and $\{u_n\}_{n\geq 1}$ satisfying the conditions of Definition 2.10 with respect to u. We get

$$\begin{aligned}
&\int_\Omega F^{p-1}(\nabla u_n)F_\xi(\nabla u_n)\cdot\nabla(u_n - u)\,dx \\
&\leq \int_\Omega F^{p-1}(\nabla u_n)F_\xi(\nabla u_n)\cdot\nabla(u_n - u)\,dx \\
&\qquad - \mu\int_\Omega \Big(F^{q-1}(\nabla u_n) - F^{q-1}(\nabla u_n)\Big)F_\xi(\nabla u_n)\cdot\nabla(u_n - u)\,dx \\
&= \int_\Omega \Big(F^{p-1}(\nabla u_n) - \mu F^{q-1}(\nabla u_n)\Big)F_\xi(\nabla u_n)\cdot\nabla(u_n - u)\,dx \\
&\qquad + \mu\int_\Omega F^{q-1}(\nabla u_n)F_\xi(\nabla u_n)\cdot\nabla(u_n - u)\,dx
\end{aligned}$$

by the monotonicity of $-Q_q$ and so

$$\limsup_{n\to\infty}\int_\Omega F^{p-1}(\nabla u_n)F_\xi(\nabla u_n)\cdot\nabla(u_n - u)\,dx \leq 0.$$

Then $u_n \to u$ strongly in $W^{1,p}(\Omega)$ (see, e.g., [12, Proposition 2.72]). The continuity of A (Lemma 3.1), shows $A(u_n) \to A(u)$ in $W^{-1,p'}(\Omega)$ and condition (II) of Definition 2.10, shows $A(u) = 0$. This shows that

$$\int_\Omega F^{p-1}(\nabla u)F_\xi(\nabla u) \cdot \nabla v \, dx = \int_\Omega f(x, \phi \star u(x), \nabla(\phi \star u(x)))v(x)dx$$

for all $v \in W_0^{1,p}(\Omega)$, which means u is a weak solution to problem (1). □

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