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Kirchhoff-type problem involving the fractional p-Laplacian via Young measures

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Abstract. In this paper, we study the local existence of weak solutions for a Kirchhoff-type problem involving the fractional p-Laplacian. Under some conditions on the main functions, we obtain the existence of weak solutions by using the Galerkin method combined with the theory of Young measures. In addition, an example is given to illustrate the main results.

1. Introduction

In this paper, suppose that Ω is a bounded open domain of $\mathbb{R}^n$ $(n \geqslant 2)$ and T is a real positive number, we deal with the following initial boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} + M(\|u\|^p_{W_0})(-\Delta)^s_p u = f(x,t,u) & \text{in } Q_T = \Omega \times (0,T), \\ u = 0 & \text{in } (\mathbb{R}^n \backslash \Omega) \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $0 < s < 1$ and $2 < p$ are real numbers, W_0 will be defined later, $u : \Omega \times (0,T) \to \mathbb{R}^m$, $m \in \{0,1,2...\}$ is a vector-valued function and the functions f and M satisfies the following hypothesis:
$(H1)$ $f : \Omega \times (0,T) \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function satisfies

$$|f(x,t,r)| \leq \alpha_0 \left(1 + |r|^{q-1}\right),$$
$$F_t(x,t,r) \geq \alpha_1 \left(-1 - |r|^q\right),$$

for all $(x,t,r) \in \Omega \times (0,T) \times \mathbb{R}^m$, with α_0, α_1 are a positive constants, F_t is the derivative of F and $F(x,t,r) = \int_0^r f(x,t,l)dl$.
$(H2)$ $M : \mathbb{R}^+ \to \mathbb{R}^+$is a continuous function such that there exist constants m_0, m_1 with $0 < m_0 \leq m_1$ and $\beta \geq 1$ such that $m_0 s^{\beta-1} \leq M(s) \leq m_1 s^{\beta-1}$ for all $s > 0$.
The fractional p-Laplacian operator $(-\Delta)^s_p u$ is defined as follows:

$$(-\Delta)^s_p u(x,t) = P.V \int_{\mathbb{R}^n} \frac{|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))}{|x-y|^{n+ps}} dy, \quad x \in \mathbb{R}^n,$$

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where $P.V$, which stands for "in the principal value sense," is a frequently used abbreviation. For more information on this operator see [10].

Recently, there has been a lot of interest in the systematic study of problems involving fractional Laplacian due to their frequency in practical real-world applications, such as continuum mechanics, minimal surfaces, conversation laws, population dynamics, image processing, finance, and many others, see for example [4, 10, 17] and the references therein. The interest in studying problems like (1) relies not only on mathematical purposes but also on their contributions to the modeling of many physical and biological phenomena, for more details see [11, 21, 26, 27]. In [16], a stationary Kirchhoff variational equation was first proposed by Fiscella and Valdinoci as a model to study the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. Indeed, the stationary problem of (1) is a fractional version of a model, the so-called stationary Kirchhoff equation, which was introduced by Kirchhoff in [18] as a model to study elastic string vibrations.

Recently, many authors investigated the existence results for the problem (1), for example, Pan et al. [22] studied the existence of a global solution by using the Galerkin method and potential well theory. They assumed the function M to satisfy $M(s) = s^{\lambda-1}$, $p < q < p_s^*$, $1 < p < \frac{n}{s}$, $\lambda \in [1, \frac{p_s^*}{p})$ and $f(u) = |u|^{q-1}u$. Based On the sub-differential approach, the authors in [24] established the well-posedness of solutions for problem (1) where f is independent of u. For the interested reader, we refer also to [8].

Concerning the fractional Laplacian, the authors in [21] obtained a nonnegative local weak solution to the following problem

$$\begin{cases} u_t + M\left([u]_s^2\right)(-\Delta)^s u = |u|^{p-2}u & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{in } \left(\mathbb{R}^N \backslash \Omega\right) \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2}$$

Moreover, by combining the Galerkin method with the potential well theory they proved also an estimate for the lower and upper bounds of the blow-up time.

When $M \equiv 1$, problem (1) reduces to the fractional p-Laplacian problem:

$$\begin{cases} u_t + (-\Delta)_p^s u = f(x,t,u) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 & \text{in } \left(\mathbb{R}^N \backslash \Omega\right) \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3}$$

The problem (3) has been studied by many researchers, for example In [1], the authors have studied the problem (1) with f depends only on x and t and prove the existence results with suitable regularity if $(f, u_0) \in L^1(\Omega_T) \times L^1(\Omega)$ and has a nonnegative entropy solution if f_0, u_0 are nonnegative. The same author in [2] proved the asymptotic behavior result of entropy solutions when the right-hand side does not depend on time, see also [25].

It is worth mentioning that problem (1) can be regard as a fractional version of the initial-boundary value problem of the following equation:

$$\begin{cases} \frac{\partial u}{\partial t} - M\left(\int_\Omega |Du|^p dx\right) \operatorname{div}(|Du|^{p-2}Du) = f(x,t,u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases} \tag{4}$$

In [28] studied the existence of local solutions for problem (4) using the Galerkin method and the properties of Sobolev space. Based on the theory of Young measures, the existence of weak solutions to (4) was proved by Balaadich in [7].

Motivated by all of the results above, especially [7], we study the local existence of a weak solution to Eq. (1) using Galerkin approximation and the theory of Young measures. To the best of our knowledge, this is the first paper that treats the problem (1) by such a theory. We suggest to the readers to consult [5, 6, 13] which treats some elliptic and parabolic systems by the Young measures theory.

This article is organized into four sections:
(2) we give some background information on fractional Sobolev spaces and a review of the Young measures theory.
(3) Under some assumptions, we obtain the existence of weak solutions using the Galerkin approximation and Young measures.
(4) is devoted to illustrating the feasibility of the hypotheses with an example.

2. Preliminaries and notations

In this section, we first recall some necessary results which will be used in the next section. Let $1 < p < \infty$, $s \in (0,1)$ and we define p_s^* the fractional critical exponent by:

$$p_s^* = \begin{cases} \infty \text{ if } ps \geq n, \\ np/(n-ps) \text{ if } ps < n. \end{cases}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, $Q_\Omega = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (C\Omega \times C\Omega)$, $Q_\tau = \Omega \times (0,\tau)$ for all $\tau \in (0,T]$ and $C\Omega = \mathbb{R}^n \setminus \Omega$. It is clear that $\Omega \times \Omega$ is strictly contained in Q_Ω. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega;\mathbb{R}^m)$ and

$$\iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega;\mathbb{R}^m)} + \left(\iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx \right)^{\frac{1}{p}}.$$

And the closed linear subspace

$$W_0 = \{u \in W : u = 0 \text{ a.e. in } C\Omega\}.$$

In W_0, we may also use the norm

$$\|u\|_{W_0} = \left(\iint_{Q_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dydx \right)^{\frac{1}{p}}.$$

It is known that $(W_0, \|\cdot\|_{W_0})$ is a uniformly convex reflexive Banach space (see [28]). The Poincaré's inequality [9] will be used below: there exists $C_r > 0$ such that

$$\|\phi\|_{L^r(\Omega,\mathbb{R}^m)} \leq C_r \|\phi\|_{W_0} \quad \text{for all} \quad \phi \in W_0 \quad \text{and for all} \quad r \in [1, p_s^*]. \tag{5}$$

In the sequel, let $p < \frac{n}{s}$ and $C_i, i = 1,2,...$ be positive constants that vary from line to line, and are independent of the terms involved in any limit process.
We note the following functional space $L^p(0,T;W_0)$, which is a separable and reflexive Banach space endowed with the norm

$$\|u\|_{L^p(0,T;W_0)} = \left(\int_0^T \|u\|_{W_0}^p dt \right)^{\frac{1}{p}}.$$

Lemma 2.1. *[15] $C_0^\infty(\Omega;\mathbb{R}^m)$ is a space of infinitely differentiable functions with compact support on Ω which is dense in W_0.*

Lemma 2.2. *[14] The following embedding $W_0 \hookrightarrow L^r(\Omega;\mathbb{R}^m)$ is compact for all $r \in [1, p_s^*)$, and continuous for all $r \in [1, p_s^*]$.*

In the following, $C_0(\mathbb{R}^m)$ stands for the space of continuous functions on $\mathbb{R}^m$ with compact support with regard to the $\|.\|_\infty$-norm. The space of signed Radon measures with finite mass is noted $\mathcal{M}(\mathbb{R}^m)$. The corresponding duality is given by

$$\langle \mu, \rho \rangle = \int_{\mathbb{R}^m} \rho(\lambda) d\mu(\lambda).$$

Definition 2.3. *[6] Let a bounded sequence noted by* $\{z_j\}_{j\geq 1}$ *in* $L^\infty(\Omega;\mathbb{R}^m)$. *Then there exists a subsequence* $\{z_k\} \subset \{z_j\}$ *and a Borel probability measure* μ_x *on* $\mathbb{R}^m$ *for almost every* $x \in \Omega$, *such that for a.e.* $\rho \in C(\mathbb{R}^m)$ *we have* $\rho(z_k) \rightharpoonup^* \bar{\rho}$ *weakly in* $L^\infty(\Omega)$, *where* $\bar{\rho}(x) = \langle \mu_x, \rho \rangle = \int_{\mathbb{R}^m} \rho(\lambda) d\mu_x(\lambda)$ *for a.e.* $x \in \Omega$.

Lemma 2.4. *[19] Let* $\Omega \subset \mathbb{R}^n$ *be Lebesgue measurable (not necessarily bounded) and* z_j *from* Ω *to* $\mathbb{R}^m$, *with* $j \in \mathbb{N}$, *be a sequence of Lebesgue measurable functions. Then there exist a subsequence* z_k *and a family* $\{\mu_x\}_{x\in\Omega}$ *of non-negative Radon measures on* $\mathbb{R}^m$, *such that*

- ***i)*** $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^m)} := \int_{\mathbb{R}^m} d\mu_x(\lambda) \leq 1$ *for almost* $x \in \Omega$.
- ***ii)*** $\rho(z_k) \rightharpoonup^* \bar{\rho}$ *weakly in* $L^\infty(\Omega)$ *for all* $C_0(\mathbb{R}^m)$, *where* $\bar{\rho} = \langle \mu_x, \rho \rangle$.
- ***iii)*** *If for all* $M > 0$

$$\lim_{N\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_M(0) : |z_k(x)| \geq N\}| = 0, \tag{6}$$

then $\|\mu_x\| = 1$ *for a.e.* $x \in \Omega$, *and for any measurable* $\Omega' \subset \Omega$ *we have* $\rho(z_k) \rightharpoonup \bar{\rho} = \langle \mu_x, \rho \rangle$ *weakly in* $L^1(\Omega')$ *for continuous function* ρ *provided the sequence* $\rho(z_k)$ *is weakly precompact in* $L^1(\Omega')$.

3. Local existence of weak solutions

In this section, we will define a weak solution to the problem (1) and prove the main results. We start with the following definition:

Definition 3.1. *A function* $u \in L^p(0,T;W_0) \cap C(0,T;L^2(\Omega,\mathbb{R}^m))$ *is called a weak solution of* (1), *if* $\frac{\partial u}{\partial t} \in L^2(Q_T;\mathbb{R}^m)$ *and*

$$\int_{Q_T} \frac{\partial u}{\partial t} \phi dxdt + \int_0^T M(\|u\|^p_{W_0}) \langle u, \phi \rangle_{W_0} dt = \int_{Q_T} f(x,t,u)\phi dxdt,$$

holds for all $\phi \in C^1(0,T;C_0^\infty(\Omega))$. *Where*

$$\langle u, \phi \rangle_{W_0} := \iint_{Q_\Omega} \frac{|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))}{|x-y|^{n+ps}} (\phi(x,t) - \phi(y,t)) dxdy.$$

Theorem 3.2. *If* $u_0 \in W_0$, $2 < q < \frac{(2+p)p_s^* - 2p}{p_s^*} < p_s^*$ *and suppose that* $(H1) - (H2)$ *are satisfied, then there exists a constant* $T_0 > 0$ *such that problem* (1) *has at least one weak solution as* $T < T_0$.

Proof. The proof is divided into three assertions:
Assertion 1: Galerkin approximation

Similar to that in [28], we take a sequence $\{w_j\}_{j\geq 1} \subset C_0^\infty(\Omega;\mathbb{R}^m)$, such that $C_0^\infty(\Omega;\mathbb{R}^m) \subset \overline{\bigcup_{k\geq 1} U_k}^{C_1(\bar{\Omega})}$, where $\{w_j\}_{j\geq 1}$ is an orthonormal basis in $L^2(\Omega;\mathbb{R}^m)$ and $U_k = \text{span}\{w_1, \ldots, w_k\}$.

Lemma 3.3. *For the function* $u_0 \in W_0$, *there exists a subsequence* $\xi_k \in U_k$ *such that* $\xi_k \to u_0$ *in* W_0 *as* $k \to \infty$.

Proof. Since $u_0 \in W_0$, we can find a sequence v_k in $C_0^\infty(\Omega;\mathbb{R}^m)$ such that $v_k \to u_0$ in W_0. Since $\{v_k\} \subset C_0^\infty(\Omega;\mathbb{R}^m) \subset \bigcup_{N\geq 1} \overline{U_N}^{C^1(\bar{\Omega};\mathbb{R}^m)}$, there exists a sequence $\left\{v_k^i\right\} \subset \bigcup_{N\geq 1} U_N$ such that $v_k^i \to v_k$ in $C^1(\bar{\Omega};\mathbb{R}^m)$ as i tends to ∞. For $\frac{1}{2^k}$, there exists $i_k \geq 1$ such that $\left\|v_k^{i_k} - v_k\right\|_{C^1(\bar{\Omega})} \leq \frac{1}{2^k}$. Therefore

$$\left\|v_k^{i_k} - u_0\right\|_{W_0} \leq C_1 \left\|v_k^{i_k} - v_k\right\|_{C^1(\bar{\Omega})} + \|v_k - u_0\|_{W_0}.$$

Hence $v_k^{i_k} \to u_0$ in W_0 as k tends to ∞. We denote $u_k = v_k^{i_k}$, since $u_k \in \bigcup_{N\geq 1} U_N$, there exists U_{N_k} such that $u_k \in U_{N_k}$, without loss of generality, we assume that $U_{N_1} \subset U_{N_2}$ as $N_1 \leq N_2$. We suppose that $N_1 > 1$ and define ξ_k as follows:

$$\begin{cases} \xi_k(x) = 0 & k = 1,\dots,N_1 - 1,\\ \xi_k(x) = u_1 & k = N_1,\dots,N_2 - 1,\\ \xi_k(x) = u_2 & k = N_2,\dots,N_3 - 1,\\ \dots & \dots \end{cases}$$

we obtain the desired sequence $\{\xi_k\}$ and $\xi_k \to u_0$ in W_0 as k tends to ∞. □

We define the function $R_k : [0,T) \times \mathbb{R}^k \to \mathbb{R}^k$ where k is fixed:

$$(R(t,\varsigma))_i = M\left(\|\sum_{j=1}^k (\varsigma(t))_j\, w_j(x)\|_{W_0}^p\right)\langle \sum_{j=1}^k \left(\varsigma_j(t)\right)_j w_j(x), w_i\rangle_{W_0}$$

for $\varsigma \in \mathbb{R}^k$ and $i = 1,\dots,k$. The function $R(t,\varsigma)$ is continuous in t and ς.
Now, we shall construct the approximating solutions for (1) as follows:

$$u_k(x,t) = \sum_{j=1}^k \left(b_j(t)\right)_j w_j(x),$$

where unknown functions $(b(t))_j$ are determined by the following system of ODE

$$\begin{cases} b'(t) + R_k(t,b(t)) = S_k(t,b(t)), \quad 0 < t < T,\\ b(0) = \psi_k(0), \end{cases} \tag{7}$$

where

$$(S_k(t,b))_i = \int_\Omega f(x,t,\sum_{j=1}^k b_j w_j) w_i dx,$$

$$(\psi_k(0))_i = \int_\Omega \xi_k(x) w_i dx$$

and

$$\xi_k(x) \to u_0 \text{ in } W_0 \text{ as } k \to \infty \text{ where } \xi_k(x) \in U_k.$$

We multiply Equation (7) by $b(t)$, we get

$$b'b + R_k(t,b)b = S_k(t,b)b. \tag{8}$$

(H2) implies that

$$\begin{aligned} R_k(t,b)b &= M\left(\|\sum_{j=1}^k \left(b_j(t)\right)_j w_j(x)\|_{W_0}^p\right)\langle \sum_{j=1}^k \left(b_j(t)\right)_j w_j, \sum_{i=1}^k (b_i(t))_i\, w_i\rangle_{W_0}\\ &\geqslant m_0\left(\|\sum_{j=1}^k \left(b_j(t)\right)_j w_j(x)\|_{W_0}^p\right)^\beta \end{aligned}$$

According to $(H1)$, the following inequality holds

$$\begin{aligned} S_k(t,b)b &\leq \alpha_0 \int_\Omega \left|\sum_{j=1}^k b_j w_j\right|^q + \left|\sum_{j=1}^k b_j w_j\right| dx \\ &\leq \alpha_0 \int_\Omega \left|\sum_{j=1}^k b_j w_j\right|^q dx + \alpha_0 C_2 \int_\Omega \left|\sum_{j=1}^k b_j w_j\right|^2 dx. \end{aligned} \tag{9}$$

Since $2 < q < p_s^*$, using the interpolation inequality (see [3, Theorem 2.11]) and (5), we get

$$\int_\Omega \left|\sum_{j=1}^k b_j w_j\right|^q dx \leq \left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m)}^{\theta q} \left\|\sum_{j=1}^k b_j w_j\right\|_{L^{p_s^*}(\Omega;\mathbb{R}^m)}^{(1-\theta)q} \leq C_{p_s^*} \left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m)}^{\theta q} \left\|\sum_{j=1}^k b_j w_j\right\|_{W_0}^{(1-\theta)q}, \tag{10}$$

where $\theta \in (0,1)$ satisfies

$$\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{p_s^*}.$$

We observe that

$$(1-\theta)q = \frac{p_s^*(q-2)}{p_s^*-2} < p < p_s^*$$

and

$$\lambda := \frac{p\theta q}{p-(1-\theta)q} = \frac{2p(p_s^*-q)}{p_s^*(p-q+2)-2p} > 2.$$

For any $\epsilon \in (0,1)$, the Young inequality implies

$$\left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m)}^{\theta q} \left\|\sum_{j=1}^k b_j w_j\right\|_{W_0}^{(1-\theta)q} \leq \epsilon \left\|\sum_{j=1}^k b_j w_j\right\|_{W_0}^{p} + C(\epsilon) \left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m)}^{\lambda}. \tag{11}$$

Then, the inequality (10) is transformed into the following inequality

$$\int_\Omega \left|\sum_{j=1}^k b_j w_j\right|^q dx \leq C_{p_s^*}\epsilon \left\|\sum_{j=1}^k b_j w_j\right\|_{W_0}^{p} + C(\epsilon) \left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m)}^{\lambda}. \tag{12}$$

Plugging inequalities (9), (10) and (12) into (8), we deduce that

$$\frac{1}{2}\frac{d|b(t)|^2}{dt} + m_0 \left\|\sum_{j=1}^k b_j w_j\right\|_{W_0}^{p} \leq C_{p_s^*}\alpha_0\epsilon \left\|\sum_{j=1}^k b_j w_j\right\|_{W_0}^{p} + \alpha_0 C(\epsilon) \left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m)}^{\lambda} + \alpha_0 \left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m)}^{2}.$$

By choosing $\epsilon = \frac{m_0}{2\alpha_0 C_{p_s^*}}$, we get

$$\frac{1}{2}\frac{d|b(t)|^2}{dt} + \frac{m_0}{2} \left\|\sum_{j=1}^k b_j w_j\right\|_{W_0}^{p} \leq \alpha_0 C(\epsilon) \left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m)}^{\lambda} + \alpha_0 \left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m)}^{2}, \tag{13}$$

It follows that

$$\frac{d|b(t)|^2}{dt} \leq 2C_3 \left(\left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m))}^{\lambda} + \left\|\sum_{j=1}^k b_j w_j\right\|_{L^2(\Omega;\mathbb{R}^m))}^{2} \right). \tag{14}$$

Denote $z(t) = |b(t)|^2$, then

$$\frac{dz(t)}{dt} \leq 2C_3\left(z(t)^{\frac{\lambda}{2}} + z(t)\right). \tag{15}$$

Integrating (15) from 0 to t, and using that

$$z(0) = |b(0)|^2 = \int_\Omega \xi_k^2(x)dx \leq C_4,$$

we can conclude that

$$z(t) \leq \exp(2C_3 t)\left(C_4^{1-\frac{\alpha}{2}} - \exp(C_3(\alpha-2)t)\right)^{\frac{2}{2-\alpha}}, \text{ as } t < \frac{\ln(C_4^{1-\frac{\alpha}{2}})}{C_3(\alpha-2)}.$$

For $0 < T < T_0 = \frac{\ln(C_4^{1-\frac{\alpha}{2}})}{C_3(\alpha-2)}$, we obtain that $|b(t)| \leq C(T)\ \forall t \in [0,T]$, where

$$C(T) = \exp(2C_3 T)\left(C_4^{1-\frac{\alpha}{2}} - \exp(C_3(\alpha-2)T)\right)^{\frac{2}{2-\alpha}}.$$

Putting

$$\mathcal{J}_k = \max_{(t,b)\in[0,T]\times B(b(0),2C(T))} |S_k - R_k(t,b)| \quad \text{and} \quad \beta_k = \min\left\{T, \frac{2C(T)}{\mathcal{J}_k}\right\},$$

where $B(b(0), 2C(T))$ is the ball of center $b(0)$ and radius $2C(T)$. By [12, Peano theorem], we know that problem (7) has a C^1 solution on $[0,\beta_k]$. We put $l_1 = \beta_k$ and $b(l_1)$ be an initial value, then we can repeat the above process and get C^1 solution on $[l_1,l_2]$ with $l_2 = l_1 + \beta_k$, which means the existence of an interval $[l_{i-1}, l_{i-2}] \subset [0,T]$, such that (7) has a solution on $[l_{i-1}, l_{i-2}]$ where $l_i = l_{i-1} + \beta_k, i = 1,\ldots,N-1, l_N = T$. Therefore, we get a solution $b_k(t) \in C^1([0,T])$. As a result, we get the desired Galerkin approximation solution.

Assertion 2: A priori estimates

By (7), we have

$$\int_\Omega \frac{\partial u_k}{\partial t} w_i dx + M\left(\|u_k\|_{W_0}^p\right)\langle u_k, w_i\rangle_{W_0} = \int_\Omega f(x,t,u_k)\, w_i dx, \tag{16}$$

where $1 \leq i \leq k$ and $t \in [0,T]$ $(T < T_0)$.

Multiplying (16) by $(b(t))_i$ $\left(\text{resp. by } \frac{d}{dt}(b(t))_i\right)$ and summing with respect to i from 1 to k, we arrive at (integrating with respect to t from 0 to τ ($\tau \in (0,T]$))

$$\int_{Q_\tau} \frac{\partial u_k}{\partial t} u_k dxdt + \int_0^\tau M\left(\|u_k\|_{W_0}^p\right)\|u_k\|_{W_0} dt = \int_{Q_\tau} f(x,t,u_k)\, u_k dxdt, \tag{17}$$

$$\int_\Omega \left|\frac{\partial u_k}{\partial t}\right|^2 dx + M\left(\|u_k\|_{W_0}^p\right)\langle u_k, \frac{\partial u_k}{\partial t}\rangle_{W_0} = \int_\Omega f(x,t,u_k)\frac{\partial u_k}{\partial t}dx. \tag{18}$$

According to (13) and $\beta \geq 1$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_k(x,t)^2 dx + \frac{m_0}{2}\left(\|u_k\|_{W_0}^p\right) \leq C_5\left(\left(\int_\Omega |u_k|^2\, dx\right)^{\frac{\lambda}{2}} + \int_\Omega |u_k|^2\, dx\right),$$

similar to the estimation of $b(t)$, we have

$$\int_\Omega |u_k(x,t)|^2\, dx \leq C(T),\ \forall t \in [0,T] \quad (T < T_0). \tag{19}$$

Moreover

$$\|u_k\|_{L^p(0,T;W_0)} \leq C_6. \tag{20}$$

Also, we get

$$\|u_k\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))} \leq C_7. \tag{21}$$

By virtue of (18) and $(H1)$, it yields

$$\begin{aligned} \int_\Omega \left|\frac{\partial u_k}{\partial t}\right|^2 dx + \frac{m_0}{p}\left(\|u_k\|_{W_0}^p\right)^{\beta-1}\frac{d}{dt}\|u_k\|_{W_0}^p - \frac{d}{dt}\int_\Omega F(x,t,u_k)\,dx &\leq -\int_\Omega F_t(x,t,u_k)\,dx \\ &\leq \alpha_1\int_\Omega |u_k|^q\,dx + \alpha_1. \end{aligned} \tag{22}$$

From the fact

$$\frac{m_0}{p}\left(\|u_k\|_{W_0}^p\right)^{\beta-1}\frac{d}{dt}\|u_k\|_{W_0}^p = \frac{m_0}{p\beta}\frac{d}{dt}\left(\|u_k\|_{W_0}^p\right)^\beta, \tag{23}$$

plugging (23) into (22) and since $\beta \geqslant 1$, we deduce

$$\int_\Omega \left|\frac{\partial u_k}{\partial t}\right|^2 dx + \frac{d}{dt}\left(\frac{m_0}{p\beta}\|u_k\|_{W_0}^p - \int_\Omega F(x,t,u_k)\,dx\right) \leq \alpha_1\left(\int_\Omega |u_k|^q dx + 1\right). \tag{24}$$

By using the same technique in (11) and using (19) to the term in the right-hand side in (24), we get

$$\begin{aligned} &\int_\Omega \left|\frac{\partial u_k}{\partial t}\right|^2 dx + \frac{d}{dt}\left(\frac{m_0}{p\beta}\|u_k\|_{W_0}^p - \int_\Omega F(x,t,u_k)\,dx\right) \\ &\qquad \leq \alpha_1\epsilon C_{p_s^*}\|u_k(x,t)\|_{W_0}^p + \alpha_1 C(\epsilon)\left(\int_\Omega |u_k|^2 dx\right)^{\frac{\lambda}{2}} + \alpha_1 \\ &\qquad \leq C_8\left(\|u_k(x,t)\|_{W_0}^p + 1\right). \end{aligned} \tag{25}$$

Integrating (25) with respect to t from 0 to τ ($\tau \in (0,T]$) and using the strong convergence of $u_k(x,0) \to u_0(x)$ in W_0, we get

$$\int_{Q_\tau} \left|\frac{\partial u_k}{\partial t}\right|^2 dxdt + \frac{m_0}{p\beta}\|u_k(x,\tau)\|_{W_0}^p \leq C_9\left(\int_0^\tau \|u_k(x,t)\|_{W_0}^p dt + 1\right) + \int_\Omega F(x,\tau,u_k)\,dx. \tag{26}$$

By assumption $(H1)$ and interpolation inequality using in (11), we get

$$\int_\Omega F(x,\tau,u_k)\,dx \leq \alpha_1\epsilon C_{p_s^*}\|u_k(x,\tau)\|_{W_0}^p + \alpha_1 C(\epsilon)\left(\int_\Omega |u_k|^2 dx\right)^{\frac{\lambda}{2}}. \tag{27}$$

Plugging (27) in (26), we arrive at

$$\int_{Q_\tau} \left|\frac{\partial u_k}{\partial t}\right|^2 dxdt + \frac{m_0}{p\beta}\|u_k(x,\tau)\|_{W_0}^p \leq C_9\left(\int_0^\tau \|u_k(x,t)\|_{W_0}^p dt + 1\right) + \alpha_1\epsilon C_{p_s^*}\|u_k(x,\tau)\|_{W_0}^p + \alpha_1 C(\epsilon)\left(\int_\Omega |u_k|^2 dx\right)^{\frac{\lambda}{2}}.$$

By choosing $\epsilon = \frac{m_0}{2\alpha_1 p C_{p_s^*}}$, we get

$$\int_{Q_\tau} \left|\frac{\partial u_k}{\partial t}\right|^2 dxdt + \frac{m_0}{2p\beta}\|u_k(x,\tau)\|_{W_0}^p \leq C_{10}\left(\int_0^\tau \|u_k(x,t)\|_{W_0}^p dt + 1\right).$$

The Gronwall inequality implies that $\int_0^\tau \|u_k(x,t)\|_{W_0}^p dt \leq C_{11}$ for each $\tau \in [0,T]$. Therefore

$$\int_{Q_\tau} \left|\frac{\partial u_k}{\partial t}\right|^2 dxdt + \frac{m_0}{2p\beta}\|u_k\|_{W_0}^p \leq C_{12}, \tag{28}$$

We finally get

$$\left\|\frac{\partial u_k}{\partial t}\right\|_{L^2(Q_T)} + \|u_k\|_{L^\infty(0,T;W_0)} \leq C_{13}. \tag{29}$$

The assumption $(H1)$ imply that

$$\left\|f(x,t,u_k)\right\|_{L^{q'}(Q_T)} \leq C_{14}. \tag{30}$$

Assertion 3 : Passage to the limit

By virtue of (20), (21), (29), and (30), there exists a subsequence of (u_k) (still denoted by (u_k)) such that

$$\begin{cases} u_k \rightharpoonup u \text{ in } L^p(0,T;W_0), \\ u_k \rightharpoonup^* u \text{ in } L^\infty\left(0,T;L^2(\Omega;\mathbb{R}^m)\right) \cap L^\infty(0,T;W_0), \\ \frac{\partial u_k}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(Q_T;\mathbb{R}^m) \\ f(x,t,u_k) \rightharpoonup \chi \text{ in } L^{q'}(Q_T,\mathbb{R}^m). \end{cases} \tag{31}$$

[20, Theorem 5.1] and (31) implies that $u_k \to u$ in $L^p(0,T,L^2(\Omega;\mathbb{R}^m))$ and a.e. on Q_T (for a subsequence), and [20, Lemma 1.3] implies that $f(x,t,u) = \chi$. We can conclude from the continuity in $(H1)$,

$$f(x,t,u_k)\,u_k \to f(x,t,u)u \text{ a.e. in } Q_T.$$

Using the Vitali Theorem, we get

$$\lim_{k\to\infty}\int_{Q_T} f(x,t,u_k)\,u_k dxdt = \int_{Q_T} f(x,t,u)udxdt.$$

By $\int_\Omega u_k(x,T)^2 dx \leq C_{15}$, there exist a subsequence (still labelled by k) of $(u_k(x,T))$ and a function $\hat{u}$ in $L^2(\Omega;\mathbb{R}^m)$ such that $u_k(x,T) \rightharpoonup \hat{u}$ in $L^2(\Omega;\mathbb{R}^m)$. Then, for any $b(t) \in C^1([0,T])$ and $\phi \in C_0^\infty(\Omega)$, such that

$$\int_{Q_T} \frac{\partial u_k}{\partial t} b\phi dxdt = \int_\Omega u_k(x,T)b(T)\phi dx - \int_\Omega u_k(x,0)b(0)\phi dx - \int_{Q_T} u_k \frac{\partial b}{\partial t}\phi dxdt.$$

Tending k to ∞, we get

$$\int_\Omega (\hat{u} - u(x,T))\,b(T)\phi dx - \int_\Omega (u_0(x) - u(x,0))\,b(0)\phi dx = 0.$$

We choosing $b(T) = 1, b(0) = 0$ or $b(T) = 0, b(0) = 1$, by the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$, we have $\hat{u} = u(x,T)$ and $u_0(x) = u(x,0)$ and $u_k(x,T) \rightharpoonup u(x,T)$ in $L^2(\Omega)$.

As stated in the introduction, Young measures is the tool we use to prove the existence of a weak solution. To identify the weak limit, we consider the following lemma:

Lemma 3.4. *Suppose that* (20) *holds. Then, the Young measure* $\mu_{(x,y,t)}$ *generated by* $\frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n}{p}+s}} \in L^p(Q_\Omega \times (0,T);\mathbb{R}^m)$ *has the following properties:*

1) $\left\|\mu_{(x,y,t)}\right\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ *for a.e.* $(x,y,t) \in Q_\Omega \times (0,T)$, *i.e.* $\mu_{(x,y,t)}$ *is a probability measure.*

2) $\left\langle \mu_{(x,y,t)}, id \right\rangle = \int_{\mathbb{R}^m} \lambda d\mu_{(x,y,t)}(\lambda)$ *is the weak* L^1*-limit of* $\frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n}{p}+s}}$.

3) $\left\langle \mu_{(x,y,t)}, id \right\rangle = \frac{u(x,t)-u(y,t)}{|x-y|^{\frac{n}{p}+s}}$ *for a.e.* $(x,y,t) \in Q_\Omega \times (0,T)$.

Proof. 1) For simplicity reasons, we consider

$$v_k(x,y,t) = \frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n}{p}+s}} \in L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right). \tag{32}$$

We know that for any $G > 0$, $(\Omega \cap B_G)^2 \subseteq \Omega \times \Omega \subsetneq Q_\Omega$, where B_G is the ball centered in 0 with radius G. Let $N \in \mathbb{R}$ such that

$$Q_N \equiv \left\{(x,y,t) \in \Omega \cap B_G \times \Omega \cap B_G \times (0,T) : \left|v_k(x,y,t)\right| \geq N\right\}.$$

Using (20), we get

$$\begin{aligned} \|v_k\|_{L^p(Q_\Omega \times (0,T);\mathbb{R}^m)} &= \left(\int_0^T \iint_{Q_\Omega} \frac{|u_k(x,t)-u_k(y,t)|^p}{|x-y|^{n+ps}} dxdydt\right)^{\frac{1}{p}} \\ &= \|u_k\|_{L^p(0,T;W_0)} \leq G. \end{aligned}$$

Consequently, there exists $C_{16} \geq 0$ such that

$$C_{16} \geq \iint_{Q_\Omega \times (0,T)} \left|v_k(x,y,t)\right|^p dxdy \geq \iint_{Q_N} \left|v_k(x,y,t)\right|^p dxdy \geq N^p \left|Q_N\right|, \tag{33}$$

where $|Q_N|$ is the Lebesgue measure of Q_N. According to equation (33), the sequence (v_k) satisfies the equation (6). Hence, a Young measure noted by $\mu_{(x,y,t)}$ is generated by v_k such that $\left\|\mu_{(x,y,t)}\right\|_{\mathcal{M}(\mathbb{R}^m)} = 1$ for a.e. $(x,y,t) \in Q_\Omega \times (0,T)$.

2) By (20), there exists a subsequence still denoted by (v_k) that converges in $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. Since $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$ is reflexive, then v_k is weakly convergent in $L^1\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. By the third assertion in Lemma 2.4, we replace the function ρ by the identity function, we then have

$$v_k \rightharpoonup \left\langle \mu_{(x,y,t)}, id \right\rangle = \int_{\mathbb{R}^m} \lambda d\mu_{(x,y,t)}(\lambda) \text{ weakly in } L^1\left(Q_\Omega \times (0,T); \mathbb{R}^m\right).$$

3) According to (20), v_k is bounded in $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$, then there exists a subsequence such that $v_k \rightharpoonup v$ in $L^p\left(Q_\Omega \times (0,T); \mathbb{R}^m\right)$. Owing to the previous arguments, we get from the uniqueness of limits that

$$\left\langle \mu_{(x,y,t)}, id \right\rangle = v(x,y,t) = \frac{u(x,t)-u(y,t)}{|x-y|^{\frac{n}{p}+s}} \quad \text{for a.e. } (x,y,t) \in Q_\Omega \times (0,T).$$

□

Now, let $\{v_k\}$ be the sequence given in (32), i.e.

$$v_k(x,y,t) = \frac{u_k(x,t)-u_k(y,t)}{|x-y|^{\frac{n+ps}{p}}}.$$

The weak convergence given in Lemma 3.4 shows that:

$$\begin{aligned} |v_k(x,y,t)|^{p-2} v_k(x,y,t) &\rightharpoonup \int_{\mathbb{R}^m} |\lambda|^{p-2} \lambda d\mu_{(x,y,t)}(\lambda) \\ &= |v(x,y,t)|^{p-2} v(x,y,t) \\ &= \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{\frac{n+ps}{p'}}} \end{aligned} \tag{34}$$

weakly in $L^1(Q_\Omega \times (0,T);\mathbb{R}^m)$. Since L^p is reflexive and $|v_k(x,y,t)|^{p-2}v_k(x,y,t)$ is bounded in $L^{p'}(Q_\Omega \times (0,T);\mathbb{R}^m)$, the sequence $|v_k(x,y,t)|^{p-2}v_k(x,y,t)$ converges in $L^{p'}(Q_\Omega \times (0,T);\mathbb{R}^m)$. Hence its weak $L^{p'}$-limit is also $|v(x,y,t)|^{p-2}v(x,y,t)$. Thus, for any $\varphi \in L^p(0,T;W_0)$ we have:

$$\frac{\varphi(x,t)-\varphi(y,t)}{|x-y|^{\frac{n+ps}{p}}} \in L^p\left(Q_\Omega \times (0,T);\mathbb{R}^m\right).$$

According to weak limit in (34), we get

$$\lim_{k\to\infty}\int_0^T \langle u_k,\varphi\rangle_{W_0}dt = \int_0^T \langle u,\varphi\rangle_{W_0}dt \tag{35}$$

for every $\varphi \in L^p(0,T;W_0)$.
Now, by using [23, proposition 1.3] and the continuity of M, we prove that $\left\{M(\|u_k\|^p_{W_0})\right\}_k$ is relatively compact in $L^1(0,T)$. Indeed, by (28) we have $M(\|u_k\|^p_{W_0}) \le c$, for all k and t. This implies that $\int_0^T M(\|u_k\|^p_{W_0})dt \le cT$ for all k. On the other hand, for any $\varepsilon > 0$, there exits $\delta = \frac{\varepsilon}{c}$ such that for any measurable subset A with $|A| < \delta$, there holds

$$\int_A M(\|u_k\|^p_{W_0})dt \le c|A| < \varepsilon$$

It follows that $\left\{M\left(\|u_k\|^p_{W_0}\right)\right\}_k$ is relatively compact in $L^1(0,T)$. Therefore, up to a subsequence, $M\left(\|u_k\|^2_{W_0}\right)$ converges to some function $\Lambda(t) \in L^1(0,T)$ for a.e. $t \in [0,T]$. Furthermore, (35) and the Lebesgue dominated convergence theorem implies that

$$\lim_{k\to\infty}\int_0^T M\left(\|u_k\|^p_{W_0}\right)\langle u_k,\phi\rangle_{W_0}dt = \int_0^T \Lambda(t)\langle u,\phi\rangle_{W_0}dt \tag{36}$$

for every $\varphi \in L^p(0,T;W_0)$.
From (16), for $\phi \in C^1(0,T;U_N), N \le k$, we have

$$\int_{Q_T}\frac{\partial u_k}{\partial t}\phi dxdt + \int_0^T M\left(\|u_k\|^p_{W_0}\right)\langle u_k,\phi\rangle_{W_0}dt = \int_{Q_T} f(x,t,u_k)\,\phi dxdt.$$

For k tend to ∞, it follows from the above results, that

$$\int_{Q_T}\frac{\partial u}{\partial t}\phi dxdt + \int_0^T \Lambda(t)\langle u,\phi\rangle_{W_0}dt = \int_{Q_T} f(x,t,u)\phi dxdt. \tag{37}$$

For all $C^1(0,T;U_N)$ $(N \in \mathbb{N})$.
Taking $u = \phi$ in (37), we get

$$\|u(x,T)\|_2^2 - \|u(x,0)\|_2^2 + \int_0^T \Lambda(t)\|u\|^p_{W_0}dt = \int_{Q_T} f(x,t,u)udxdt. \tag{38}$$

According to (17), we have

$$\|u_k(x,T)\|_2^2 - \|u_k(x,0)\|_2^2 + \int_0^T M\left(\|u_k\|^p_{W_0}\right)\|u_k\|^p_{W_0}dt = \int_{Q_T} f(x,t,u_k)u_kdxdt. \tag{39}$$

Then we have

$$\begin{aligned}&\lim_{k\to\infty}\int_0^T M\left(\|u_k\|_{W_0}^p\right)\langle u_k, u_k\rangle_{W_0}dt\\ &\quad= -\|u(x,T)\|_2^2 + \|u(x,0)\|_2^2 - \int_{Q_T} f(x,t,u)udxdt\\ &\quad= \int_0^T \Lambda(t)\langle u, u\rangle_{W_0}dt\\ &\quad= \lim_{k\to\infty}\int_0^T M\left(\|u_k\|_{W_0}^p\right)\langle u_k, u\rangle_{W_0}dt\end{aligned}$$

Thus, we deduce that

$$\lim_{k\to\infty}\int_0^T M\left(\|u_k\|_{W_0}^p\right)\langle u_k, u_k - u\rangle_{W_0}dt = 0 \tag{40}$$

Thus, there exists a subsequence still denoted by $\{u_k\}$ such that for a.e. $t \in [0,T]$

$$\lim_{k\to\infty} M\left(\|u_k\|_{W_0}^p\right)\langle u_k, u_k - u\rangle_{W_0} = 0. \tag{41}$$

Therefore, we obtain

$$M\left(\|u\|_{W_0}^p\right) = \Lambda(t) \text{ a.e. } t \in [0,T]. \tag{42}$$

Inserting (42) in (37), yields

$$\int_{Q_T} \frac{\partial u}{\partial t}\phi dxdt + \int_0^T M\left(\|u\|_{W_0}^p\right)\langle u, \phi\rangle_{W_0}dt = \int_{Q_T} f(x,t,u)\phi dxdt, \tag{43}$$

for all $\phi \in C^1(0,T;C_0^\infty(\Omega))$. Then the theorem (3.2) is proved. □

4. An example

We consider the following problem

$$\begin{cases}\frac{\partial u}{\partial t} + M(\|u\|_{W_0}^p)(-\Delta)_p^s u = a(x,t)|u|^{q-2}u & \text{in } Q_T = \Omega\times(0,T),\\ u = 0 & \text{in } C\Omega\times(0,T),\\ u(x,0) = u_0(x) & \text{in } \Omega,\end{cases} \tag{44}$$

Comparing with problem (1) where $f(x,t,u) = a(x,t)|u|^{q-2}u$, $F(x,t,u) = \frac{a(x,t)}{q}|u|^q$, and $F_t(x,t,u) \geqslant C(-|r|^q - 1)$. If $2 < q < p_s^*$, then by theorem 3.2, there exists a constant $T_0 > 0$ such tha the problem (1) has a weak solutions as $T < T_0$.

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