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Existence of solutions for a class of first order fuzzy dynamic equations on time scales

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Abstract. In this paper we investigate a class of first order fuzzy dynamic equations on arbitrary time scales for existence of solutions. We give conditions under which the considered equations have at least one and at least two solutions. To prove our main results we propose a new approach based upon recent theoretical results.

1. Introduction

The theory of dynamic equations has many interesting applications in control theory, mathematical economics, mathematical biology, engineering and technology. In some cases, there exists uncertainty, ambiguity or vague factors in such problems, and fuzzy theory and interval analysis are powerful tools for modeling these equations on time scales.

In this paper, we investigate the following class of first order fuzzy dynamic equations

$$\delta_H y = f(t, y), \quad t \in (t_0, T], \tag{1}$$

$$y(t_0) = y_0, \tag{2}$$

where

(A1) $f \in C([t_0, T] \times F(\mathbb{R}))$, $f : [t_0, T] \times F(\mathbb{R}) \to F(\mathbb{R})$, $y_0 \in F(\mathbb{R})$, $y_0 \geq \widetilde{0}$, $D(y_0, \widetilde{0}) \leq B$ for some nonnegative constant B, $t_0, T \in \mathbb{T}$, $\mathbb{T}$ is an arbitrary time scale with forward jump operator and delta differentiation operator σ and Δ, respectively.

(A2) $D(f(t, y(t)), \widetilde{0}) \leq B$ for any $t \in [t_0, T]$ and for any $y \in C_{frd}([t_0, T])$.

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Here $F(\mathbb{R})$ denotes the set of all real fuzzy numbers, $\widetilde{0}$ denotes the zero fuzzy number and δ_H denotes the first type fuzzy delta derivative on $\mathbb{T}$. With $D(\cdot,\cdot)$ it is denoted the Hausdorff distance between real fuzzy numbers.

The ordinary dynamic equations (ODE) and partial dynamic equations (PDEs) have been studied in depth for existence of solutions, uniqueness of solutions, nonuniqueness of solutions, oscillations of solutions (see [2], [4], [12] and references therein) and such concepts as boundary value problems (BVPs), initial value prob lems (IVPs), and differential operators in general are the current focus in the papers being written and published in the area.

In this paper, we will investigate the problem (1), (2) for existence of at least one solution and existence of at least two nonnegative solutions. To the best of our knowledge, there is a gap in the references for investigations of existence and nonuniqueness of the solutions of nonlinear fuzzy dynamic equations on time scales. Here, in this paper we try to fill out this gap.

This paper is organized as follows. In the next section, we give some basic definitions and facts of fuzzy dynamic calculus on time scales. In Section 3, we give some auxiliary results connected with some fixed point theorems used in this paper. In Section 4we prove existence of at least one solution for the problem (1), (2). In Section 5, we prove existence of at least two nonnegative solutions. In Section 6, we give an example to illustrate our main results.

2. Fuzzy Dynamic Calculus Essentials

In this section, we will give some basic definitions and fact of fuzzy dynamic calculus on time scales. For detailed study of fuzzy dynamic calculus on time scales we refer the reader to the book [9].

Suppose that $\mathbb{T}$ is a time scale with forward jump operator and delta differentiation operator σ and Δ, respectively. With $F(\mathbb{R})$ we will denote the space of the real fuzzy numbers and with $D(\cdot,\cdot)$ we will denote the Hausdorff distance between the real fuzzy numbers. For more details for fuzzy numbers and Hausdorff distance between the real fuzzy numbers we refer the reader to the appendix of the book [9].

Definition 2.1. *([9]) Assume that $f : \mathbb{T} \to F(\mathbb{R})$ is a fuzzy function and $t \in \mathbb{T}^\kappa$. Then f is said to be first type right fuzzy delta differentiable at t, shortly right δ_H-differentiable at t, if there exists an element $\delta_H^+ f(t) \in F(\mathbb{R})$ with the property that, for any given $\epsilon > 0$, there exists a neighbourhood $U_{\mathbb{T}}$ of t, i.e., $U_{\mathbb{T}} = (t-\delta, t+\delta) \bigcap \mathbb{T}$ for some $\delta > 0$, such that for all $t+h \in U_{\mathbb{T}}$ the H-difference $f(t+h) \ominus_H f(\sigma(t))$ exists and*

$$D\Big(f(t+h) \ominus_H f(\sigma(t)), \delta_H^+ f(t)(h-\mu(t))\Big) \le \epsilon|h-\mu(t)|$$

with $0 \le h < \delta$. In this case, $\delta_H^+ f(t)$ is said to be first type right fuzzy delta derivative of f at t, shortly right δ_H-derivative of f at t.

Definition 2.2. *([9]) Assume that $f : \mathbb{T} \to F(\mathbb{R})$ is a fuzzy function and $t \in \mathbb{T}^\kappa$. Then f is said to be first type left fuzzy delta differentiable at t, shortly left δ_H-differentiable at t, if there exists an element $\delta_H^- f(t) \in F(\mathbb{R})$ with the property that, for any given $\epsilon > 0$, there exists a neighbourhood $U_{\mathbb{T}}$ of t, i.e., $U_{\mathbb{T}} = (t-\delta, t+\delta) \bigcap \mathbb{T}$ for some $\delta > 0$, such that for all $t-h \in U_{\mathbb{T}}$ the H-difference $f(\sigma(t)) \ominus_H f(t-h)$ exists and*

$$D\Big(f(\sigma(t)) \ominus_H f(t-h), \delta_H^- f(t)(h+\mu(t))\Big) \le \epsilon(h+\mu(t))$$

with $0 \le h < \delta$.

Definition 2.3. *([9]) Let $f : \mathbb{T} \to F(\mathbb{R})$ be a fuzzy function and $t \in \mathbb{T}^\kappa$. Then f is said to be first type fuzzy delta differentiable at t, shortly δ_H-differentiable at t, if f is both first type left and right fuzzy delta differentiable at $t \in \mathbb{T}^\kappa$ and $\delta_H^- f(t) = \delta_H^+ f(t)$, and we will denote it by $\delta_H f(t)$. We call $\delta_H f(t)$ the first type fuzzy delta derivative of f at t, shortly δ_H-derivative of f at t. We say that f is first type fuzzy delta differentiable at t, shortly δ_H-differentiable at t, if its δ_H-derivative exists at t. We say that f is first type fuzzy delta differentiable on $\mathbb{T}^\kappa$, shortly δ_H-differentiable on $\mathbb{T}^\kappa$, if its δ_H-derivative exists at each $t \in \mathbb{T}^\kappa$. The fuzzy function $\delta_H f : \mathbb{T}^\kappa \to F(\mathbb{R})$ is then called first type fuzzy delta derivative, shortly δ_H-derivative of f on $\mathbb{T}^\kappa$.*

The defined δ_H-derivative has the following properties.

Theorem 2.4. *([9]) If the δ_H-derivative of f at $t \in \mathbb{T}^\kappa$ exists, then it is unique. Hence, δ_H-derivative is well-defined.*

Theorem 2.5. *([9]) Assume that $f : \mathbb{T} \to F(\mathbb{R})$ is a continuous function at $t_1 \in \mathbb{T}^\kappa$ and t_1 is right-scattered. Then f is δ_H-differentiable at t_1 and*

$$\delta_H f(t_1) = \frac{f(\sigma(t_1)) \ominus_H f(t_1)}{\mu(t_1)}.$$

Theorem 2.6. *([9]) Assume that $f : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at $t \in \mathbb{T}^\kappa$. Then f is continuous at t.*

Theorem 2.7. *([9]) Let $f : \mathbb{T} \to F(\mathbb{R})$ be a fuzzy function and let $t \in \mathbb{T}^\kappa$ be right-dense. Then f is δ_H-differentiable at t if and only if the limits*

$$\lim_{h\to 0+} \frac{f(t+h) \ominus_H f(t)}{h} \quad \text{and} \quad \lim_{h\to 0+} \frac{f(t) \ominus_H f(t-h)}{h} \tag{3}$$

exist and satisfy the relations

$$\lim_{h\to 0+} \frac{f(t+h) \ominus_H f(t)}{h} = \lim_{h\to 0+} \frac{f(t) \ominus_H f(t-h)}{h} = \delta_H f(t).$$

Theorem 2.8. *([9]) Let $f : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at $t \in \mathbb{T}^\kappa$. Then*

$$f(\sigma(t)) = f(t) + \mu(t) \cdot \delta_H f(t)$$

or

$$f(t) = f(\sigma(t)) + (-1) \cdot (\mu(t) \cdot \delta_H f(t)) .$$

Theorem 2.9. *([9]) Let $f, g : \mathbb{T} \to F(\mathbb{R})$ be δ_H-differentiable at $t \in \mathbb{T}^\kappa$. Then $f + g : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at $t \in \mathbb{T}^\kappa$ and*

$$\delta_H(f + g)(t) = \delta_H f(t) + \delta_H g(t).$$

Theorem 2.10. *([9]) Let $f : \mathbb{T} \to F(\mathbb{R})$ be δ_H-differentiable at $t \in \mathbb{T}^\kappa$. Then for any $\lambda \in \mathbb{R}$ the function $\lambda \cdot f : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at $t \in \mathbb{T}^\kappa$ and*

$$\delta_H(\lambda \cdot f)(t) = \lambda \cdot \delta_H f(t).$$

Theorem 2.11. *([9]) Let $t \in \mathbb{T}^\kappa$, $f : \mathbb{T} \to F(\mathbb{R})$ and $f_\alpha(t) = [f(t)]^\alpha$, $\alpha \in [0, 1]$. If f is δ_H-differentiable at t, then f_α is δ_H-differentiable at t and*

$$\delta_H[f(t)]^\alpha = \delta_H f_\alpha(t), \quad \alpha \in [0, 1].$$

Theorem 2.12. *([9]) Let $t \in \mathbb{T}^\kappa$, $f : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at t. Let also,*

$$[f(t)]^\alpha = \left[\underline{f}^\alpha(t), \overline{f}^\alpha(t)\right], \quad \alpha \in [0, 1].$$

Then $\underline{f}^\alpha$ and $\overline{f}^\alpha$ are Δ-differentiable at t and

$$[\delta_H f(t)]^\alpha = \left[\underline{f}^{\alpha\Delta}(t), \overline{f}^{\alpha\Delta}(t)\right], \quad \alpha \in [0, 1].$$

Theorem 2.13. *([9]) Let $t \in \mathbb{T}^\kappa$ and f, g be δ_H-differentiable at t, and*

$$I^{\alpha,1}_{f,g}(t) \le 0, \quad I^{\alpha,1}_{f^\sigma,g^\Delta}(t) \le 0, \quad I^{\alpha,1}_{f^\Delta,g}(t) \le 0,$$

$f \circ g$ is δ_H-differentiable at t. Then

$$\delta_H(f \circ g)(t) = f^\sigma(t) \circ \delta_H g(t) + \delta_H f(t) \circ g(t). \tag{4}$$

Theorem 2.14. *([9]) Let $t \in \mathbb{T}^\kappa$ and f, g be δ_H-differentiable at t, and*

$$I^{\alpha,1}_{f,g}(t) \ge 0, \quad I^{\alpha,1}_{f^\sigma,\delta_H g}(t) \ge 0, \quad I^{\alpha,1}_{\delta_H f,g}(t) \ge 0,$$

$f \circ g$ is δ_H-differentiable at t. Then

$$\delta_H(f \circ g)(t) = f^\sigma(t) \circ \delta_H g(t) + \delta_H f(t) \circ g(t).$$

Theorem 2.15. *([9]) Let $t \in \mathbb{T}^\kappa$ and f, g be δ_H-differentiable at t, and*

$$I^{\alpha,2}_{f,g}(t) \ge 0, \quad I^{\alpha,2}_{f^\sigma,\delta_H g}(t) \ge 0, \quad I^{\alpha,2}_{\delta_H f,g}(t) \ge 0,$$

$f \circledcirc g$ is δ_H-differentiable at t. Then

$$\delta_H(f \circledcirc g)(t) = f^\sigma(t) \circledcirc \delta_H g(t) + \delta_H f(t) \circledcirc g(t).$$

Now, we introduce the conception for the first type fuzzy delta integration on time scales. Let $I \subset \mathbb{T}$.

Definition 2.16. *([9]) A function $f : \mathbb{T} \to \mathbb{R}$ is called a sector of the fuzzy function $F : I \to F(\mathbb{R})$ if $f(t) \in F(t)$ for all $t \in I$. The set of all rd-continuous sectors of F on I is denoted by $S_{HF}(I)$.*

Theorem 2.17. *([9]) Let $t_0, T \in \mathbb{T}$, $t_0 < T$, $F, G : [t_0, T] \to F(\mathbb{R})$ be δ_H-integrable. Then $F + G : [t_0, T] \to F(\mathbb{R})$ is δ_H-integrable and*

$$\int_{t_0}^{T} (F(s) + G(s))\delta_H s = \int_{t_0}^{T} F(s)\delta_H s + \int_{t_0}^{T} G(s)\delta_H s. \tag{5}$$

Theorem 2.18. *([9]) Let $t_0, T \in \mathbb{T}$, $t_0 < T$, $F : [t_0, T] \to F(\mathbb{R})$ be δ_H-integrable. Then $\lambda \cdot F : [t_0, T] \to F(\mathbb{R})$ is δ_H-integrable and*

$$\int_{t_0}^{T} \lambda \cdot F(s)\delta_H s = \lambda \cdot \int_{t_0}^{T} F(s)\delta_H s$$

for any $\lambda \in \mathbb{R}$.

Theorem 2.19. *([9]) Let $t_0, T \in \mathbb{T}$, $t_0 < T$, and $F : [t_0, T] \to F(\mathbb{R})$ be δ_H-integrable. Then*

$$\int_{t_0}^{T} F(s)\delta_H s = \int_{t_0}^{t} F(s)\delta_H s + \int_{t}^{T} F(s)\delta_H s$$

for any $t \in [t_0, T]$.

Theorem 2.20. *([9]) Let $t_0, T \in \mathbb{T}$, $t_0 < T$, $F : [t_0, T] \to F(\mathbb{R})$ is rd-continuous. If $X_0 \in F(\mathbb{R})$ and*

$$f(t) = X_0 + \int_{t_0}^{t} F(s)\delta_H s, \quad t \in [t_0, T],$$

then f is δ_H-differentiable and

$$\delta_H f(t) = F(t), \quad t \in [t_0, T].$$

Theorem 2.21. *([9]) If $f : [a,b] \to F(\mathbb{R})$ is δ_H-differentiable on $[a,b]$, then*

$$\int_a^b \delta_H f(t) = f(b) \ominus_H f(a).$$

Theorem 2.22. *([9]) Let $f : [a,b] \to F(\mathbb{R})$ be δ_H-integrable. Then*

$$\int_a^b f(t)\delta_H t = (-1)\cdot \int_b^a f(t)\delta_H t.$$

Theorem 2.23 (Integration by Parts)**.** *([9]) Let $f, g : [a,b] \to F(\mathbb{R})$ be δ_H-differentiable and $f \circ g$ is also δ_H-differentiable on $[a,b]$. If*

$$I^{\alpha,1}_{f,g}(t) \leq 0, \quad I^{\alpha,1}_{f^\sigma,\delta_H g}(t) \leq 0, \quad I^{\alpha,1}_{\delta_H f,g}(t) \leq 0, \quad t \in [a,b]^\kappa, \tag{6}$$

then

$$\int_a^b f(\sigma(t)) \circ \delta_H g(t)\delta_H t \quad = \quad (f(b) \circ g(b)) \ominus_H (f(a) \circ g(a)) \ominus_H \int_a^b \delta_H f(t) \circ g(t)\delta_H t. \tag{7}$$

3. Preliminary Results

Below, assume that X is a real Banach space. Now, we will recall the definitions of compact and completely continuous mappings in Banach spaces.

Definition 3.1. *Let $K : M \subset X \to X$ be a map. We say that K is compact if $K(M)$ is contained in a compact subset of X. K is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.*

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 3.2. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0,\infty)$ is defined by*

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad diam(Y_j) \leq \delta, \quad j \in \{1,\ldots,m\}\right\},$$

where $diam(Y_j) = \sup\{\|x-y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1,\ldots,m\}$.

For the main properties of measure of noncompactness we refer the reader to [5].

Definition 3.3. *A mapping $K : X \to X$ is said to be k-set contraction if there exists a constant $k \geq 0$ such that*

$$\alpha(K(Y)) \leq k\alpha(Y)$$

for any bounded set $Y \subset X$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction(see [7]).

Proposition 3.4. *(Leray-Schauder nonlinear alternative [1]) Let C be a convex, closed subset of a Banach space E, $0 \in U \subset C$ where U is an open set. Let $f\colon \overline{U} \to C$ be a continuous, compact map. Then*

(a) *either f has a fixed point in $\overline{U}$,*

(b) *or there exist $x \in \partial U$, and $\lambda \in (0,1)$ such that $x = \lambda f(x)$.*

To prove our existence result we will use the following fixed point theorem which is a consequence of Proposition 3.4 (see [3], [8], [10] and references therein).

Theorem 3.5. *Let E be a Banach space, Y a closed, convex subset of E, U be any open subset of Y with $0 \in U$. Consider two operators T and S, where*

$$Tx = \varepsilon\, x, \ x \in \overline{U},$$

for $\varepsilon > 0$ and $S : \overline{U} \to E$ be such that

(i) *$I - S : \overline{U} \to Y$ continuous, compact and*

(ii) $\{x \in \overline{U} : x = \lambda(I - S)x, \quad x \in \partial U\} = \emptyset, \ for any\ \lambda \in \left(0, \frac{1}{\varepsilon}\right)$.

Then there exists $x^* \in \overline{U}$ such that

$$Tx^* + Sx^* = x^*.$$

Proof. We have that the operator $\frac{1}{\varepsilon}(I - S) : \overline{U} \to Y$ is continuous and compact. Suppose that there exist $x_0 \in \partial U$ and $\mu_0 \in (0, 1)$ such that

$$x_0 = \mu_0 \frac{1}{\varepsilon}(I - S)x_0,$$

that is

$$x_0 = \lambda_0 (I - S)x_0$$

where $\lambda_0 = \mu_0 \frac{1}{\varepsilon} \in \left(0, \frac{1}{\varepsilon}\right)$. This contradicts the condition (ii). From Leray-Schauder nonlinear alternative, it follows that there exists $x^* \in \overline{U}$ so that

$$x^* = \frac{1}{\varepsilon}(I - S)x^*$$

or

$$\varepsilon\, x^* + Sx^* = x^*,$$

or

$$Tx^* + Sx^* = x^*.$$

□

Definition 3.6. *Let X and Y be real Banach spaces. A map $K : X \to Y$ is called expansive if there exists a constant $h > 1$ for which one has the following inequality*

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

Now, we will recall the definition for a cone in a Banach space.

Definition 3.7. *A closed, convex set $\mathcal{P}$ in X is said to be cone if*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,*
2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$. The next result is a fixed point theorem which we will use to prove existence of at least two nonnegative global classical solutions of the IVP (1). For its proof, we refer the reader to [6] and [11].

Theorem 3.8. *Let $\mathcal{P}$ be a cone of a Banach space E; Ω a subset of $\mathcal{P}$ and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping, $S : \overline{U}_3 \to E$ is a completely continuous map and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$, $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(i) $Sx \neq (I - T)(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,

(ii) *there exists* $\epsilon \geq 0$ *such that* $Sx \neq (I - T)(\lambda x)$, *for all* $\lambda \geq 1 + \epsilon$, $x \in \partial U_2$ *and* $\lambda x \in \Omega$,

(iii) $Sx \neq (I - T)(x - \lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus U_1) \cap \Omega \text{ and } x_2 \in (\overline{U}_3 \setminus \overline{U}_2) \cap \Omega.$$

4. Existence of at Least One Solution

In the book [9], it is shown that the IVP (1), (2) is equivalent to the following integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, \xi(s))\delta_H s, \quad t \in [t_0, T].$$

In $X = C_{frd}([t_0, T])$, we introduce the norm

$$\|y\| = \sup_{t \in [t_0, T]} D(y(t), \widetilde{0}), \quad y \in C_{frd}([t_0, T]),$$

provided it exists. For $u \in X$, define the operator

$$S_1 u(t) = u(t) - y_0 - \int_{t_0}^{t} f(s, \xi(s))\delta_H s, \quad t \in [t_0, T].$$

Note, that if $u \in X$ satisfies the equation

$$S_1 u(t) = 0, \quad t \in [t_0, T],$$

then u is a solution to the IVP (1), (2). Let

$$B_1 = B(T - t_0 + 2).$$

Lemma 4.1. *Suppose $(A1)$ and $(A2)$ hold. If $u \in X$, $\|u\| \leq B$, then*

$$D(S_1 u(t), \widetilde{0}) \leq B_1, \quad t \in [t_0, T].$$

Proof. We have

$$\begin{aligned} D(S_1u(t),\widetilde{0}) &= D\left(u(t)-y_0-\int_{t_0}^{t} f(s,\xi(s))\delta_H s,\widetilde{0}\right) \\ &\le D(u(t),\widetilde{0})+D(y_0,\widetilde{0})+D\left(\int_{t_0}^{t} f(s,\xi(s))\delta_H s,\widetilde{0}\right) \\ &\le 2B+\int_{t_0}^{t} D(f(s,\xi(s)),\widetilde{0})\delta_H s \\ &\le 2B+B(T-t_0) \\ &= B(2+T-t_0) \\ &= B_1, \quad t\in[t_0,T]. \end{aligned}$$

This completes the proof. □

In addition, we suppose

(A3) there exists a positive constant A_1 such that $A_1(T-t_0)B_1 < B$.

Set $A = A_1(T-t_0)$. For $u \in X$, define the operator

$$S_2u(t) = A_1\int_{t_0}^{t} S_1u(s)\delta_H s, \quad t\in[t_0,T]. \tag{8}$$

Lemma 4.2. *Suppose* $(A1)$ -$(A3)$ *hold. If* $u\in X$ *and* $\|u\|\le B$*, then*

$$\|S_2u\| \le AB_1. \tag{9}$$

Proof. By Lemma 4.1, we have that

$$\|S_1u\|\le B_1.$$

Hence,

$$\begin{aligned} D(S_2u(t),\widetilde{0}) &= D\left(A_1\int_{t_0}^{t} S_1u(s)\delta_H s,\widetilde{0}\right) \\ &\le A_1\int_{t_0}^{t} \|S_1u\|\delta_H s \\ &\le A_1(T-t_0)B_1 \\ &= AB_1, \quad t\in[t_0,T]. \end{aligned}$$

Hence, we get (9). This completes the proof. □

Lemma 4.3. *Suppose* $(A1)$-$(A3)$ *hold. If* $u \in X$ *satisfies the equation*

$$S_2u(t) = C, \quad t \in [t_0, T], \tag{10}$$

for some nonnegative constant C, *then* u *is a solution to the IVP* (1), (2).

Proof. We δ_H-differentiate the equation (10) and we get

$$A_1S_1u(t) = 0, \quad t \in [t_0, T],$$

whereupon we find

$$S_1u(t) = 0, \quad t \in [t_0, T].$$

Therefore u is a solution to the IVP (1), (2). This completes the proof. □

Our main result in this section is as follows.

Theorem 4.4. *Suppose* $(A1)$-$(A3)$. *Then the IVP* (1), (2) *has at least one solution in* X.

Proof. Let $\widetilde{Y}$ denote the set of all equi-continuous families in X with respect to the norm $\|\cdot\|$. Let also, $Y = \overline{\overline{\widetilde{Y}}}$ and

$$U \quad = \quad \left\{u \in Y : \|u\| < B \quad \text{and if}' \quad \|u\| \geq \frac{B}{2}, \quad \text{then} \quad u(t_0) > \frac{B}{2}\right\}.$$

For $u \in \overline{U}$ and $\epsilon > 0$, define the operators

$$Tu(t) \quad = \quad \epsilon u(t),$$

$$Su(t) \quad = \quad u(t) - \epsilon u(t) - \epsilon S_2u(t), \quad t \in [t_0, T].$$

For $u \in \overline{U}$, we have

$$\begin{aligned} \|(I - S)u\| &= \|\epsilon u + \epsilon S_2u\| \\ &\leq \epsilon\|u\| + \epsilon\|S_2u\| \\ &\leq \epsilon B_1 + \epsilon AB_1. \end{aligned}$$

Thus, $S : \overline{U} \to X$ is continuous and $(I - S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $u \in \partial U$ so that

$$u = \lambda(I - S)u$$

or

$$u = \lambda\epsilon\,(u + S_2u), \tag{11}$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, using that $S_2u(t_0) = 0$ and $\|u\| \geq \frac{B}{2}$, we get $u(t_0) > \frac{B}{2}$ and

$$u(t_0) = \lambda\epsilon(u(t_0) + S_2u(t_0)) = \lambda\epsilon u(t_0),$$

whereupon $\lambda\epsilon = 1$, which is a contradiction. Consequently

$$\{u \in \overline{U} : u = \lambda_1(I - S)u,\ u \in \partial U\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 3.5, it follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore

$$\begin{aligned} u^*(t) &= Tu^*(t) + Su^*(t) \\ &= \epsilon u^*(t) + u^*(t) - \epsilon u^*(t) - \epsilon S_2 u^*(t), \quad t \in [t_0, T], \end{aligned}$$

whereupon

$$S_2 u^*(t) = 0, \quad t \in [t_0, T].$$

From here, using Lemma 4.3, we conclude that u^* is a solution to the problem (1), (2). This completes the proof. □

5. Existence of at Least Two Solutions

Assume that the constants B and A which appear in the conditions $(A1)$ and $(A3)$, respectively, satisfy the following inequalities:

(A4) $AB_1 < \frac{L}{5}$, where L is a positive constant that satisfies the following conditions:

$$r < L < R_1 \leq B,$$

with r and R_1 are positive constants.

Our main result in this section is as follows.

Theorem 5.1. *Suppose that* $(A1)$-$(A4)$ *hold. Then the problem* (1), (2) *has at least two nonnegative solutions in* X.

Proof. Let

$$\widetilde{P} = \{u \in X : u \geq 0 \quad \text{on} \quad [t_0, T]\}.$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\widetilde{P}$. For $v \in X$, define the operators

$$\begin{aligned} T_1 v(t) &= (1 + m\epsilon)v(t) - \epsilon \frac{L}{10}, \\ S_3 v(t) &= -\epsilon S_2 v(t) - m\epsilon v(t) - \epsilon \frac{L}{10}, (t) \in [t_0, T], \end{aligned}$$

where ϵ is a positive constant, $m > 0$ is large enough and the operator S_2 is given by formula (8). Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1), (2). Define

$$\begin{aligned} \Omega &= \mathcal{P}, \\ U_1 &= \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \\ U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have

   $$\|T_1 v_1 - T_1 v_2\| = (1 + m\epsilon)\|v_1 - v_2\|,$$

   whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

2. For $v \in \overline{\mathcal{P}}_{R_1}$, we get

$$\begin{aligned} \|S_3 v\| &\leq \epsilon \|S_2 v\| + m\epsilon \|v\| + \epsilon \frac{L}{10} \\ &\leq \epsilon \Big(AB_1 + mR_1 + \frac{L}{10} \Big). \end{aligned}$$

Therefore $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is completely continuous.

3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m} S_2 v_1 + \frac{L}{5m}.$$

Note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $[t_0, T]$. We have $v_2 \geq 0$ on $[t_0, T]$. Therefore $v_2 \in \Omega$ and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon S_2 v_1 - \epsilon \frac{L}{10} - \epsilon \frac{L}{10}$$

or

$$\begin{aligned} (I - T_1) v_2 &= -\epsilon m v_2 + \epsilon \frac{L}{10} \\ &= S_3 v_1. \end{aligned}$$

Consequently $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.

4. Assume that for any $v_0 \in \mathcal{P}^*$ there exist $\lambda \geq 0$ and $v \in \partial \mathcal{P}_r \cap (\Omega + \lambda v_0)$ or $v \in \partial \mathcal{P}_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$S_3 v = (I - T_1)(v - \lambda v_0).$$

Then

$$-\epsilon S_2 v - m\epsilon v - \epsilon \frac{L}{10} = -m\epsilon (v - \lambda v_0) + \epsilon \frac{L}{10}$$

or

$$-S_2 v = \lambda m v_0 + \frac{L}{5}.$$

Hence,

$$\|S_2 v\| = \left\| \lambda m v_0 + \frac{L}{5} \right\| \geq \frac{L}{5}.$$

This is a contradiction.

5. Let $\epsilon_1 = \frac{2}{5m}$. Suppose that there exist a $v_1 \in \partial \mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that

$$S_3 v_1 = (I - T_1)(\lambda_1 v_1). \qquad (12)$$

Moreover,

$$-\epsilon S_2 v_1 - m\epsilon v_1 - \epsilon \frac{L}{10} = -\lambda_1 m\epsilon v_1 + \epsilon \frac{L}{10},$$

or

$$S_2 v_1 + \frac{L}{5} = (\lambda_1 - 1) m v_1.$$

From here,

$$2\frac{L}{5} > \left\| S_2 v_1 + \frac{L}{5} \right\| = (\lambda_1 - 1)m\|v_1\| = (\lambda_1 - 1)mL$$

and

$$\frac{2}{5m} + 1 > \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 3.8 hold. Hence, the problem (1), (2) has at least two solutions u_1 and u_2 so that

$$\|u_1\| = L < \|u_2\| < R_1$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

□

6. An Example

Below, we will illustrate our main results. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $t_0 = 1$, $T = 16$, $B = 1000$ and

$$R_1 = \frac{9}{10}, \quad L = \frac{3}{5}, \quad r = \frac{2}{5}, \quad m = 10^{50}, \quad A = \frac{1}{10B_1}.$$

Then

$$B_1 \quad = \quad 1000(2 + 16 - 1) = 17000.$$

Next,

$$r < L < R_1 < B, \quad AB_1 < \frac{L}{5}.$$

i.e., $(A3)$ and $(A4)$ hold. Take

$$f(t,y) = \left(\frac{h_1(t)\left(y - \frac{1}{2} - \frac{1}{2}\alpha\right)^{\frac{1}{3}}}{1 + \left(y - \frac{1}{2} - \frac{1}{2}\alpha\right)^{\frac{2}{3}}}, \frac{h_2(t)\left(y - \frac{3}{2} + \frac{1}{2}\alpha\right)}{1 + \left(y - \frac{3}{2} + \frac{1}{2}\alpha\right)^2} \right), \quad t \in [1,16], \quad \alpha \in [0,1],$$

where

$$h_1(t) \quad = \quad \begin{cases} 7t(1 + t^2), & t \in [1,4], \\ 0, & t \in [8,16], \end{cases}$$

$$h_2(t) \quad = \quad \begin{cases} \frac{3(1+t^4)}{t}, & t \in [1,4], \\ 0, & t \in [8,16]. \end{cases}$$

We have that

$$D(f(t,y), \widetilde{0}) \le 1000, \quad t \in [t_0, T].$$

Next, let $y_0 = \left(\frac{1}{2}, 1, \frac{3}{2}\right)$. Then $(A1)$ and $(A2)$ hold. Therefore for the problem

$$\delta_H y = f(t, y), \quad t \in [t_0, T],$$
$$y(t_0) = \left(\frac{1}{2}, 1, \frac{3}{2}\right)$$

are fulfilled all conditions of Theorem 4.4 and Theorem 5.1. Two nonnegative solutions of the considered problem are as follows

$$y_1(t) = \left[\frac{1}{2} + \frac{1}{2}\alpha, \frac{3}{2} - \frac{1}{2}\alpha\right], \quad t \in [1, 16], \quad \alpha \in [0, 1],$$

and

$$y_2(t) = \begin{cases} \left[t^3 + \frac{1}{2} + \frac{1}{2}\alpha, t^2 + \frac{3}{2} - \frac{1}{2}\alpha\right], & t \in [1, 4], \quad \alpha \in [0, 1], \\ \left[\frac{1}{2} + \frac{1}{2}\alpha, \frac{3}{2} - \frac{1}{2}\alpha\right], & t \in [8, 16], \quad \alpha \in [0, 1]. \end{cases}$$

Then, we have

1. for $\alpha = \frac{1}{5}$

$$y_1(t) = \left[\frac{3}{5}, \frac{7}{5}\right], \quad t \in [1, 16],$$

and

$$y_2(t) = \begin{cases} \left[t^3 + \frac{3}{5}, t^2 + \frac{7}{5}\right], & t \in [1, 4], \\ \left[\frac{3}{5}, \frac{7}{5}\right], & t \in [8, 16]. \end{cases}$$

2. for $\alpha = \frac{1}{4}$

$$y_1(t) = \left[\frac{5}{8}, \frac{11}{8}\right], \quad t \in [1, 16],$$

and

$$y_2(t) = \begin{cases} \left[t^3 + \frac{5}{8}, t^2 + \frac{11}{8}\right], & t \in [1, 4], \\ \left[\frac{5}{8}, \frac{11}{8}\right], & t \in [8, 16]. \end{cases}$$

3. for $\alpha = \frac{3}{4}$

$$y_1(t) = \left[\frac{7}{8}, \frac{9}{8}\right], \quad t \in [1, 16],$$

and

$$y_2(t) = \begin{cases} \left[t^3 + \frac{7}{8}, t^2 + \frac{9}{8}\right], & t \in [1, 4], \\ \left[\frac{7}{8}, \frac{9}{8}\right], & t \in [8, 16]. \end{cases}$$

In the first four figures are shown y_1 for $\alpha = \frac{1}{5}$, $\alpha = \frac{1}{4}$, $\alpha = \frac{3}{4}$ and $\alpha = 1$, respectively. In the second four figures are shown y_2 for $\alpha = \frac{1}{5}$, $\alpha = \frac{1}{4}$, $\alpha = \frac{3}{4}$ and $\alpha = 1$, respectively.

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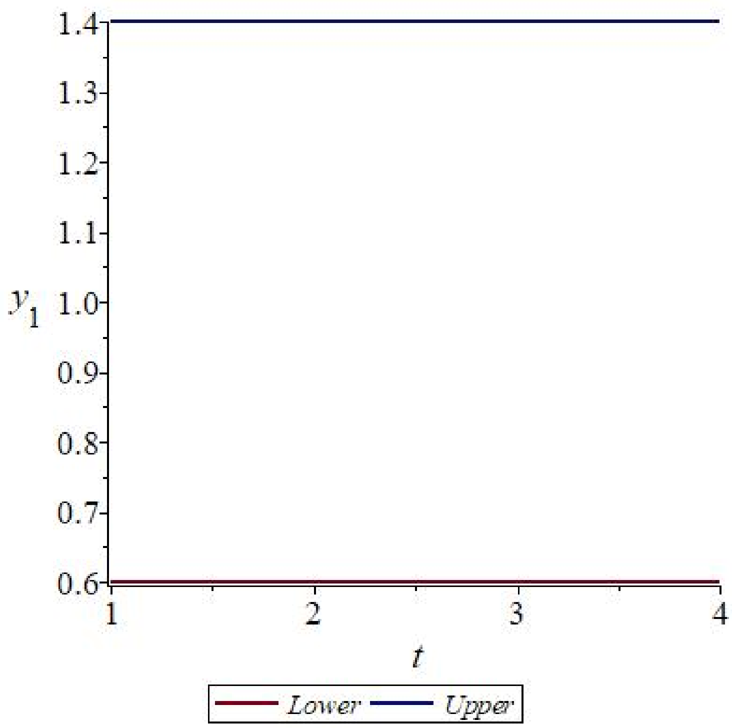

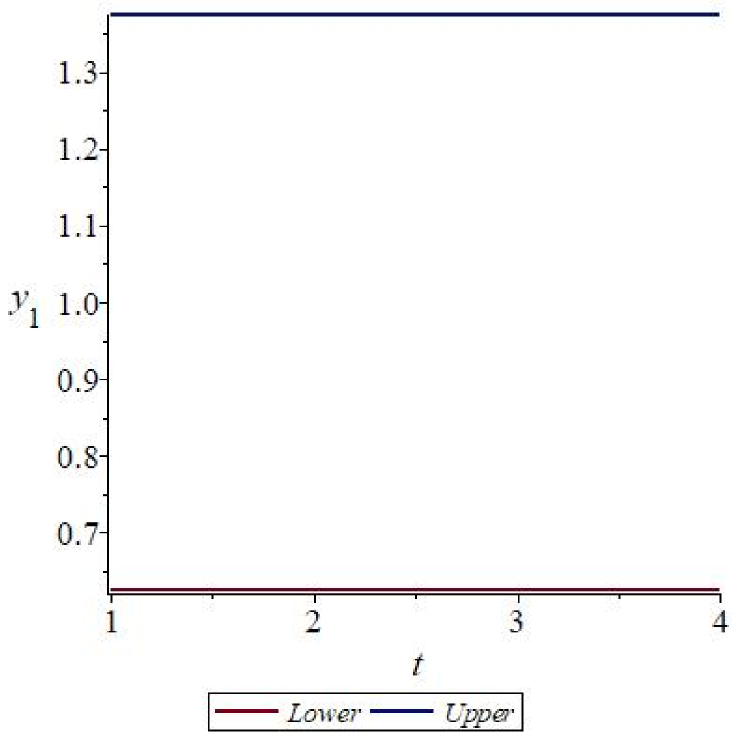
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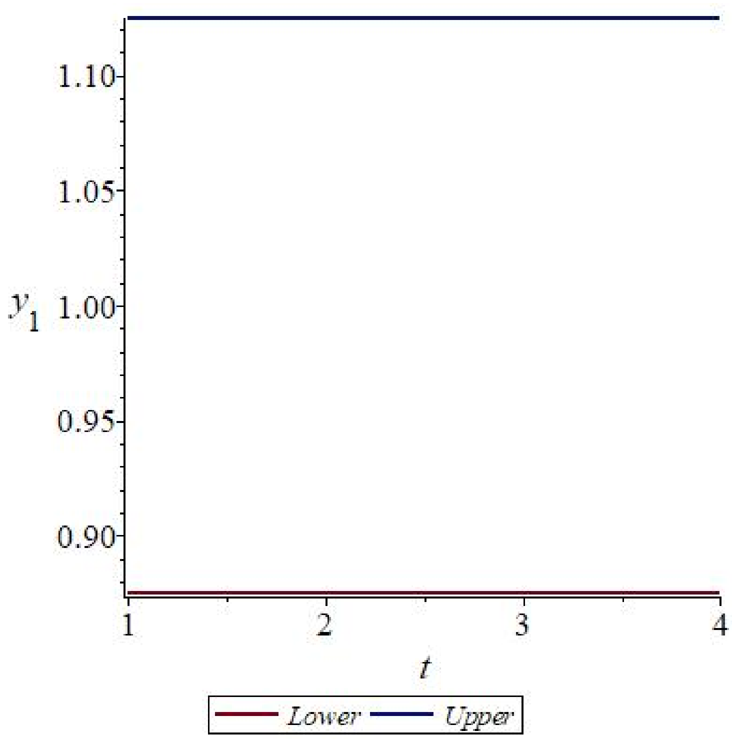
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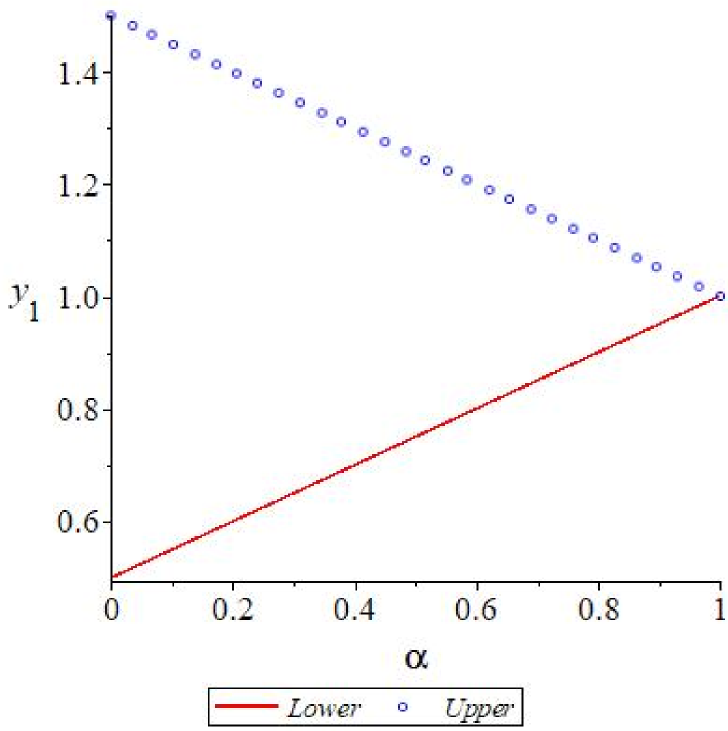
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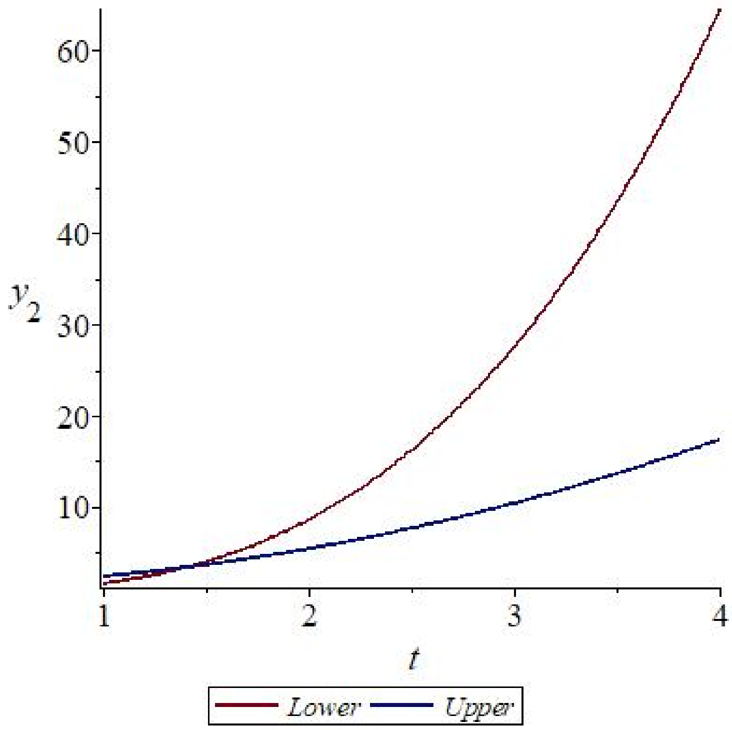
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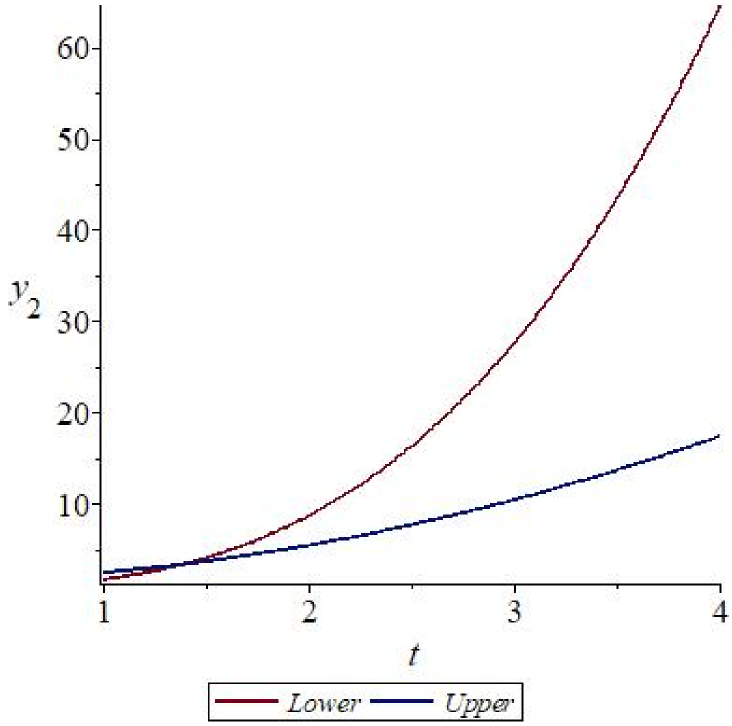
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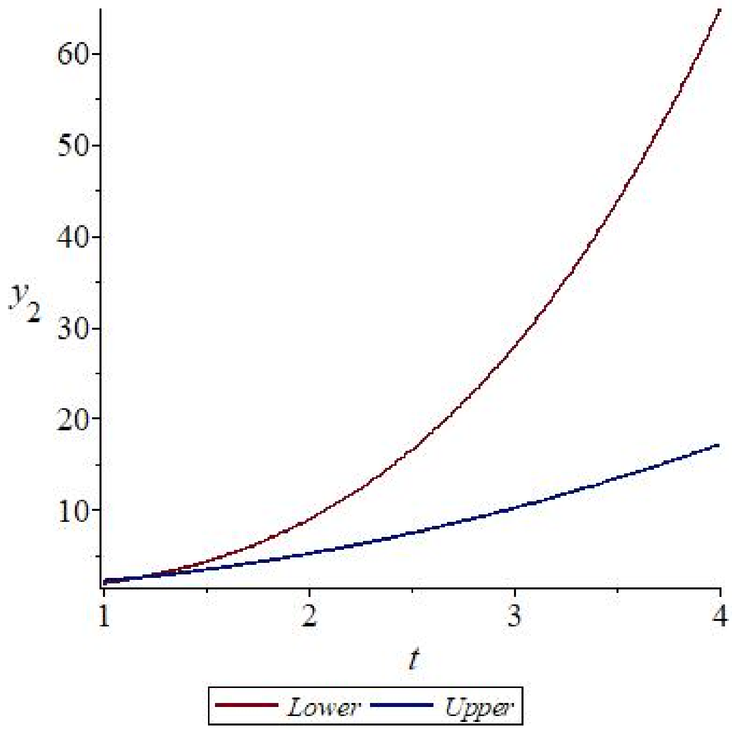
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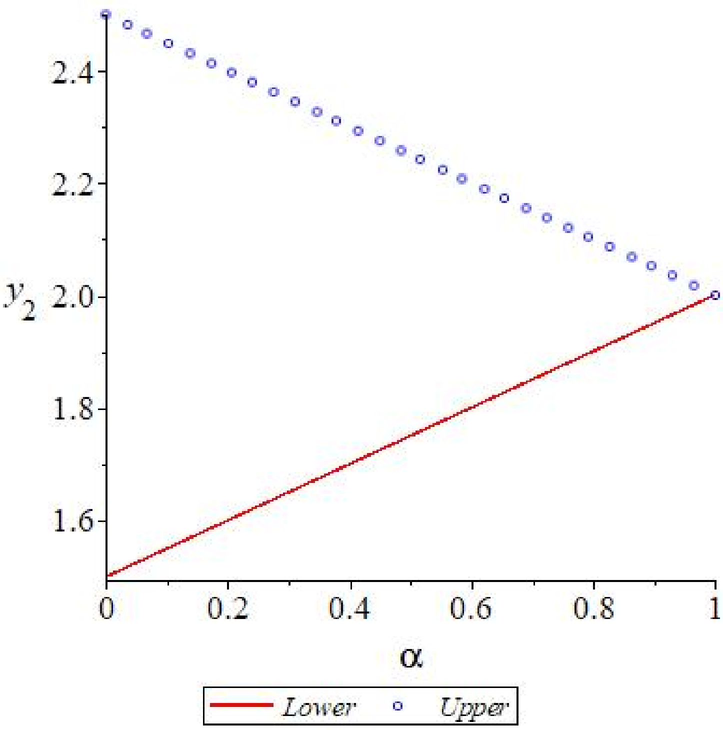
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