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Generating functions of generalized Gaussian polynomials

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Abstract. In this paper, we first define new generalization for Gaussian polynomials $\{GW_n(x)\}_{n\geq 0}$ and then we obtain the Binet's formula to find the n^{th} general term of generalized Gaussian polynomials $\{GW_n(x)\}_{n\geq 0}$. After that, the ordinary generating functions and the explicit formulas of generalized Gaussian polynomials and (p,q)-Fibonacci-like numbers are obtained. Considering the sequence of generalized Gaussian polynomials, we give Binet's formulas, explicit formulas and ordinary generating functions of Gaussian Pell and Gaussian Pell Lucas polynomials, Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials, Gaussian Fibonacci and Gaussian Lucas polynomials. Also, we present and prove certain ordinary generating functions for the products of (p,q)-Fibonacci-like numbers with these Gaussian polynomials and the products of (p,q)-Fibonacci-like numbers with Gaussian Fibonacci numbers, Gaussian Lucas numbers, Gaussian Jacobsthal numbers, Gaussian Jacobsthal Lucas numbers, Gaussian Pell numbers and Gaussian Pell Lucas numbers.

1. Introduction and preliminaries

In mathematics, orthogonal polynomials consist of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. The most widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases the Gegenbauer polynomials, the Chebyshev polynomials, and the Legendre polynomials, cf. [11, 12]. Recent works including the symmetric properties of some known special polynomials, e.g., Bernoulli polynomials, Euler polynomials, Genocchi polynomials and others, have been extensively investigated. For details (see [3, 24]).

The theory of derivatives and integrals of arbitrary real or complex order that is well know as fractional calculus came to the fore in the past decades. The paramount importance of this topic lied in the fact that its definite and prevalent applications were extended to describe several phenomena in different fields of applied science and engineering.

Fractional calculus is characterized by the presence of several different fractional defnitions. All of them include integral operators with different regularity properties. Among the well-known definitions

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that are widely employed in the context of scientific research and modelings, we distinguish the follow ing Riemann-Liouville's, Caputo's, Grünwald-Letnikov's, Hadamard definitions, ect.

Many problems are modelled by FDEs [16, 21]. Obtaining exact solutions for FDEs is not an easy task. Therefore, it is crucial to use numerical and analytical methods to attain approximate solutions for these equations. Spectral meth ods are among the most significant methods that are applied to solve FDEs since they enable to attain global solutions that are characterized by quick con vergence. For example, Multi-term fractional differential equations have been investigated in [2] using the collocation approach based on using GFPs. In [6, 7], the tau algorithm, which is based on using GFPs, has been used to evaluate a coupled system of Caputo fractional differential equations and solving homo geneous boundary value problem of fractional B-T differential equation. The author also in [38] has introduced new operational matrices of fractional deriva tives of Fermat and generalized Fibonacci polynomials to treat some types of fractional differential equation.

Hussain et al. [20] introduced a family of the Daftardar-Jeffery polynomials are incorporated in the homotopy of the optimal homotopy asymptotic method (OHAM) for solving the generalized Hirota-Satsuma coupled system of Korteweg-de Vries equations. In 2012, Chu et al. [14] defined the second dual form of the Hamy symmetric function $H_n(x, r)$, the purpose of this travail is to prove that $H_n^{**}(x, r)$ is Schur concave, and Schur multiplicatively and harmonic convex in $\mathbb{R}_+^n$, and presented some applications in inequalities. Chu et al. [15], by using the symmetric functions concept, they presented results about that both $F_n(x, r)$ and $\phi_n(x, r)$ are Schur multiplicative and harmonic convexities of the complete symmetric function.

There has been much work done in the study of bifurcation and chaos in some well-known generating functions; however, the bifurcation and chaotic behavior of generating functions associated with Chebyshev polynomials have not been thoroughly investigated. In fact, there are many distinct families of polynomials known as Chebyshev polynomials which can be applied in approximation theory, quadrature rules, etc. [12, 13]. The complex representation of these polynomials allows the derivation of many identities involving special generating functions of exponential, bilinear, and mixed type [10, 11]. It should be noted that most of the developed chaotic functions have been applied to generate pseudo-random noise generators (PRNG). Accordingly, chaos maps were implemented using both digital and analog electronics. Digital implementations consist of approximating the chaotic response of a system by solving ordinary differential equations [23]. In 2021, Louzzani et.al [23] looked for a novel chaos based generating function of the Chebyshev polynomials and its applications in image encryption.

For p and q positive real numbers, the (p, q) -Fibonacci-like numbers $\{S_{p,q,n}\}_{n\geq 0}$ can be defined as:

$$S_{p,q,n} = pS_{p,q,n-1} + qS_{p,q,n-2}, \text{ for } n \geq 2,$$

with the initial conditions $S_{p,q,0} = 2$ and $S_{p,q,1} = 2p$ (see [35]). Special cases of (p, q)-Fibonacci-like numbers are k -Fibonacci-like numbers $\{S_{k,1,n}\}_{n\geq 0} = \{S_{k,n}\}_{n\geq 0} = \{2, 2k, 2k^2 + 2, 2k^3 + 4k, 2k^4 + 6k^2 + 2, 2k^5 + 8k^3 + 6k, \ldots\}$ and Fibonacci-like numbers $\{S_{1,1,n}\}_{n\geq 0} = \{S_n\}_{n\geq 0} = \{2, 2, 4, 6, 10, 16, 26, \ldots\}$ (for more information of these numbers see the papers [26, 34]). The well-known Binet's formula for (p, q)-Fibonacci-like numbers is given by

$$S_{p,q,n} = 2\frac{t_1^{n+1} - t_2^{n+1}}{t_1 - t_2},$$

where

$$t_1 = \frac{p + \sqrt{p^2 + 4q}}{2} \text{ and } t_2 = \frac{p - \sqrt{p^2 + 4q}}{2},$$

are the roots of characteristic equation $t^2 - pt - q = 0$. Then, we have: $t_1 + t_2 = p$, $t_1t_2 = -q$ and $t_1 - t_2 = \sqrt{p^2 + 4q}$.

In literature, there have been so many studies of the sequences of Gaussian numbers. A Gaussian number z is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, a, $b \in \mathbb{Z}$.

The set of these numbers is denoted by $\mathbb{Z}[i]$. Gaussian numbers were investigated in 1832 by Gauss. In 1963, Horadam [19] examined Fibonacci numbers on the complex plane and established some interesting properties about them. Then, Jordan in [22] studied on Gaussian Fibonacci and Gaussian Lucas numbers. In 2013, the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers are defined and studied by Asci and Gurel in [4]. Next, Halici and Oz [18] introduced Gaussian Pell and Gaussian Pell Lucas numbers. In the same work, authors (Halici and Oz) gave ordinary generating function, Binet's formula and some important identities involving the Gaussian Pell and Gaussian Pell Lucas numbers. In the Table 1, we give the recurrence relations and ordinary generating functions of these Gaussian numbers.

Table 1: Recurrence relations and generating functions of some Gaussian numbers.

Gaussian numbers	**Recurrence relation**	**Ordinary generating function $g(z)$**
Gaussian Fibonacci GF_n	$\begin{cases} GF_0 = i,\ GF_1 = 1 \\ GF_n = GF_{n-1} + GF_{n-2},\ n \geq 2 \end{cases}$	$\dfrac{i + (1-i)z}{1 - z - z^2}$
Gaussian Lucas GL_n	$\begin{cases} GL_0 = 2 - i,\ GL_1 = 1 + 2i \\ GL_n = GL_{n-1} + GL_{n-2},\ n \geq 2 \end{cases}$	$\dfrac{2 - i + (3i-1)z}{1 - z - z^2}$
Gaussian Jacobsthal GJ_n	$\begin{cases} GJ_0 = \frac{i}{2},\ GJ_1 = 1 \\ GJ_n = GJ_{n-1} + 2GJ_{n-2},\ n \geq 2 \end{cases}$	$\dfrac{i + (2-i)z}{2 - 2z - 4z^2}$
Gaussian Jacobsthal Lucas Gj_n	$\begin{cases} Gj_0 = 2 - \frac{i}{2},\ Gj_1 = 1 + 2i \\ Gj_n = Gj_{n-1} + 2Gj_{n-2},\ n \geq 2 \end{cases}$	$\dfrac{4 - i + (5i-2)z}{2 - 2z - 4z^2}$
Gaussian Pell GP_n	$\begin{cases} GP_0 = i,\ GP_1 = 1 \\ GP_n = 2GP_{n-1} + GP_{n-2},\ n \geq 2 \end{cases}$	$\dfrac{i + (1-2i)z}{1 - 2z - z^2}$
Gaussian Pell Lucas GQ_n	$\begin{cases} GQ_0 = 2 - 2i,\ GQ_1 = 2 + 2i \\ GQ_n = 2GQ_{n-1} + GQ_{n-2},\ n \geq 2 \end{cases}$	$\dfrac{2 - 2i + (6i-2)z}{1 - 2z - z^2}$

In [5], the authors defined the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials. They proved generating function, Binet's formula and explicit formula of these polynomials, and studied some properties of them by using the matrix methods. After that, Yadav presented the generalizations of Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials in [36]. In 2018, Halici and Oz introduced the Gaussian Pell polynomials [17]. The year after, Yagmur in [37] defined the Gaussian Pell Lucas polynomials and proved some properties of them.

In the rest, we introduce certain definitions and results of the symmetric functions. (For more details, please see [9, 29, 31, 32]).

Definition 1.1. *[25] For any natural numbers k and n, the complete homogeneous symmetric function of degree k in n variables $a_1, a_2, \ldots, a_n$ is defined by:*

$$h_k(a_1, a_2, \ldots, a_n) = \sum_{i_1+i_2+\ldots+i_n=k} a_1^{i_1} a_2^{i_2} \ldots a_n^{i_n}, \text{ with } i_1, i_2, \ldots, i_n \geq 0.$$

Remark 1.2. *Set $h_0(a_1, a_2, \ldots, a_n) = 1$, by usual convention. For $k < 0$, we set$h_k(a_1, a_2, \ldots, a_n) = 0$.*

Definition 1.3. *[1, 33] Let A and P be any two alphabets. We define $S_n(A - P)$ by the following form:*

$$\sum_{n=0}^{\infty} S_n(A-P)z^n = \frac{\prod_{p \in P}(1 - pz)}{\prod_{a \in A}(1 - az)}, \tag{1}$$

with the condition $S_n(A - P) = 0$ for $n < 0$.

Equation (1) can be rewritten in the following form:

$$\sum_{n=0}^{\infty} S_n(A-P)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)z^n\right),$$

where

$$S_n(A-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(A).$$

Definition 1.4. *[28] Let n be positive integer and $A = \{a_1, a_2\}$ an alphabet. Then, the n^{th} symmetric function $S_n(a_1 + a_2)$ is defined by:*

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$\begin{aligned} S_0(A) &= S_0(a_1 + a_2) = 1, \\ S_1(A) &= S_1(a_1 + a_2) = a_1 + a_2, \\ S_2(A) &= S_2(a_1 + a_2) = a_1^2 + a_1 a_2 + a_2^2, \\ &\vdots \end{aligned}$$

The structure of this paper is arranged in the following way: The next section is devoted to defining the generalized Gaussian polynomials sequence. In particular, Gaussian Jacobsthal polynomials, Gaussian Jacobsthal Lucas polynomials, Gaussian Pell polynomials, Gaussian Pell Lucas polynomials, Gaussian Fibonacci polynomials and Gaussian Lucas polynomials are defined. In the rest of section 2, we give the Binet's formula for generalized Gaussian polynomials, and we present the generating functions of generalized Gaussian polynomials and (p, q)-Fibonacci-like numbers. In section 3, we calculate certain novel ordinary generating functions of the products for (p, q)-Fibonacci-like numbers with Gaussian numbers. By utilizing the symmetric functions, the new generating functions of the products of (p, q) -Fibonacci-like numbers with Gaussian polynomials are presented in section 4.

Notation: In the rest of this paper, the (p, q) -Fibonacci-like numbers, k-Fibonacci-like numbers and Fibonacci-like numbers will be denoted by $l_{p,q,n}$, $l_{k,n}$ and l_n instead of $S_{p,q,n}$, $S_{k,n}$ and S_n respectively, because the symmetric function is denoted by $S_n(A)$.

2. Main results

Now, we introduce the generalized Gaussian polynomials sequence and we obtain the Binet's formula of them. Also, we give the ordinary generating functions and explicit formulas of generalized Gaussian polynomials and (p, q)-Fibonacci-like numbers.

2.1. The generalized Gaussian polynomials

The new generalization of Gaussian polynomials is given in the following definition.

Definition 2.1. *The generalized Gaussian polynomials sequence $\{GW_n(x)\}_{n\geq 0}$ is given by the following recurrence relation:*

$$GW_n(x) = (ax + b)\, GW_{n-1}(x) + (cx + d)\, GW_{n-2}(x),\ n \geq 2, \tag{2}$$

with $GW_0(x) = \beta x + \alpha$, $GW_1(x) = \lambda x + \gamma$ and $\{a, b, c, d, \alpha, \beta, \lambda, \gamma\} \in \mathbb{C}$.

The following corollary gives some special cases of the Definition 2.1.

Corollary 2.2. *Particular cases of generalized Gaussian polynomials* $\{GW_n(x)\}_{n\geq 0}$ *are:*

- **Case1:** If we take $a=d=\beta=\lambda=0$, $b=\gamma=1$, $c=2$ and $\alpha=\frac{i}{2}$, then we get the **Gaussian Jacobsthal** polynomials, known as:

$$\begin{cases} GJ_0(x)=\frac{i}{2}, GJ_1(x)=1 \\ GJ_n(x)=GJ_{n-1}(x)+2xGJ_{n-2}(x),\ n\geq 2 \end{cases},$$

or

$$\{GJ_n(x)\}_{n\geq 0}=\{\tfrac{i}{2},\ 1,\ 1+xi,\ 2x+1+ix,\ 4x+1+ix(2x+1),\ldots\}.$$

- **Case 2:** If we take $a=d=\beta=0, b=\gamma=1, c=2$, $\alpha=2-\frac{i}{2}$ and $\lambda=2i$, then we get the **Gaussian Jacobsthal Lucas** polynomials, known as:

$$\begin{cases} Gj_0(x)=2-\frac{i}{2}, Gj_1(x)=1+2ix \\ Gj_n(x)=Gj_{n-1}(x)+2xGj_{n-2}(x),\ n\geq 2 \end{cases},$$

or

$$\{Gj_n(x)\}_{n\geq 0}=\{2-\tfrac{i}{2},\ 1+2xi,\ 4x+1+ix,\ 6x+1+xi(4x+1),\ 8x^2+8x+1+xi(6x+1),\ldots\}.$$

- **Case 3:** If we take $b=c=\beta=\lambda=0$, $d=\gamma=1$, $a=2$ and $\alpha=i$, then we get the **Gaussian Pell** polynomials, known as:

$$\begin{cases} GP_0(x)=i, GP_1(x)=1 \\ GP_n(x)=2xGP_{n-1}(x)+GP_{n-2}(x),\ n\geq 2 \end{cases},$$

or

$$\{GP_n(x)\}_{n\geq 0}=\{i,\ 1,\ 2x+i,\ 4x^2+1+2ix,\ 8x^3+4x+i\left(4x^2+1\right),\ldots\}.$$

- **Case 4:** If we take $b=c=0$, $d=1$, $a=\alpha=\lambda=2$, $\beta=-2i$ and $\gamma=2i$, then we get the **Gaussian Pell Lucas** polynomials, known as:

$$\begin{cases} GQ_0(x)=2-2xi, GQ_1(x)=2x+2i \\ GQ_n(x)=2xGQ_{n-1}(x)+GQ_{n-2}(x),\ n\geq 2 \end{cases},$$

or

$$\{GQ_n(x)\}_{n\geq 0}=\{2-2ix,\ 2x+2i,\ 4x^2+2+2xi,\ 8x^3+6x+2i\left(2x^2+1\right),\ 16x^4+16x^2+2+2ix\left(4x^2+3\right),\ldots\}.$$

- **Case 5:** If we take $b=c=\beta=\lambda=0$, $\alpha=i$, and $a=d=\gamma=1$, then we get the **Gaussian Fibonacci** polynomials, known as:

$$\begin{cases} GF_0(x)=i, GF_1(x)=1 \\ GF_n(x)=xGF_{n-1}(x)+GF_{n-2}(x),\ n\geq 2 \end{cases},$$

or

$$\{GF_n(x)\}_{n\geq 0}=\{i,\ 1,\ x+i,\ x^2+1+ix,\ x^3+2x+i\left(x^2+1\right),\ldots\}.$$

- **Case 6:** If we take $b=c=0$, $d=a=\lambda=1$, $\alpha=2$, $\beta=-i$ and $\gamma=2i$, then we get the **Gaussian Lucas** polynomials, known as:

$$\begin{cases} GL_0(x)=2-xi, GL_1(x)=x+2i \\ GL_n(x)=xGL_{n-1}(x)+GL_{n-2}(x),\ n\geq 2 \end{cases},$$

or

$$\{GL_n(x)\}_{n\geq 0}=\{2-ix,\ x+2i,\ x^2+2+xi,\ x^3+3x+i\left(x^2+2\right),\ x^4+4x^2+2+ix\left(x^2+3\right),\ldots\}.$$

Recurrence relationship (2) involve the characteristic equation:

$$t^2 - (ax+b)\,t - (cx+d) = 0,$$

which has two characteristic roots:

$$t_1 = \frac{ax+b+\sqrt{a^2x^2+2abx+b^2+4cx+4d}}{2} \text{ and } t_2 = \frac{ax+b-\sqrt{a^2x^2+2abx+b^2+4cx+4d}}{2},$$

characteristic roots verify the properties:

$$t_1 + t_2 = ax+b,\ t_1t_2 = -(cx+d) \text{ and } t_1 - t_2 = \sqrt{a^2x^2+2abx+b^2+4cx+4d}.$$

The next theorem gives the Binet's formula for generalized Gaussian polynomials.

Theorem 2.3. *The n^{th} term of the generalized Gaussian polynomials is given by:*

$$GW_n(x) = \frac{At_1^n - Bt_2^n}{t_1 - t_2}, \tag{3}$$

with $A = \gamma + \lambda x - (\alpha + \beta x)\,t_2$ and $B = \gamma + \lambda x - (\alpha + \beta x)\,t_1$.

Proof. From the theory of difference equation we know the general term of generalized Gaussian polynomials can be expressed in the following form:

$$GW_n(x) = C_1t_1^n + C_2t_2^n,$$

where C_1 and C_2 are the coefficients. Plugging the general solution in the initial conditions $\{GW_0(x), GW_1(x)\}$ gives the system:

$$\begin{cases} C_1 + C_2 = \beta x + \alpha \\ C_1t_1 + C_2t_2 = \lambda x + \gamma \end{cases}.$$

By these equalities:

$$\begin{cases} C_1 = \frac{\gamma+\lambda x-(\alpha+\beta x)t_2}{t_1-t_2} = \frac{A}{t_1-t_2} \\ C_2 = -\frac{\gamma+\lambda x-(\alpha+\beta x)t_1}{t_1-t_2} = -\frac{B}{t_1-t_2} \end{cases}.$$

Therefore, we get:

$$GW_n(x) = \frac{At_1^n - Bt_2^n}{t_1 - t_2}.$$

This completes the proof. □

The special cases of the Binet's formula for generalized Gaussian polynomials are listed in the Table 2.

Table 2: Binet's formulas for some Gaussian polynomials

a	b	c	d	α	β	γ	λ	Roots (t_1 and t_2)	Binet's formula ($GW_n(x)$)
0	1	2	0	$\frac{i}{2}$	0	1	0	$t_{1,2} = \frac{1\pm\sqrt{8x+1}}{2}$	$GJ_n(x) = \frac{t_1^n - t_2^n}{t_1 - t_2} - \frac{i}{2}\left(\frac{t_2t_1^n - t_1t_2^n}{t_1 - t_2}\right)$
0	1	2	0	$2 - \frac{i}{2}$	0	1	$2i$	$t_{1,2} = \frac{1\pm\sqrt{8x+1}}{2}$	$Gj_n(x) = t_1^n + t_2^n - \frac{i}{2}\left(t_2t_1^n + t_1t_2^n\right)$
2	0	0	1	i	0	1	0	$t_{1,2} = x \pm \sqrt{x^2+1}$	$GP_n(x) = \frac{t_1^n - t_2^n}{t_1 - t_2} - i\left(\frac{t_2t_1^n - t_1t_2^n}{t_1 - t_2}\right)$
2	0	0	1	2	$-2i$	$2i$	2	$t_{1,2} = x \pm \sqrt{x^2+1}$	$GQ_n(x) = t_1^n + t_2^n - i\left(t_2t_1^n + t_1t_2^n\right)$
1	0	0	1	i	0	1	0	$t_{1,2} = \frac{x\pm\sqrt{x^2+4}}{2}$	$GF_n(x) = \frac{t_1^n - t_2^n}{t_1 - t_2} - i\left(\frac{t_2t_1^n - t_1t_2^n}{t_1 - t_2}\right)$
1	0	0	1	2	$-i$	$2i$	1	$t_{1,2} = \frac{x\pm\sqrt{x^2+4}}{2}$	$GL_n(x) = t_1^n + t_2^n - i\left(t_2t_1^n + t_1t_2^n\right)$

2.2. The ordinary generating functions and explicit formulas of generalized Gaussian polynomials and (p, q)-Fibonacci-like numbers

The following proposition is one of the key tools of the proof of our main results. It has been proved in [12].

Proposition 2.4. *Given an alphabet $A = \{a_1, -a_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])z^n = \frac{1}{1 - (a_1 - a_2)z - a_1a_2z^2}. \tag{4}$$

The Eq. (5) is a special case of Proposition 2.4.

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])z^n = \frac{z}{1 - (a_1 - a_2)z - a_1a_2z^2}. \tag{5}$$

Now, we will give the generating function for the generalized Gaussian polynomials.

Choosing a_1 and a_2 such that $\begin{cases} a_1 - a_2 = ax + b \\ a_1a_2 = cx + d \end{cases}$ and substituting in (4) and (5), we obtain:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])z^n = \frac{1}{1 - (ax + b)z - (cx + d)z^2}, \tag{6}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])z^n = \frac{z}{1 - (ax + b)z - (cx + d)z^2}, \tag{7}$$

respectively.

Multiplying the equation (6) by $(\alpha + \beta x)$ and adding it to the equation obtained by (7) multiplied by $\left(\gamma - b\alpha + (\lambda - \alpha a - b\beta)x - a\beta x^2\right)$, then we obtain:

$$\sum_{n=0}^{\infty} \left(\begin{array}{c} (\alpha + \beta x) S_n(a_1 + [-a_2]) \\ +\left(\gamma - b\alpha + (\lambda - a\alpha - b\beta)x - a\beta x^2\right) S_{n-1}(a_1 + [-a_2]) \end{array} \right) z^n = \frac{\alpha + \beta x + \left(\gamma - b\alpha + (\lambda - a\alpha - b\beta)x - a\beta x^2\right)z}{1 - (ax + b)z - (cx + d)z^2},$$

and we have the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$, the new ordinary generating function of generalized Gaussian polynomials is given by:*

$$\sum_{n=0}^{\infty} GW_n(x)z^n = \frac{\alpha + \beta x + \left(\gamma - b\alpha + (\lambda - a\alpha - b\beta)x - a\beta x^2\right)z}{1 - (ax + b)z - (cx + d)z^2}, \tag{8}$$

with

$$GW_n(x) = (\alpha + \beta x) S_n(a_1 + [-a_2]) + \left(\gamma - b\alpha + (\lambda - a\alpha - b\beta)x - a\beta x^2\right) S_{n-1}(a_1 + [-a_2]).$$

Proof. Let $g(x,z)$ be the ordinary generating function for the sequence of generalized Gaussian polynomials, then:

$$g(x,z) = \sum_{n=0}^{\infty} GW_n(x) z^n.$$

By using the initial values $GW_0(x)$ and $GW_1(x)$, we obtain:

$$\begin{aligned} g(x,z) &= GW_0(x) + GW_1(x) z + \sum_{n=2}^{\infty} GW_n(x) z^n \\ &= GW_0(x) + GW_1(x) z + \sum_{n=2}^{\infty} ((ax+b) GW_{n-1}(x) + (cx+d) GW_{n-2}(x)) z^n \\ &= GW_0(x) + GW_1(x) z + (ax+b) z \sum_{n=1}^{\infty} GW_n(x) z^n + (cx+d) z^2 \sum_{n=0}^{\infty} GW_n(x) z^n \\ &= GW_0(x) + (GW_1(x) - (ax+b) GW_0(x)) z \\ &+ (ax+b) z \sum_{n=0}^{\infty} GW_n(x) z^n + (cx+d) z^2 \sum_{n=0}^{\infty} GW_n(x) z^n \\ &= \alpha + \beta x + \left(\gamma - b\alpha + (\lambda - a\alpha - b\beta)x - a\beta x^2\right) z + \left((ax+b)z + (cx+d)z^2\right) g(x,z). \end{aligned}$$

Hence, we obtain:

$$\left(1 - (ax+b)z - (cx+d)z^2\right) g(x,z) = \alpha + \beta x + \left(\gamma - b\alpha + (\lambda - a\alpha - b\beta)x - a\beta x^2\right) z.$$

Therefore:

$$g(x,z) = \sum_{n=0}^{\infty} GW_n(x) z^n = \frac{\alpha + \beta x + \left(\gamma - b\alpha + (\lambda - a\alpha - b\beta)x - a\beta x^2\right) z}{1 - (ax+b)z - (cx+d)z^2}.$$

Thus, this completes the proof. □

- By putting $a = d = \beta = \lambda = 0$, $b = \gamma = 1$, $c = 2$ and $\alpha = \frac{i}{2}$ in the relationship (8), we can state the following corollary.

Corollary 2.6. *For $n \in \mathbb{N}$, the ordinary generating function of Gaussian Jacobsthal polynomials is given by:*

$$\sum_{n=0}^{\infty} GJ_n(x) z^n = \frac{i + (2-i)z}{2 - 2z - 4xz^2}, \tag{9}$$

with

$$GJ_n(x) = \frac{i}{2} S_n(a_1 + [-a_2]) + \left(1 - \frac{i}{2}\right) S_{n-1}(a_1 + [-a_2]).$$

- By putting $a = d = \beta = 0, b = \gamma = 1$, $c = 2$, $\alpha = 2 - \frac{i}{2}$ and $\lambda = 2i$ in the relationship (8), we can state the following corollary.

Corollary 2.7. *For $n \in \mathbb{N}$, the ordinary generating function of Gaussian Jacobsthal Lucas polynomials is given by:*

$$\sum_{n=0}^{\infty} Gj_n(x) z^n = \frac{4 - i + ((4x+1)i - 2) z}{2 - 2z - 4xz^2}, \tag{10}$$

with

$$Gj_n(x) = \left(2 - \frac{i}{2}\right) S_n(a_1 + [-a_2]) + \left(\left(2x + \frac{1}{2}\right) i - 1\right) S_{n-1}(a_1 + [-a_2]).$$

- By putting $b = c = \beta = \lambda = 0$, $d = \gamma = 1$, $a = 2$ and $\alpha = i$ in the relationship (8), we can state the following corollary.

Corollary 2.8. *For $n \in \mathbb{N}$, the ordinary generating function of Gaussian Pell polynomials is given by:*

$$\sum_{n=0}^{\infty} GP_n(x) z^n = \frac{i + (1 - 2ix) z}{1 - 2xz - z^2}, \text{ with } GP_n(x) = iS_n(a_1 + [-a_2]) + (1 - 2ix) S_{n-1}(a_1 + [-a_2]). \tag{11}$$

- By putting $b = c = 0$, $d = 1$, $a = \alpha = \lambda = 2$, $\beta = -2i$ and $\gamma = 2i$ in the relationship (8), we can state the following corollary.

Corollary 2.9. *For $n \in \mathbb{N}$, the ordinary generating function of Gaussian Pell Lucas polynomials is given by:*

$$\sum_{n=0}^{\infty} GQ_n(x) z^n = \frac{2 - 2ix + \left(i\left(2 + 4x^2\right) - 2x\right) z}{1 - 2xz - z^2}, \tag{12}$$

with

$$GQ_n(x) = (2 - 2ix) S_n(a_1 + [-a_2]) + \left(i\left(2 + 4x^2\right) - 2x\right) S_{n-1}(a_1 + [-a_2]).$$

- By putting $b = c = \beta = \lambda = 0$, $\alpha = i$, and $a = d = \gamma = 1$, in the relationship (8), we can state the following corollary.

Corollary 2.10. *For $n \in \mathbb{N}$, the ordinary generating function of Gaussian Fibonacci polynomials is given by:*

$$\sum_{n=0}^{\infty} GF_n(x) z^n = \frac{i + (1 - ix) z}{1 - xz - z^2}, \tag{13}$$

with

$$GF_n(x) = iS_n(a_1 + [-a_2]) + (1 - ix) S_{n-1}(a_1 + [-a_2]).$$

- By putting $b = c = 0$, $d = a = \lambda = 1$, $\alpha = 2$, $\beta = -i$ and $\gamma = 2i$, in the relationship (8), we can state the following corollary.

Corollary 2.11. *For $n \in \mathbb{N}$, the ordinary generating function of Gaussian Lucas polynomials is given by:*

$$\sum_{n=0}^{\infty} GL_n(x) z^n = \frac{2 - ix + \left(i\left(x^2 + 2\right) - x\right) z}{1 - xz - z^2}, \tag{14}$$

with

$$GL_n(x) = (2 - ix) S_n(a_1 + [-a_2]) + \left(i\left(x^2 + 2\right) - x\right) S_{n-1}(a_1 + [-a_2]).$$

Now, we will give the generating function for (p, q) -Fibonacci-like numbers.

By the substitution $\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = q \end{cases}$ in (4) and (5) we obtain:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])z^n = \frac{1}{1 - pz - qz^2}. \tag{15}$$

Multiplying the equation (15) by (2), we obtain:

$$\sum_{n=0}^{\infty} 2S_n(a_1 + [-a_2])z^n = \frac{2}{1 - pz - qz^2},$$

and we have the following theorem.

Theorem 2.12. *For $n \in \mathbb{N}$, the new ordinary generating function of (p, q) -Fibonacci-like numbers is given by:*

$$\sum_{n=0}^{\infty} l_{p,q,n} z^n = \frac{2}{1 - pz - qz^2}, \text{ with } l_{p,q,n} = 2S_n(a_1 + [-a_2]). \tag{16}$$

Proof. The (p, q)-Fibonacci-like numbers can be considered as the coefficients of the formal power series:

$$g(z) = \sum_{n=0}^{\infty} l_{p,q,n} z^n.$$

Using the initial condition, we get:

$$\begin{aligned}
g(z) &= l_{p,q,0} + l_{p,q,1} z + \sum_{n=2}^{\infty} l_{p,q,n} z^n \\
&= l_{p,q,0} + l_{p,q,1} z + \sum_{n=2}^{\infty} \left(p l_{p,q,n-1} + q l_{p,q,n-2}\right) z^n \\
&= l_{p,q,0} + l_{p,q,1} z + pz \sum_{n=1}^{\infty} l_{p,q,n} z^n + qz^2 \sum_{n=0}^{\infty} l_{p,q,n} z^n \\
&= l_{p,q,0} + \left(l_{p,q,1} - p l_{p,q,0}\right) z + pz \sum_{n=0}^{\infty} l_{p,q,n} z^n + qz^2 \sum_{n=0}^{\infty} l_{p,q,n} z^n \\
&= 2 + \left(pz + qz^2\right) g(z).
\end{aligned}$$

Hence, we obtain:

$$\left(1 - pz - qz^2\right) g(z) = 2.$$

Therefore:

$$g(z) = \sum_{n=0}^{\infty} l_{p,q,n} z^n = \frac{2}{1 - pz - qz^2}.$$

Thus, this completes the proof. □

And we deduce the following theorem.

Theorem 2.13. *The explicit formula of (p, q)-Fibonacci-like numbers is:*

$$l_{p,q,n} = 2\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} p^{n-2j} q^j. \tag{17}$$

Proof. The generating function for (p, q)-Fibonacci-like numbers is:

$$g(z) = \sum_{n=0}^{\infty} l_{p,q,n} z^n = \frac{2}{1 - pz - qz^2}.$$

Then, we get:

$$\begin{aligned}\sum_{n=0}^{\infty} l_{p,q,n} z^n &= \frac{2}{1 - (pz + qz^2)} \\ &= 2\sum_{n=0}^{\infty} (pz + qz^2)^n \\ &= 2\sum_{n=0}^{\infty}\sum_{j=0}^{n} \binom{n}{j} p^{n-j} q^j z^{n+j} \\ &= \sum_{n=0}^{\infty} 2\sum_{j=0}^{n} \binom{n}{j} p^{n-j} q^j z^{n+j}.\end{aligned}$$

Writing (n) instead of $(n + j)$, we obtain:

$$\sum_{n=0}^{\infty} l_{p,q,n} z^n = \sum_{n=0}^{\infty} \left(2\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} p^{n-2j} q^j \right) z^n.$$

Comparing of the coefficients of z^n, we deduce:

$$l_{p,q,n} = 2\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} p^{n-2j} q^j.$$

Thus, this completes the proof. □

Theorem 2.14. *The explicit formula of generalized Gaussian polynomials is given by:*

$$\begin{aligned}GW_n(x) &= (\alpha + \beta x) \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (ax + b)^{n-2j} (cx + d)^j \\ &+ \left(\gamma - b\alpha + (\lambda - a\alpha - b\beta) x - a\beta x^2\right) \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} (ax + b)^{n-2j-1} (cx + d)^j.\end{aligned} \tag{18}$$

Proof. We have:

$$\sum_{n=0}^{\infty} GW_n(x) z^n = \frac{\alpha + \beta x + \left(\gamma - b\alpha + (\lambda - a\alpha - b\beta) x - a\beta x^2\right) z}{1 - (ax + b) z - (cx + d) z^2}.$$

The result can be proved by the same method given in the Theorem 4, thus we omit the proof. □

The special cases of Eq. (18) are listed in the Table 3.

Table 3: Explicit formulas for some Gaussian polynomials.

a	b	c	d	α	β	γ	λ	Explicit formula
0	1	2	0	$\frac{i}{2}$	0	1	0	$GJ_n(x)=\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-j-1}{j}(2x)^j+i\sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-j-2}{j}2^jx^{j+1}$
0	1	2	0	$2-\frac{i}{2}$	0	1	$2i$	$Gj_n(x)=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n}{n-j}\binom{n-j}{j}(2x)^j+i\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{n-1}{n-j-1}\binom{n-j-1}{j}2^jx^{j+1}$
2	0	0	1	i	0	1	0	$GP_n(x)=\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-j-1}{j}(2x)^{n-2j-1}+i\sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-j-2}{j}(2x)^{n-2j-2}$
2	0	0	1	2	$-2i$	$2i$	2	$GQ_n(x)=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n}{n-j}\binom{n-j}{j}(2x)^{n-2j}+i\sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\frac{n-1}{n-j-1}\binom{n-j-1}{j}(2x)^{n-2j-1}$
1	0	0	1	i	0	1	0	$GF_n(x)=\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-j-1}{j}x^{n-2j-1}+i\sum_{j=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-j-2}{j}x^{n-2j-2}$
1	0	0	1	2	$-i$	$2i$	1	$GL_n(x)=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n}{n-j}\binom{n-j}{j}x^{n-2j}+i\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{n-1}{n-j-1}\binom{n-j-1}{j}x^{n-2j-1}$

3. Ordinary generating functions of the products of (p,q)-Fibonacci-like numbers with Gaussian numbers

This theorem is well-known from [27]. So we give them without proof.

Theorem 3.1. *Given two alphabets $A=\{a_1,a_2,a_3,...,a_k\}$ and $E=\{e_1,e_2\}$, we have: for all $k\in\mathbb{N}_0$ and $l\in\{0,1\}$.*

$$\sum_{n=0}^{\infty}S_n(A)S_{n-l}(E)z^n=\frac{S_{-l}(E)-e_1^{1-l}e_2^{1-l}z^{2-l}\sum\limits_{n=0}^{\infty}S_{n-l+2}(-A)S_n(E)z^n}{\left(\sum\limits_{n=0}^{\infty}S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)e_2^nz^n\right)}. \tag{19}$$

For $A=\{a_1,a_2\}$, $E=\{e_1,e_2\}$ and $l\in\{0,1\}$ in the Theorem 3.1 we deduce the following lemmas.

Lemma 3.2. *Given two alphabets $E=\{e_1,e_2\}$ and $A=\{a_1,a_2\}$, then we have:*

$$\sum_{n=0}^{\infty}S_n(A)S_n(E)z^n=\frac{1-a_1a_2e_1e_2z^2}{\left(\sum\limits_{n=0}^{\infty}S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)e_2^nz^n\right)}. \tag{20}$$

Lemma 3.3. *Given two alphabets $E=\{e_1,e_2\}$ and $A=\{a_1,a_2\}$, then we have:*

$$\sum_{n=0}^{\infty}S_n(A)S_{n-1}(E)z^n=\frac{(a_1+a_2)z-a_1a_2(e_1+e_2)z^2}{\left(\sum\limits_{n=0}^{\infty}S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)e_2^nz^n\right)}. \tag{21}$$

In this part, we now derive the new ordinary generating functions of the products of (p,q)-Fibonacci-like numbers with Gaussian Fibonacci and Gaussian Lucas numbers, Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers, Gaussian Pell and Gaussian Pell Lucas numbers.

We consider the following sets:

$A=\{a_1,-a_2\}$ and $E=\{e_1,-e_2\}$.

By changing a_2 to $(-a_2)$ and e_2 to $(-e_2)$ in Eqs. (20) and (21), it becomes

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])\,S_n(e_1+[-e_2])\,z^n = \frac{1-a_1a_2e_1e_2z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}, \tag{22}$$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{(a_1-a_2)z+a_1a_2(e_1-e_2)z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}. \tag{23}$$

This case consists of three related parts. **Firstly**, the substitutions:

$$\begin{cases} a_1-a_2=p \\ a_1a_2=q \end{cases} \text{ and } \begin{cases} e_1-e_2=1 \\ e_1e_2=1 \end{cases},$$

in the Eqs. (22) and (23), we obtain:

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])\,S_n(e_1+[-e_2])\,z^n = \frac{1-qz^2}{1-pz-(p^2+3q)z^2-pqz^3+q^2z^4}, \tag{24}$$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])\,S_{n-1}(e_1+[-e_2])\,z^n = \frac{pz+qz^2}{1-pz-(p^2+3q)z^2-pqz^3+q^2z^4}. \tag{25}$$

Therefore, we state the following theorems.

Theorem 3.4. *For $n \in \mathbb{N}$, we have:*

$$\sum_{n=0}^{\infty} l_{p,q,n}GF_nz^n = \frac{2i+2p(1-i)z+2q(1-2i)z^2}{1-pz-(p^2+3q)z^2-pqz^3+q^2z^4}, \tag{26}$$

which is the novel ordinary generating function of the product of (p,q)-Fibonacci-like numbers with Gaussian Fibonacci numbers $\left(l_{p,q,n}GF_n\right)$.

Proof. Recall that, we have $GF_n = iS_n(e_1+[-e_2]) + (1-i)\,S_{n-1}(e_1+[-e_2])$, (see [30]). Then, according Eq. (16) we can get:

$$\begin{aligned}\sum_{n=0}^{\infty} l_{p,q,n}GF_nz^n &= \sum_{n=0}^{\infty} 2S_n(a_1+[-a_2])\left(iS_n(e_1+[-e_2])+(1-i)\,S_{n-1}(e_1+[-e_2])\right)z^n \\ &= 2i\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\ &\quad + 2(1-i)\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n,\end{aligned}$$

by using the relationships (24) and (25), we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty} l_{p,q,n}GF_nz^n &= \frac{2i\left(1-qz^2\right)}{1-pz-(p^2+3q)z^2-pqz^3+q^2z^4} \\ &\quad + \frac{2(1-i)\left(pz+qz^2\right)}{1-pz-(p^2+3q)z^2-pqz^3+q^2z^4} \\ &= \frac{2i+2p(1-i)z+2q(1-2i)z^2}{1-pz-(p^2+3q)z^2-pqz^3+q^2z^4}.\end{aligned}$$

Thus, this completes the proof. □

Theorem 3.5. *Let $n \in \mathbb{N}$, the novel ordinary generating function of $\left(l_{p,q,n}GL_n\right)$ is as follows:*

$$\sum_{n=0}^{\infty} l_{p,q,n}GL_n z^n = \frac{2(2-i)+2p(3i-1)z+2q(4i-3)z^2}{1-pz-(p^2+3q)z^2-pqz^3+q^2z^4}. \tag{27}$$

Proof. Recall that, we have $l_{p,q,n} = 2S_n(a_1+[-a_2])$ and $GL_n = (2-i)S_n(e_1+[-e_2]) + (3i-1)S_{n-1}(e_1+[-e_2])$, (see [30]). We prove this result by the same method given in the Theorem 3.4. □

Corollary 3.6. *Putting $p = k$ and $q = 1$ in the Eqs.* (26) *and* (27) *gives the new ordinary generating functions of the products of k-Fibonacci-like numbers with Gaussian Fibonacci and Gaussian Lucas numbers as follows:*

$$\sum_{n=0}^{\infty} l_{k,n}GF_n z^n = \frac{2i+2k(1-i)z+2(1-2i)z^2}{1-kz-(k^2+3)z^2-kz^3+z^4},$$

$$\sum_{n=0}^{\infty} l_{k,n}GL_n z^n = \frac{2(2-i)+2k(3i-1)z+2(4i-3)z^2}{1-kz-(k^2+3)z^2-kz^3+z^4}.$$

By putting $k = 1$ in the Corollary 3.6, we obtain the Table 4.

Table 4: New ordinary generating functions of the products of Fibonacci-like numbers with Gaussian Fibonacci and Gaussian Lucas numbers.

Coefficient of z^n	Ordinary generating function
l_nGF_n	$\frac{2i+2(1-i)z+2(1-2i)z^2}{1-z-4z^2-z^3+z^4}$
l_nGL_n	$\frac{2(2-i)+2(3i-1)z+2(4i-3)z^2}{1-z-4z^2-z^3+z^4}$

Secondly, the substitutions:

$$\begin{cases} a_1-a_2=p \\ a_1a_2=q \end{cases} \text{ and } \begin{cases} e_1-e_2=1 \\ e_1e_2=2 \end{cases},$$

in the Eqs. (22) and (23), we get:

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{1-2qz^2}{1-pz-(2p^2+5q)z^2-2pqz^3+4q^2z^4}, \tag{28}$$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{pz+qz^2}{1-pz-(2p^2+5q)z^2-2pqz^3+4q^2z^4}. \tag{29}$$

Thus, we derive the following results.

Theorem 3.7. *For $n \in \mathbb{N}$, we have:*

$$\sum_{n=0}^{\infty} l_{p,q,n}GJ_n z^n = \frac{i+p(2-i)z+q(2-3i)z^2}{1-pz-(2p^2+5q)z^2-2pqz^3+4q^2z^4}, \tag{30}$$

which represents the novel ordinary generating function of the product of (p,q)-Fibonacci-like numbers with Gaussian Jacobsthal numbers $\left(l_{p,q,n}GJ_n\right)$.

Proof. By [30] we have $GJ_n = \frac{i}{2}S_n(e_1 + [-e_2]) + \left(1 - \frac{i}{2}\right)S_{n-1}(e_1 + [-e_2])$. Then, according Eq. (16) we can see that:

$$\sum_{n=0}^{\infty} l_{p,q,n} GJ_n z^n = \sum_{n=0}^{\infty} 2S_n(a_1 + [-a_2])\left(\frac{i}{2}S_n(e_1 + [-e_2]) + \left(1 - \frac{i}{2}\right)S_{n-1}(e_1 + [-e_2])\right)z^n$$
$$= i\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n$$
$$+ (2-i)\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n.$$

According to the relationships (28) and (29), we obtain:

$$\sum_{n=0}^{\infty} l_{p,q,n} GJ_n z^n = \frac{i\left(1 - 2qz^2\right)}{1 - pz - (2p^2 + 5q)z^2 - 2pqz^3 + 4q^2z^4}$$
$$+ \frac{(2-i)\left(pz + qz^2\right)}{1 - pz - (2p^2 + 5q)z^2 - 2pqz^3 + 4q^2z^4}$$
$$= \frac{i + p\,(2-i)\,z + q\,(2-3i)\,z^2}{1 - pz - (2p^2 + 5q)z^2 - 2pqz^3 + 4q^2z^4}.$$

This completes the proof. □

Theorem 3.8. *For any natural number n, the novel ordinary generating function of* $\left(l_{p,q,n}Gj_n\right)$ *is found as:*

$$\sum_{n=0}^{\infty} l_{p,q,n} Gj_n z^n = \frac{4 - i + p\,(5i-2)\,z + q\,(7i-10)\,z^2}{1 - pz - (2p^2 + 5q)z^2 - 2pqz^3 + 4q^2z^4}. \tag{31}$$

Proof. We have $l_{p,q,n} = 2S_n(a_1 + [-a_2])$ and $Gj_n = \left(2 - \frac{i}{2}\right)S_n(e_1 + [-e_2]) + \left(\frac{5i}{2} - 1\right)S_{n-1}(e_1 + [-e_2])$, (see [30]). The result can be proved by the same method given in the Theorem 3.7. □

Corollary 3.9. *Taking $p = k$ and $q = 1$ in the Eqs.* (30) *and* (31) *gives the new ordinary generating functions of the products of k-Fibonacci-like numbers with Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers as follows:*

$$\sum_{n=0}^{\infty} l_{k,n} GJ_n z^n = \frac{i + k\,(2-i)\,z + (2-3i)\,z^2}{1 - kz - (2k^2 + 5)z^2 - 2kz^3 + 4z^4},$$

$$\sum_{n=0}^{\infty} l_{k,n} Gj_n z^n = \frac{4 - i + k\,(5i-2)\,z + (7i-10)\,z^2}{1 - kz - (2k^2 + 5)z^2 - 2kz^3 + 4z^4}.$$

By putting $k = 1$ in the Corollary 3.9, we obtain the Table 5.

Table 5: New ordinary generating functions of the products of Fibonacci-like numbers with Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers.

Coefficient of z^n	Ordinary generating function
$l_n GJ_n$	$\dfrac{i+(2-i)z+(2-3i)z^2}{1-z-7z^2-2z^3+4z^4}$
$l_n Gj_n$	$\dfrac{4-i+(5i-2)z+(7i-10)z^2}{1-z-7z^2-2z^3+4z^4}$

Thirdly, the substitutions:

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = q \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 2 \\ e_1 e_2 = 1 \end{cases},$$

in the Eqs. (22) and (23). Then one has:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{1 - qz^2}{1 - 2pz - (p^2 + 6q)z^2 - 2pqz^3 + q^2z^4}, \tag{32}$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n = \frac{pz + 2qz^2}{1 - 2pz - (p^2 + 6q)z^2 - 2pqz^3 + q^2z^4}. \tag{33}$$

Therefore, we state the following theorems.

Theorem 3.10. *For $n \in \mathbb{N}$, we have:*

$$\sum_{n=0}^{\infty} l_{p,q,n} GP_n z^n = \frac{2i + 2p(1-2i)z + 2q(2-5i)z^2}{1 - 2pz - (p^2 + 6q)z^2 - 2pqz^3 + q^2z^4}, \tag{34}$$

which is the novel ordinary generating function of the product of (p, q)-Fibonacci-like numbers with Gaussian Pell numbers $\left(l_{p,q,n} GP_n\right)$.

Proof. By [30], we have $GP_n = iS_n(e_1 + [-e_2]) + (1-2i)S_{n-1}(e_1 + [-e_2])$. Then, according Eq. (16) we can see that:

$$\begin{aligned}
\sum_{n=0}^{\infty} l_{p,q,n} GP_n z^n &= \sum_{n=0}^{\infty} 2S_n(a_1 + [-a_2])\left(iS_n(e_1 + [-e_2]) + (1-2i)S_{n-1}(e_1 + [-e_2])\right) z^n \\
&= 2i \sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
&\quad + 2(1-2i) \sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
&= \frac{2i\left(1 - qz^2\right)}{1 - 2pz - (p^2 + 6q)z^2 - 2pqz^3 + q^2z^4} \\
&\quad + \frac{2(1-2i)\left(pz + 2qz^2\right)}{1 - 2pz - (p^2 + 6q)z^2 - 2pqz^3 + q^2z^4},
\end{aligned}$$

after necessary calculations we get:

$$\sum_{n=0}^{\infty} l_{p,q,n} GP_n z^n = \frac{2i + 2p(1-2i)z + 2q(2-5i)z^2}{1 - 2pz - (p^2 + 6q)z^2 - 2pqz^3 + q^2z^4}.$$

Thus, this completes the proof. □

Theorem 3.11. *Let $n \geq 0$, we have the novel ordinary generating function of $\left(l_{p,q,n}GQ_n\right)$ as follows:*

$$\sum_{n=0}^{\infty} l_{p,q,n}GQ_n z^n = \frac{4(1-i)+4p(3i-1)z+4q(7i-3)z^2}{1-2pz-(p^2+6q)z^2-2pqz^3+q^2z^4}. \tag{35}$$

Proof. We know that:

$$l_{p,q,n} = 2S_n(a_1+[-a_2]) \text{ and } GQ_n = (2-2i)\,S_n(e_1+[-e_2])+(6i-2)\,S_{n-1}(e_1+[-e_2]), \text{ (see [30])}.$$

The proof is similar to the proof of the Theorem 3.10. □

Corollary 3.12. *Taking $p = k$ and $q = 1$ in the Eqs.* (34) *and* (35) *gives the new ordinary generating functions of the products of k-Fibonacci-like numbers with Gaussian Pell and Gaussian Pell Lucas numbers as follows:*

$$\sum_{n=0}^{\infty} l_{k,n}GP_n z^n = \frac{2i+2k(1-2i)z+2(2-5i)z^2}{1-2kz-(k^2+6)z^2-2kz^3+z^4},$$

$$\sum_{n=0}^{\infty} l_{k,n}GQ_n z^n = \frac{4(1-i)+4k(3i-1)z+4(7i-3)z^2}{1-2kz-(k^2+6)z^2-2kz^3+z^4}.$$

By putting $k = 1$ in the Corollary 3.12, we obtain the Table 6.

Table 6: New ordinary generating functions of the products of Fibonacci-like numbers with Gaussian Pell and Gaussian Pell Lucas numbers.

Coefficient of z^n	Ordinary generating function
l_nGP_n	$\frac{2i+2(1-2i)z+2(2-5i)z^2}{1-2z-7z^2-2z^3+z^4}$
l_nGQ_n	$\frac{4(1-i)+4(3i-1)z+4(7i-3)z^2}{1-2z-7z^2-2z^3+z^4}$

4. The novel ordinary generating functions of several products

Here, we introduce and prove our theorems. Also we calculate the novel ordinary generating functions for the products of (p,q)-Fibonacci-like numbers with Gaussian Pell polynomials, Gaussian Pell Lucas polynomials, Gaussian Jacobsthal polynomials, Gaussian Jacobsthal Lucas polynomials, Gaussian Fibonacci polynomials and Gaussian Lucas polynomials.

Firstly, let us now consider the following conditions for Eqs. (22) and (23):

$$\begin{cases} a_1 - a_2 = p \\ a_1a_2 = q \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 1 \\ e_1e_2 = 2x \end{cases}.$$

Then it yields:

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])\,S_n(e_1+[-e_2])\,z^n = \frac{1-2qxz^2}{1-pz-(2x(p^2+2q)+q)z^2-2pqxz^3+4q^2x^2z^4}, \tag{36}$$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{pz+qz^2}{1-pz-(2x(p^2+2q)+q)z^2-2pqxz^3+4q^2x^2z^4}. \tag{37}$$

Therefore, we state the following theorems.

Theorem 4.1. *Let $n \in \mathbb{N}$, the novel ordinary generating function of $\left(l_{p,q,n}GJ_n(x)\right)$ is as follows:*

$$\sum_{n=0}^{\infty} l_{p,q,n}GJ_n(x)z^n = \frac{i + p(2-i)z + q(2 - i(2x+1))z^2}{1 - pz - (2x(p^2+2q)+q)z^2 - 2pqxz^3 + 4q^2x^2z^4}. \tag{38}$$

Proof. By using the relationships (9) and (16), we obtain:

$$\begin{aligned}
\sum_{n=0}^{\infty} l_{p,q,n}GJ_n(x)z^n &= \sum_{n=0}^{\infty} 2S_n(a_1+[-a_2])\left(\frac{i}{2}S_n(e_1+[-e_2]) + \left(1-\frac{i}{2}\right)S_{n-1}(e_1+[-e_2])\right)z^n \\
&= i\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&+ (2-i)\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&= \frac{i\left(1-2qxz^2\right)}{1 - pz - (2x(p^2+2q)+q)z^2 - 2pqxz^3 + 4q^2x^2z^4} \\
&+ \frac{(2-i)\left(pz+qz^2\right)}{1 - pz - (2x(p^2+2q)+q)z^2 - 2pqxz^3 + 4q^2x^2z^4} \\
&= \frac{i + p(2-i)z + q(2-i(2x+1))z^2}{1 - pz - (2x(p^2+2q)+q)z^2 - 2pqxz^3 + 4q^2x^2z^4}.
\end{aligned}$$

This completes the proof. □

Theorem 4.2. *For any natural number n, the novel ordinary generating function of $\left(l_{p,q,n}Gj_n(x)\right)$ is found as:*

$$\sum_{n=0}^{\infty} l_{p,q,n}Gj_n(x)z^n = \frac{4 - i + p((4x+1)i-2)z + q(i(6x+1)-8x-2)z^2}{1 - pz - (2x(p^2+2q)+q)z^2 - 2pqxz^3 + 4q^2x^2z^4}. \tag{39}$$

Proof. Since:

$$l_{p,q,n} = 2S_n(a_1+[-a_2]) \text{ and } Gj_n(x) = \left(2-\frac{i}{2}\right)S_n(e_1+[-e_2]) + \left(\left(2x+\frac{1}{2}\right)i - 1\right)S_{n-1}(e_1+[-e_2]).$$

We prove this result by the same method given in the Theorem 4.1 . □

Corollary 4.3. *Putting $p = k$ and $q = 1$ in the Eqs.* (38) *and* (39) *gives the new ordinary generating functions of the products of k-Fibonacci-like numbers with Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials as follows:*

$$\sum_{n=0}^{\infty} l_{k,n}GJ_n(x)z^n = \frac{i + k(2-i)z + (2-i(2x+1))z^2}{1 - kz - (2x(k^2+2)+1)z^2 - 2kxz^3 + 4x^2z^4},$$

$$\sum_{n=0}^{\infty} l_{k,n}Gj_n(x)z^n = \frac{4 - i + k((4x+1)i-2)z + (i(6x+1)-8x-2)z^2}{1 - kz - (2x(k^2+2)+1)z^2 - 2kxz^3 + 4x^2z^4}.$$

By putting $k = 1$ in the Corollary 4.3, we obtain the Table 7.

Table 7: New ordinary generating functions of the products of Fibonacci-like numbers with Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials.

Coefficient of z^n	Ordinary generating function
$l_n GJ_n(x)$	$\dfrac{i+(2-i)z+(2-i(2x+1))z^2}{1-z-(6x+1)z^2-2xz^3+4x^2z^4}$
$l_n Gj_n(x)$	$\dfrac{4-i+((4x+1)i-2)z+(i(6x+1)-8x-2)z^2}{1-z-(6x+1)z^2-2xz^3+4x^2z^4}$

Secondly, let us now consider the following conditions for Eqs. (22) and (23):

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = q \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 2x \\ e_1 e_2 = 1 \end{cases}.$$

Then it gives:

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])\, S_n(e_1+[-e_2])\, z^n = \frac{1-qz^2}{1-2pxz-(4qx^2+p^2+2q)z^2-2pqxz^3+q^2z^4}, \tag{40}$$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2]) S_{n-1}(e_1+[-e_2]) z^n = \frac{pz+2qxz^2}{1-2pxz-(4qx^2+p^2+2q)z^2-2pqxz^3+q^2z^4}. \tag{41}$$

Therefore, we state the following theorems.

Theorem 4.4. *Let $n \geq 0$, we have the novel ordinary generating function of $\left(l_{p,q,n} GP_n(x)\right)$ as follows:*

$$\sum_{n=0}^{\infty} l_{p,q,n} GP_n(x)\, z^n = \frac{2i+2p(1-2ix)z+2q\left(2x-i\left(4x^2+1\right)\right)z^2}{1-2pxz-(4qx^2+p^2+2q)z^2-2pqxz^3+q^2z^4}. \tag{42}$$

Proof. By using the relationships (11) and (16), we obtain:

$$\begin{aligned}
\sum_{n=0}^{\infty} l_{p,q,n} GP_n(x)\, z^n &= \sum_{n=0}^{\infty} 2S_n(a_1+[-a_2])\left(iS_n(e_1+[-e_2])+(1-2ix)\,S_{n-1}(e_1+[-e_2])\right) z^n \\
&= 2i\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&+ 2(1-2ix)\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&= \frac{2i\left(1-qz^2\right)}{1-2pxz-(4qx^2+p^2+2q)z^2-2pqxz^3+q^2z^4} \\
&+ \frac{2(1-2ix)\left(pz+2qxz^2\right)}{1-2pxz-(4qx^2+p^2+2q)z^2-2pqxz^3+q^2z^4} \\
&= \frac{2i+2p(1-2ix)z+2q\left(2x-i\left(4x^2+1\right)\right)z^2}{1-2pxz-(4qx^2+p^2+2q)z^2-2pqxz^3+q^2z^4}.
\end{aligned}$$

This completes the proof. □

Theorem 4.5. *For any natural number n, the novel ordinary generating function of* $\left(l_{p,q,n}GQ_n(x)\right)$ *is found as:*

$$\sum_{n=0}^{\infty} l_{p,q,n}GQ_n(x)z^n = \frac{4(1-ix)+4p\left(i\left(1+2x^2\right)-x\right)z+4q\left(ix\left(4x^2+3\right)-2x^2-1\right)z^2}{1-2pxz-(4qx^2+p^2+2q)z^2-2pqxz^3+q^2z^4}. \tag{43}$$

Proof. We have:

$$l_{p,q,n} = 2S_n(a_1+[-a_2]) \text{ and } GQ_n(x) = (2-2ix)S_n(e_1+[-e_2]) + \left(i\left(2+4x^2\right)-2x\right)S_{n-1}(e_1+[-e_2]).$$

The result can be proved by the same method given in the Theorem 4.4, thus we omit the proof. □

Corollary 4.6. *Putting $p = k$ and $q = 1$ in the Eqs.* (42) *and* (43) *gives the new ordinary generating functions of the products of k-Fibonacci-like numbers with Gaussian Pell and Gaussian Pell Lucas polynomials as follows:*

$$\sum_{n=0}^{\infty} l_{k,n}GP_n(x)z^n = \frac{2i+2k(1-2ix)z+2\left(2x-i\left(4x^2+1\right)\right)z^2}{1-2kxz-(4x^2+k^2+2)z^2-2kxz^3+z^4},$$

$$\sum_{n=0}^{\infty} l_{k,n}GQ_n(x)z^n = \frac{4(1-ix)+4k\left(i\left(1+2x^2\right)-x\right)z+4\left(ix\left(4x^2+3\right)-2x^2-1\right)z^2}{1-2kxz-(4x^2+k^2+2)z^2-2kxz^3+z^4}.$$

By putting $k = 1$ in the Corollary 4.6, we obtain the Table 8.

Table 8: New ordinary generating functions of the products of Fibonacci-like numbers with Gaussian Pell and Gaussian Pell Lucas polynomials.

Coefficient of z^n	**Ordinary generating function**
$l_nGP_n(x)$	$\frac{2i+2(1-2ix)z+2\left(2x-i\left(4x^2+1\right)\right)z^2}{1-2xz-(4x^2+3)z^2-2xz^3+z^4}$
$l_nGQ_n(x)$	$\frac{4(1-ix)+4\left(i\left(1+2x^2\right)-x\right)z+4\left(ix\left(4x^2+3\right)-2x^2-1\right)z^2}{1-2xz-(4x^2+3)z^2-2xz^3+z^4}$

Thirdly, let us now consider the following conditions for Eqs. (22) and (23):

$$\begin{cases} a_1-a_2=p \\ a_1a_2=q \end{cases} \text{ and } \begin{cases} e_1-e_2=x \\ e_1e_2=1 \end{cases}.$$

Then it yields:

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{1-qz^2}{1-pxz-(qx^2+p^2+2q)z^2-pqxz^3+q^2z^4}, \tag{44}$$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{pz+qxz^2}{1-pxz-(qx^2+p^2+2q)z^2-pqxz^3+q^2z^4}. \tag{45}$$

Therefore, we state the following theorems.

Theorem 4.7. *Let $n \in \mathbb{N}$, the novel ordinary generating function of* $\left(l_{p,q,n}GF_n(x)\right)$ *is as follows:*

$$\sum_{n=0}^{\infty} l_{p,q,n}GF_n(x)z^n = \frac{2i+2p(1-ix)z+2q\left(x-i\left(x^2+1\right)\right)z^2}{1-pxz-(qx^2+p^2+2q)z^2-pqxz^3+q^2z^4}. \tag{46}$$

Proof. By using the relationships (13) and (16), we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty} l_{p,q,n} GF_n(x) z^n &= \sum_{n=0}^{\infty} 2S_n(a_1+[-a_2])\left(iS_n(e_1+[-e_2])+(1-ix)S_{n-1}(e_1+[-e_2])\right)z^n \\ &= 2i\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\ &+ 2(1-ix)\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\ &= \frac{2i\left(1-qz^2\right)}{1-pxz-(qx^2+p^2+2q)z^2-pqxz^3+q^2z^4} \\ &+ \frac{2(1-ix)\left(pz+qxz^2\right)}{1-pxz-(qx^2+p^2+2q)z^2-pqxz^3+q^2z^4} \\ &= \frac{2i+2p(1-ix)z+2q\left(x-i\left(x^2+1\right)\right)z^2}{1-pxz-(qx^2+p^2+2q)z^2-pqxz^3+q^2z^4}.\end{aligned}$$

This completes the proof. □

Theorem 4.8. *For any natural number n, the novel ordinary generating function of $\left(l_{p,q,n}GL_n(x)\right)$ is found as:*

$$\sum_{n=0}^{\infty} l_{p,q,n} GL_n(x) z^n = \frac{2(2-ix)+2p\left(i\left(x^2+2\right)-x\right)z+2q\left(ix\left(x^2+3\right)-x^2-2\right)z^2}{1-pxz-(qx^2+p^2+2q)z^2-pqxz^3+q^2z^4}. \quad (47)$$

Proof. Since:

$$l_{p,q,n} = 2S_n(a_1+[-a_2]) \text{ and } GL_n(x) = (2-ix)S_n(e_1+[-e_2]) + \left(i\left(x^2+2\right)-x\right)S_{n-1}(e_1+[-e_2]).$$

The proof is similar to the proof of the Theorem 4.7. □

Corollary 4.9. *Putting $p = k$ and $q = 1$ in the Eqs.* (46) *and* (47) *gives the new ordinary generating functions of the products of k-Fibonacci-like numbers with Gaussian Fibonacci and Gaussian Lucas polynomials as follows:*

$$\sum_{n=0}^{\infty} l_{k,n} GF_n(x) z^n = \frac{2i+2k(1-ix)z+2\left(x-i\left(x^2+1\right)\right)z^2}{1-kxz-(x^2+k^2+2)z^2-kxz^3+z^4},$$

$$\sum_{n=0}^{\infty} l_{k,n} GL_n(x) z^n = \frac{2(2-ix)+2k\left(i\left(x^2+2\right)-x\right)z+2\left(ix\left(x^2+3\right)-x^2-2\right)z^2}{1-kxz-(x^2+k^2+2)z^2-kxz^3+z^4}.$$

By putting $k = 1$ in the Corollary 4.9, we obtain the Table 9.

Table 9: New ordinary generating functions of the products of Fibonacci-like numbers with Gaussian Fibonacci and Gaussian Lucas polynomials.

Coefficient of z^n	Ordinary generating function
$l_n GF_n(x)$	$\frac{2i+2(1-ix)z+2\left(x-i\left(x^2+1\right)\right)z^2}{1-xz-(x^2+3)z^2-xz^3+z^4}$
$l_n GL_n(x)$	$\frac{2(2-ix)+2\left(i\left(x^2+2\right)-x\right)z+2\left(ix\left(x^2+3\right)-x^2-2\right)z^2}{1-xz-(x^2+3)z^2-xz^3+z^4}$

5. Conclusion

In this paper, we introduced a new generalization of Gaussian polynomials. We also gave some results including Binet's formula, explicit formula and ordinary generating function for generalized Gaussian polynomials. Considering these results, we gave Binet's formulas, explicit formulas and ordinary generating functions of Gaussian Pell, Gaussian Pell Lucas, Gaussian Jacobsthal, Gaussian Jacobsthal Lucas, Gaussian Fibonacci and Gaussian Lucas polynomials. Moreover, by using the symmetric functions we have obtained the novel ordinary generating functions for the products of (p,q)-Fibonacci-like numbers with some Gaussian numbers and polynomials. We have summarized the sections as follows:

- **In Section 1,** we presented some preliminaries on (p,q)-Fibonacci-like numbers, Gaussian numbers and symmetric functions.
- **In Section 2,** we organized this section in two parts, in part 1, we introduced novel generalization of polynomials which is called generalized Gaussian polynomials and we gave the general solution of them. In part 2, we have obtained the ordinary generating functions, symmetric functions and explicit formulas of generalized Gaussian polynomials and (p,q)-Fibonacci-like numbers.
- **In Section 3,** by considering the Theorem 3.1, we have obtained several ordinary generating functions such as: $\sum\limits_{n=0}^{\infty} l_{p,q,n}GJ_nz^n$, $\sum\limits_{n=0}^{\infty} l_{p,q,n}GQ_nz^n$, $\sum\limits_{n=0}^{\infty} l_{p,q,n}GF_nz^n$, $\sum\limits_{n=0}^{\infty} l_{k,n}Gj_nz^n$, $\sum\limits_{n=0}^{\infty} l_nGP_nz^n$, $\sum\limits_{n=0}^{\infty} l_nGL_nz^n$, ...etc.
- **In Section 4,** by making use the symmetric functions and the products of (p,q)-Fibonacci-like numbers with Gaussian polynomials, we have derived some new theorems and corollaries on ordinary generating functions.

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