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Blow up of solutions for a fourth-order reaction-diffusion equation in variable-exponent Sobolev spaces

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Abstract. This work deals with a fourth order reaction-diffusion equation with variable exponents. Firstly, we investigate the finite time blow-up of solutions for positive initial energy. Later, we establish by utilizing a technique differential inequalities an upper limit on the blow-up time.

1. Introduction

1.1. Setting of the problem:

In this paper, we consider the following fourth order reaction-diffusion equation with variable exponents

$$\begin{cases} z_t - \Delta z + \Delta^2 z + \Delta^2 z_t = z^{q(x)}, & x \in \Omega,\ t > 0, \\ z(x,t) = \frac{\partial z}{\partial v}(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ z(x,0) = z_0(x), & x \in \Omega, \end{cases} \tag{1}$$

here Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$, $(n \geq 1)$, and the initial value $z_0 \in W_0^{2,q(\cdot)}(\Omega)$, the exponent $q(\cdot)$ is given measurable function on Ω satisfying

$$\begin{cases} 2 < q_1 \leq q(x) \leq q_2 < \infty, & n \leq 4, \\ 2 < q_1 \leq q(x) \leq q_2 < \frac{n}{n-4}, & n \geq 5, \end{cases} \tag{2}$$

here

$$q_1 = ess \inf_{x\in\Omega} q(x) \text{ and } q_2 = ess \sup_{x\in\Omega} q(x). \tag{3}$$

For each points z and v in the bounded domain Ω with $|z-v| < \frac{1}{2}$, there exists a constant $A > 0$ so that the subsequent inequality yields:

$$\left|q(z) - q(v)\right| \leq -\frac{A}{\ln|z-v|}. \tag{4}$$

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1.2. Literature overview:

Wu et al. [39] considered the following semilinear parabolic equation with variable exponent

$$u_t - \Delta u = u^{q(x)}.$$

They demonstrated the blow up of solutions. Later, many authors studied the blow up of solutions the same equation under different conditions [5, 18, 38].

Di et al. [9] has investigated the following problem

$$z_t - \upsilon \Delta z_t - div\left(|\nabla z|^{m(x)-2} \nabla z\right) = |z|^{p(x)-2} z, \tag{5}$$

with Dirichlet boundary condition. The authors obtained both an upper bound and a lower bound for blow-up. It is clear that when $\upsilon = 1$, $m(x) = 2$, $p(x) = p$, the expression (5) becomes the subsequent pseudo-parabolic equation

$$z_t - \Delta z - \Delta z_t = |z|^{p-2} z. \tag{6}$$

As for (5), there are many results related to asymptotic behavior [20, 40], the uniqueness and existence [2, 35] of solutions, blow-up [22, 40] property and other aspects.

Qu et al. [31] examined the subsequent fourth- order parabolic equation

$$u_t + \Delta^2 u = u^{p(x)}.$$

They demonstrated the asymptotic behavior of solutions. Also, Liu [21] proved both the local existence and blow up of solutions the same equation.

Han [16] investigated the fourth-order parabolic equation

$$u_t + \Delta^2 u - \nabla f(\nabla u) = h(x, t, u).$$

The author showed that global existence and finite-time blow-up of solutions are obtained under different conditions for the initial data.

Abita [1] consider a semilinear pseudo-parabolic equation

$$u_t - \Delta u_t - \Delta u = u^{p(x)}.$$

The author has demonstrated that nonnegative classical solutions experience finite-time blow-up when initiated with arbitrary positive energy and appropriate large initial values.

Also, the existence, blow up and decay of solutions was studied by many authors for the equation with variable exponents, see for instance [3, 4, 6, 7, 11–13, 15, 26–28, 30, 33, 34, 36, 37, 41].

The equation with variable exponents arises in many branches in sciences such as image processing, electrorheological fluids and nonlinear elasticity theory [8, 10, 32]. The fourth-order equation has its origin in the canonical model introduced by Petrovsky [24, 25]. This type equıations arises in many branches in sciences such as acoustics, geophysics, ocean acoustics and optics [14].

This article is organized as follows: In part 2, we present various materials, including notations, hypotheses, and auxiliary formulas. In part 3, we prove the blow up in finite time T . In part 4, obtained an upper bound for the blow-up time with utilize the technique of differential inequalities.

2. Preliminaries

This part we give some preliminary facts and definitions about the Lebesgue spaces and Sobolev spaces with variable exponents (see [10, 29]). Consider a measurable function $q : \Omega \to [1, \infty]$ where Ω represents a bounded domain in $\mathbb{R}^n$. The Lebesgue space with a variable exponent $q(\cdot)$ is defined as follows:

$$L^{q(x)}(\Omega) = \left\{z : \Omega \to \mathbb{R},\ z \text{ is measurable and } \rho_{q(\cdot)}(\lambda z) < \infty,\ \text{for some } \lambda > 0\right\},$$

where

$$\rho_{q(\cdot)}(z) = \int_{\Omega} |z|^{q(x)}\, dx.$$

Also endowed with the Luxemburg-type norm

$$\|z\|_{q(x)} = \inf\left\{\lambda > 0 : \int_{\Omega} \left|\frac{z}{\lambda}\right|^{q(x)} dx \le 1\right\},$$

$L^{q(x)}(\Omega)$ is a Banach space.

The Sobolev space with variable exponent $W^{m,q(x)}(\Omega)$ is defined as

$$W^{m,q(x)}(\Omega) = \left\{z \in L^{q(x)}(\Omega) : D^{\alpha}z \in L^{q(x)}(\Omega),\ |\alpha| \le m\right\}.$$

Sobolev space with variable exponent is a Banach space with respect to the norm

$$\|z\|_{H_0^2(\Omega)} = \sqrt{\|z\|_2^2 + \|\nabla z\|_2^2 + \|\Delta z\|_2^2}.$$

Lemma 2.1. *[10] Suppose that Ω be a bounded domain of $\mathbb{R}^n$, $q(.) \subset (1,\infty)$ is a measurable function on $\overline{\Omega}$ so*

$$\min\left\{\|z\|_{q(\cdot)}^{q_1}, \|z\|_{q(\cdot)}^{q_2}\right\} \le \rho_{q(\cdot)}(z) \le \max\left\{\|z\|_{q(\cdot)}^{q_1}, \|z\|_{q(\cdot)}^{q_2}\right\},$$

for any $z \in L^{q(\cdot)}(\Omega)$.

Lemma 2.2. *[10] Assume that Ω be a bounded domain of $\mathbb{R}^n$, $q(\cdot)$ is a measurable function on $\overline{\Omega}$ complies with (2) and (3), so*

$$\|z\|_{q(\cdot)+1} \le K\|\Delta z\|_2, \text{ for all } z \in H_0^2(\Omega), \tag{7}$$

here the optimal constant of Sobolev embedding denoted as K depends on $q_{1,2}$ and $|\Omega|$.

3. Blow up

In this part, we derive an upper bound for the blow-up time concerning the problem (1) under certain conditions on the variable exponents $q(\cdot)$ and initial data. Initially, a local existence result is obtained for the problem (1), laterly through the application of the maximal principle the local existence in time, uniqueness and regularity of solutions of problem [14].

Theorem 3.1. *Assuming $z_0 \in H_0^2(\Omega)$, $T = T(z_0) > 0$ so that the problem defined by equation (1) exhibits a distinct classical solution that remains nonnegative throughout.*

$$z \in L^{\infty}\left([0,T]; H_0^2(\Omega)\right),\ z_t \in L^2\left([0,T]; H_0^2(\Omega)\right),$$

and

(i) $\forall v \in H_0^2(\Omega)$, $t \in (0,T)$

$$(z_t, v) + (\nabla z, \nabla v) + (\Delta z, \Delta v) + (\Delta z_t, \Delta v) = \left(z^{q(\cdot)}, v\right) \text{ for all } v \in H_0^2(\Omega).$$

(ii) $z(x,0) = z_0(x) \ge 0$ in $H_0^2(\Omega)$.

The main results proof depends on the following two lemmas:

Lemma 3.2. *[17, 19]* ***(Concavity methods)****. Suppose that $\beta > 0$, let $\psi(t) \geq 0$ be weakly twice-differentiable on $(0,\infty)$ such that $\psi(0) > 0$, $\psi'(0) > 0$ and*

$$\psi''(t)\,\psi(t) - (1+\beta)\,(\psi'(t))^2 \geq 0,$$

for all $t \in (0,\infty)$. Then there exists a $T > 0$ such that

$$\lim_{t\to T^-} \psi(t) = \infty,$$

and

$$T \leq \frac{\psi(0)}{\beta\psi'(0)}.$$

Lemma 3.3. *The associated energy corresponding to problem (1) denoted as $J : H_0^2(\Omega) \cap L^{q(\cdot)+1}(\Omega) \to \mathbb{R}$, is defined as follows:*

$$J(t) = \frac{1}{2}\|\nabla z\|^2 + \frac{1}{2}\|\Delta z\|^2 - \int_\Omega \frac{1}{q(x)+1}|z|^{q(x)+1}\,dx, \tag{8}$$

moreover

$$J'(t) = -\|z_t\|^2 - \|\Delta z_t\|^2 \leq 0, \tag{9}$$

here, the inequality $J(t) \leq J(0)$ is satisfied

$$J(0) = \frac{1}{2}\|\nabla z_0\|^2 + \frac{1}{2}\|\Delta z_0\|^2 - \int_\Omega \frac{1}{q(x)+1}|z_0|^{q(x)+1}\,dx.$$

Proof. Multiplying the first equation of (1) by z_t and integrating over Ω, utilizing integrating by parts, we get

$$\int_\Omega z_t z_t dx + \int_\Omega \nabla z \nabla z_t dx + \int_\Omega \Delta z \Delta z_t dx + \int_\Omega \Delta z_t \Delta z_t dx = \int_\Omega z^{q(x)} z_t dx,$$

$$\int_\Omega z_t^2 dx + \frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla z|^2\,dx + \frac{d}{dt}\frac{1}{2}\int_\Omega |\Delta z|^2\,dx + \int_\Omega |\Delta z_t|^2\,dx = \frac{d}{dt}\left(\frac{1}{q(x)+1}\int_\Omega |z|^{q(x)+1}\,dx\right),$$

$$\frac{dJ(t)}{dt} + \int_\Omega z_t^2 dx + \int_\Omega |\Delta z_t|^2\,dx = 0.$$

And obviously, we have equation (9).

Integrating the last inequality over the interval $(0,t)$, we obtain

$$J(t) = J(z_0) - \int_0^t \|z_t(\cdot,s)\|^2_{H_0^2(\Omega)}\,ds. \tag{10}$$

□

Let $T_{\max}$ denote the maximal existence time of the solution z,

$$T_{\max} = \sup\{T > 0 : z(\cdot, t) \text{ exist on } [0, T]\} < \infty.$$

Definition 3.4. *[23] Suppose that problem (1) is well posed in $H_0^2(\Omega)$ and that $z_0 \in H_0^2(\Omega)$. The solution of (1) is global if $T_{\max} = \infty$. Blow-up is said to occur for problem (1) if*

$$T_{\max} < \infty \text{ and } \lim_{t \to T_{\max}} \|z(t)\|_{H_0^2(\Omega)} = \infty.$$

Our result regarding blow-up is stated.

Theorem 3.5. *No solution exists for problem (1) with initial data satisfying $J(z_0) > 0$, that is*

$$\|z_0\|^2_{H_0^2(\Omega)} > 2\left(1 + K^2\right)\alpha\left(\frac{q_1 + 1}{q_1 - 1}\right)J(z_0), \tag{11}$$

can exist for all time if $q(.)$ satisfy (2) and (4). However there exists a $T_1 \leq T_{\max}$ such that $\lim_{t \to T_1} \int_0^t \|z(s)\|^2_{H_0^2(\Omega)} ds = +\infty$ meaning that the solution z blows up in finite time in the $H_0^2(\Omega)$-norm.

Where α is to be chosen such that

$$1 < \alpha < \frac{q_1 - 1}{2(q_1 + 1)(1 + K^2)J(z_0)} \|z_0\|^2_{H_0^2(\Omega)},$$

and K be the optimal constant of Poincaré inequality, which is

$$\|z\|_2 \leq K\|\Delta z\|_2. \tag{12}$$

Proof. To establish the theorem we initially suppose that z exists on $\Omega \times [0, \infty)$ i.e. $T_{\max} = +\infty$ and later demonstrate that this assumption leads to a contradiction. We choose a function $\psi(t)$ of the following form for $0 < t < \infty$,

$$\psi(t) = \|z\|_2^2 + \|\nabla z\|_2^2 + \|\Delta z\|_2^2,$$

we have

$$\begin{aligned} \psi'(t) &= 2(z, z_t) + 2(\nabla z, \nabla z_t) + 2(\Delta z, \Delta z_t) \\ &= -2\left(\|\nabla z\|_2^2 + \|\Delta z\|_2^2 - \int_\Omega |z|^{q(x)+1} dx\right). \end{aligned} \tag{13}$$

State 1. $J(t) \geq 0$, for every $t > 0$. By utilizing equations (10) and (13) we get

$$\begin{aligned} \psi'(t) &= -2\left(\|\nabla z\|_2^2 + \|\Delta z\|_2^2 - \int_\Omega |z|^{q(x)+1} dx\right) \\ &\geq -2(q_1 + 1)\left(\frac{1}{2}\|\nabla z\|_2^2 + \frac{1}{2}\|\Delta z\|_2^2 - \frac{1}{q(x)+1}\int_\Omega |z|^{q(x)+1} dx\right) \\ &\quad + (q_1 - 1)\|\nabla z\|_2^2 + (q_1 - 1)\|\Delta z\|_2^2 \\ &= -2(q_1 + 1)J(t) + (q_1 - 1)\|\nabla z\|_2^2 + (q_1 - 1)\|\Delta z\|_2^2 \\ &= 2(q_1 + 1)(\alpha - 1)J(t) - 2(q_1 + 1)\alpha J(t) \\ &\quad + (q_1 - 1)\|\nabla z\|_2^2 + (q_1 - 1)\|\Delta z\|_2^2 \\ &\geq -2(q_1 + 1)\alpha J(z_0) + 2(q_1 + 1)\int_0^t \|z_t(\cdot, s)\|^2_{H_0^2(\Omega)} ds \\ &\quad + (q_1 - 1)\|\nabla z\|_2^2 + (q_1 - 1)\|\Delta z\|_2^2. \end{aligned} \tag{14}$$

Utilizing equation (12), to obtain

$$\psi(t) = \|z\|_2^2 + \|\nabla z\|_2^2 + \|\Delta z\|_2^2 \leq \left(1 + K^2\right) \|\Delta z\|_2^2 . \tag{15}$$

By substituting equation (15) into equation (14), we have

$$\psi'(t) \geq -2(q_1+1)\alpha J(z_0) + 2(q_1+1)\int_0^t \|z_t(\cdot,s)\|_{H_0^2(\Omega)}^2 ds + \frac{(q_1-1)}{(1+K^2)}\psi(t). \tag{16}$$

Then

$$\begin{aligned}\psi'(t) - \frac{(q_1-1)}{(1+K^2)}\psi(t) &\geq -2\alpha(q_1+1)J(z_0) + 2(q_1+1)\int_0^t \|z_t(\cdot,s)\|_{H_0^2(\Omega)}^2 ds \\ &\geq -2\alpha(q_1+1)J(z_0).\end{aligned}$$

Integrating clearly, we obtain

$$\psi(t) \geq 2\frac{\left(1+K^2\right)}{(q_1-1)}\alpha(q_1+1)J(z_0)\left(1 - e^{\frac{(q_1-1)}{(1+K^2)}t}\right) + \psi(0)e^{\frac{(q_1-1)}{(1+K^2)}t}, \tag{17}$$

for every $t > 0$, we get

$$\eta'(t) = \psi(t) \text{ and } \eta''(t) = \psi'(t).$$

By substituting equation (17) into equation (16), we find

$$\eta''(t) \geq 2\alpha(q_1+1)\int_0^t \|z_t(\cdot,s)\|_{H_0^2(\Omega)}^2 ds + \left(\frac{(q_1-1)}{(1+K^2)}\psi(0) - 2\alpha(q_1+1)J(z_0)\right)e^{\frac{(q_1-1)}{(1+K^2)}t}. \tag{18}$$

Now, define the auxiliary function φ as follows:

$$\varphi(t) = \eta^2(t) + \varepsilon^{-1}\psi(0)\eta(t) + \beta,$$

where $\varepsilon > 0$ is chosen to be sufficiently small such that

$$0 < \varepsilon < \frac{1}{2\alpha(q_1+1)(1+K^2)\psi(0)}\left(\frac{(q_1-1)}{(1+K^2)}\psi(0) - 2\alpha(q_1+1)J(z_0)\right),$$

and $\beta > 0$ is chosen to be sufficiently large, so that

$$4\varepsilon^2\beta > \psi^2(0). \tag{19}$$

Thus

$$\varphi'(t) = \left(2\eta(t) + \varepsilon^{-1}\psi(0)\right)\eta'(t), \tag{20}$$

$$\varphi''(t) = \left(2\eta(t) + \varepsilon^{-1}\psi(0)\right)\eta''(t) + 2(\eta'(t))^2. \tag{21}$$

From equation (20), we get

$$\begin{aligned}(\varphi'(t))^2 &= \left(2\eta(t) + \varepsilon^{-1}\psi(0)\right)^2(\eta'(t))^2 \\ &= \left(4\eta^2(t) + \varepsilon^{-2}\psi^2(0) + 4\varepsilon^{-1}\eta(t)\psi(0)\right)(\eta'(t))^2 \\ &= \left(4\eta^2(t) + 4\varepsilon^{-1}\eta(t)\psi(0) + 4\beta - \delta\right)(\eta'(t))^2 \\ &= (4\varphi(t) - \delta)(\eta'(t))^2,\end{aligned} \tag{22}$$

here $\delta = 4\beta - \varepsilon^{-2}\psi^2(0) > 0$, laterly

$$(\varphi'(t))^2 + \delta(\eta'(t))^2 = 4\varphi(t)(\eta'(t))^2. \tag{23}$$

Observe that

$$\begin{aligned}\int_0^t ((z_t(\cdot,s), z) + (\Delta z_t(\cdot,s), \Delta z))\,ds &= \frac{1}{2}\int_0^t \left(\frac{d}{ds}\|z\|^2_{H_0^2(\Omega)}\right) ds \\ &= \frac{1}{2}\|z(t)\|^2_{H_0^2(\Omega)} - \frac{1}{2}\|z_0\|^2_{H_0^2(\Omega)}.\end{aligned}$$

Thus

$$\|z(t)\|^2_{H_0^2(\Omega)} = \|z_0\|^2_{H_0^2(\Omega)} + 2\int_0^t\int_\Omega z_t(\cdot,s)\,z(s)\,dxds + 2\int_0^t\int_\Omega \Delta z_t(\cdot,s)\,\Delta z(s)\,dxds.$$

Utilizing the Hölder and Young's inequality, along with given that $\|z_t\|_2 \le K\|\Delta z_t\|_2$, we obtain

$$\begin{aligned}(\eta'(t))^2 &= \|z(t)\|^4_{H_0^2(\Omega)} \\ &= \begin{pmatrix} \|z(t)\|^2_{H_0^2(\Omega)} + +2\int_0^t\int_\Omega z_t(\cdot,s)\,z(s)\,dxds \\ +2\int_0^t\int_\Omega \nabla z_t(\cdot,s)\,\nabla z(s)\,dxds + 2\int_0^t\int_\Omega \Delta z_t(\cdot,s)\,\Delta z(s)\,dxds \end{pmatrix}^2 \\ &\le \begin{pmatrix} \|z(t)\|^2_{H_0^2(\Omega)} + 2\int_0^t\int_\Omega |z_t(\cdot,s)|\,|z(s)|\,dxds \\ +2\int_0^t\int_\Omega |\nabla z_t(\cdot,s)|\,|\nabla z(s)|\,dxds + 2\int_0^t\int_\Omega |\Delta z_t(\cdot,s)|\,|\Delta z(s)|\,dxds \end{pmatrix}^2 \\ &\le \begin{pmatrix} \|z(t)\|^2_{H_0^2(\Omega)} + 2\left(\int_0^t \|z\|_2^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t \|z_t(\cdot,s)\|_2^2\,ds\right)^{\frac{1}{2}} \\ +2\left(\int_0^t \|\nabla z\|_2^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t \|\nabla z_t(\cdot,s)\|_2^2\,ds\right)^{\frac{1}{2}} \\ +2\left(\int_0^t \|\Delta z\|_2^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t \|\Delta z_t(\cdot,s)\|_2^2\,ds\right)^{\frac{1}{2}} \end{pmatrix}^2,\end{aligned}$$

we get

$$\begin{aligned}
\left(\eta'(t)\right)^2 &= \|z_0\|^4_{H_0^2(\Omega)} + 4\|z_0\|^2_{H_0^2(\Omega)} \left(\int_0^t \|z\|_2^2\, ds\right)^{\frac{1}{2}} \left(\int_0^t \|z_t(\cdot,s)\|_2^2\, ds\right)^{\frac{1}{2}} \\
&\quad +4\|z_0\|^2_{H_0^2(\Omega)} \left(\int_0^t \|\nabla z\|_2^2\, ds\right)^{\frac{1}{2}} \left(\int_0^t \|\nabla z_t(\cdot,s)\|_2^2\, ds\right)^{\frac{1}{2}} \\
&\quad +4\|z_0\|^2_{H_0^2(\Omega)} \left(\int_0^t \|\Delta z\|_2^2\, ds\right)^{\frac{1}{2}} \left(\int_0^t \|\Delta z_t(\cdot,s)\|_2^2\, ds\right)^{\frac{1}{2}} \\
&\quad +4\int_0^t \|z\|_2^2\, ds \int_0^t \|z_t(\cdot,s)\|_2^2\, ds + 4\int_0^t \|\nabla z\|_2^2\, ds \int_0^t \|\nabla z_t(\cdot,s)\|_2^2\, ds \\
&\quad +4\int_0^t \|\Delta z\|_2^2\, ds \int_0^t \|\Delta z_t(\cdot,s)\|_2^2\, ds,
\end{aligned}$$

that is

$$\begin{aligned}
\left(\eta'(t)\right)^2 &\leq \|z_0\|^4_{H_0^2(\Omega)} + 4(1+K)\|z_0\|^2_{H_0^2(\Omega)} \left(\int_0^t \|\Delta z_t(\cdot,s)\|_2^2\, ds\right)^{\frac{1}{2}} \\
&\quad \times\left(\left(\int_0^t \|z\|_2^2\, ds\right)^{\frac{1}{2}} + \left(\int_0^t \|\nabla z\|_2^2\, ds\right)^{\frac{1}{2}} + \left(\int_0^t \|\Delta z\|_2^2\, ds\right)^{\frac{1}{2}}\right) \\
&\quad +4\left(\int_0^t \|z\|_2^2\, ds + \int_0^t \|\nabla z\|_2^2\, ds + \int_0^t \|\Delta z\|_2^2\, ds\right) \\
&\quad \times\left(\int_0^t \|z_t(\cdot,s)\|_2^2\, ds + \int_0^t \|\nabla z_t(\cdot,s)\|_2^2\, ds + \int_0^t \|\Delta z_t(\cdot,s)\|_2^2\, ds\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\left(\eta'(t)\right)^2 &\leq \|z_0\|^4_{H_0^2(\Omega)} + 2\varepsilon^{-1}\|z_0\|^2_{H_0^2(\Omega)} \int_0^t \|\Delta z_t(\cdot,s)\|_2^2\, ds \\
&\quad +4(1+K)^2\varepsilon\|z_0\|^2_{H_0^2(\Omega)}\eta(t) + 4\eta(t)\int_0^t \|z_t(\cdot,s)\|_2^2\, ds, \qquad (24)
\end{aligned}$$

where we apply the established algebraic inequality

$$\left(M^{\frac{1}{2}} + N^{\frac{1}{2}}\right) \leq 2(M+N), \text{ for } M>0, N>0.$$

From (21) and (23), we have

$$\begin{aligned} 2\varphi''(t)\varphi(t) &= 2\left(2\eta(t)+\varepsilon^{-1}\psi(0)\right)\eta''(t)\varphi(t)+4\left(\eta'(t)\right)^2\varphi(t) \\ &= 2\left(2\eta(t)+\varepsilon^{-1}\psi(0)\right)\eta''(t)\varphi(t)+\left(\varphi'(t)\right)^2+\delta\left(\varphi'(t)\right)^2. \end{aligned} \tag{25}$$

At present, from (18), (22), (24) and (25), we can express

$$\begin{aligned} & 2\varphi''(t)\varphi(t)-(1+\alpha)\left(\varphi'(t)\right)^2 \\ = \; & 2\left(2\eta(t)+\varepsilon^{-1}\psi(0)\right)\eta''(t)\varphi(t)+\delta\left(\eta'(t)\right)^2-\alpha\left(\varphi'(t)\right)^2 \\ = \; & 2\left(2\eta(t)+\varepsilon^{-1}\psi(0)\right)\eta''(t)\varphi(t)+\delta\left(\eta'(t)\right)^2-\alpha\left(4\varphi(t)-\delta\right)\left(\eta'(t)\right)^2 \\ = \; & 2\left(2\eta(t)+\varepsilon^{-1}\psi(0)\right)\eta''(t)\varphi(t)-4\alpha\varphi(t)\left(\eta'(t)\right)^2+\delta(1+\alpha)\left(\eta'(t)\right)^2 \\ \geq \; & 2\varphi(t)\left(2\eta(t)+\varepsilon^{-1}\psi(0)\right)\begin{pmatrix} 2(q_1+1)\alpha\int_0^t \|z_t(.,s)\|^2_{H_0^2(\Omega)}ds \\ +\left(\frac{(q_1-1)}{(1+K^2)}\psi(0)-2(q_1+1)\alpha J(z_0)\right)e^{\frac{(q_1-1)}{(1+K^2)}t} \end{pmatrix} \\ & -4\alpha\varphi(t)\begin{pmatrix} \|z_0\|^4_{H_0^2(\Omega)}+2\varepsilon^{-1}\|z_0\|^2_{H_0^2(\Omega)}\int_0^t\|\Delta z_t(.,s)\|_2^2ds \\ +4(1+K)^2\varepsilon\|z_0\|^2_{H_0^2(\Omega)}\eta(t)+4\eta(t)\int_0^t\|z_t(.,s)\|_2^2ds \end{pmatrix}. \end{aligned}$$

By choosing the values of α and ε considering the given that

$$e^{\frac{q_1-1}{1+K^2}t}\geq 1,\ q_1+1>2,\ \varphi<0,\ (1+K)^2>1$$

and

$$\int_0^t\|z_t(.,s)\|_2^2ds\geq\int_0^t\|\Delta z_t(.,s)\|_2^2ds,$$

we get

$$\begin{aligned} & 2\varphi''(t)\varphi(t)-(1+\alpha)\left(\varphi'(t)\right)^2 \\ \geq \; & 4\alpha\varphi(t)\left(2\eta(t)+\varepsilon^{-1}\psi(0)\right)\left((q_1+1)\int_0^t\|z_t(.,s)\|_2^2ds+2\varepsilon(q_1+1)(1+K)^2\psi(0)\right) \\ & -4\alpha\varphi(t)\begin{pmatrix} \|z_0\|^4_{H_0^2(\Omega)}+2\varepsilon^{-1}\|z_0\|^2_{H_0^2(\Omega)}\int_0^t\|\Delta z_t(.,s)\|_2^2ds \\ +4(1+K)^2\varepsilon\|z_0\|^2_{H_0^2(\Omega)}\eta(t)+4\eta(t)\int_0^t\|z_t(.,s)\|_2^2ds \end{pmatrix} \\ \geq \; & 0. \end{aligned}$$

In this case, we will show that T cannot be infinite; in other words, it is established that no nonnegative classical weak solution can exist for all time.

From Lemma 3.2, we can conclude that there exists a $0<t_1<+\infty$ so that $\varphi(t)\to\infty$ as $t\to t_1$, here

$$0<t_1<\frac{2\varphi(0)}{(\alpha-1)\varphi'(0)}=\frac{2\beta\varepsilon}{(\alpha-1)\|z_0\|^4_{H_0^2(\Omega)}}<+\infty.$$

At present, by examining the continuity of φ with aspect to ψ, we can deduce that there exists a $T_1 \leq t_1$ so that $\lim_{t\to T_1} \int_0^t \|z(s)\|^2_{H_0^2(\Omega)} ds = +\infty \Rightarrow \lim_{t\to T_1} \sup \|z(t)\|^2_{H_0^2(\Omega)} = +\infty$. Thus, z discontinuing at some finite time $T_1 \leq T_{\max}$, this implies z not exist for every time. In other words z blows up at a time T_1, and cannot persist as finite for every t, this leads to the nonexistence conclusion stated in the theorem. Consequently, ψ blows up at time T_1 in $H_0^2(\Omega)$-norm, which contradicts $T_{\max} = +\infty$. Therefore, for data satisfying equation (11) any solution has a finite time.

State 2. Suppose that there is $t_0 > 0$ so that $J(z(t_0)) < 0$, $(z(t_0) \neq 0)$. Therefore, consider $v(x,t) = z(x,t+t_0)$ as the solution to (1). Consequently, we have $J(v(0)) = J(z(t_0)) < 0$. From (10), we get

$$J(v(t)) \leq J(v(0)) < 0. \tag{26}$$

At present, we utilize equation (13) to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v(s)\|^2_{H_0^2(\Omega)} &= \frac{1}{2}\frac{d}{dt}\int_\Omega \left(v^2 + \Delta^2 v\right) dx \\
&= -\left(\|\nabla v\|_2^2 + \|\Delta v\|_2^2 - \int_\Omega v^{p(x)+1} dx\right) \\
&= -(q_1+1)\left(\frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2 - \frac{1}{q_1+1}\int_\Omega v^{p(x)+1} dx\right) \\
&\quad + \frac{q_1-1}{2}\int_\Omega |\nabla v|^2 dx + \frac{q_1-1}{2}\int_\Omega |\Delta v|^2 dx \\
&\geq -(q_1+1) J(v) + \frac{q_1-1}{2}\int_\Omega |\nabla v|^2 dx + \frac{q_1-1}{2}\int_\Omega |\Delta v|^2 dx.
\end{aligned} \tag{27}$$

This inequality together with (10) and $J(v(0)) < 0$ gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \left(v^2(t) + \Delta^2 v(t)\right) dx &\geq -(q_1+1) J(v(0)) + (q_1+1)\int_0^t \|v_t(\cdot,s)\|^2_{H_0^2(\Omega)} ds \\
&> (q_1+1)\int_0^t \|v_t(\cdot,s)\|^2_{H_0^2(\Omega)} ds.
\end{aligned} \tag{28}$$

Where $\beta > 0$, let's define

$$\psi(t) = \int_0^t \|v(s)\|^2_{H_0^2(\Omega)} ds + \beta.$$

Now, by utilizing equation (26) and the Poincare inequality in equation (27) we obtain

$$\begin{aligned}
\psi'' &\geq -2(q_1+1)J(v(t)) + (q_1-1)\int_\Omega |\nabla v(t)|^2 dx + (q_1-1)\int_\Omega |\Delta v(t)|^2 dx \\
&> (q_1-1)\int_\Omega |\Delta v(t)|^2 dx \\
&> \frac{1}{1+K^2}(q_1-1)\int_\Omega \left(v^2(t) + \Delta^2 v(t)\right) dx \\
&= \frac{q_1-1}{1+K^2}\psi'(t).
\end{aligned}$$

It can be easily proven by integrating the last inequality from 0 to t,

$$\psi'(t) > \psi'(0)\, e^{\frac{q_1-1}{1+K^2}t}. \tag{29}$$

By utilizing the Hölder inequality, (28) and as a matter of fact

$$\|v\|_2 \leq \|v\|_{H_0^2(\Omega)} \text{ and } \|\Delta v\|_2 \leq \|v\|_{H_0^2(\Omega)},$$

that is

$$\begin{aligned}
\psi'(t) - \psi'(0) &= \int_0^t \psi''(s)\,ds = 2\int_0^t\int_\Omega v_t(\cdot,s)\,v(s)\,dxds + 2\int_0^t\int_\Omega \Delta v_t(\cdot,s)\,\Delta v(s)\,dxds \\
&\leq 2\left(\int_0^t \|v\|_2^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t \|v_t(\cdot,s)\|_2^2\,ds\right)^{\frac{1}{2}} \\
&\quad +2\left(\int_0^t \|\Delta v\|_2^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t \|\Delta v_t(\cdot,s)\|_2^2\,ds\right)^{\frac{1}{2}} \\
&\leq 4\left(\int_0^t \|v(s)\|_{H_0^2(\Omega)}^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t \|v_t(\cdot,s)\|_{H_0^2(\Omega)}^2\,ds\right)^{\frac{1}{2}} \\
&\leq \left(\frac{8}{q_1+1}\right)^{\frac{1}{2}}\left(\int_0^t \|v(s)\|_{H_0^2(\Omega)}^2\,ds\right)^{\frac{1}{2}}\left(\frac{d}{dt}\|v(t)\|_{H_0^2(\Omega)}\right)^{\frac{1}{2}} \\
&\leq \left(\frac{8}{q_1+1}\right)^{\frac{1}{2}}\left(\int_0^t \|v(s)\|_{H_0^2(\Omega)}^2\,ds + \beta\right)^{\frac{1}{2}}\left(\frac{d}{dt}\|v(t)\|_{H_0^2(\Omega)}\right)^{\frac{1}{2}} \\
&= \left(\frac{8}{q_1+1}\right)^{\frac{1}{2}}(\psi(t))^{\frac{1}{2}}(\psi''(t))^{\frac{1}{2}}.
\end{aligned} \tag{30}$$

Thus, (30) implies that

$$\frac{q_1+1}{8}\left(\psi'(t) - \psi'(0)\right)^2 \leq \psi(t)\,\psi''(t), \tag{31}$$

letting $\mu = \frac{q_1-1}{16} > 0$, from (31) it follows that

$$\left(\frac{9}{8} + 2\mu\right)\left(1 - \frac{\psi'(0)}{\psi'(t)}\right)^2 \leq \frac{\psi(t)\psi''(t)}{(\psi'(t))^2}. \tag{32}$$

Obviously, from (29) we obtain $\lim_{t\to\infty} \psi'(t) = +\infty$, thus there is $t^* > 0$ so that for all $t \geq t^*$

$$\psi'(t) > \frac{\psi'(0)}{1 - \sqrt{\frac{\frac{9}{8}+\mu}{\frac{9}{8}+2\mu}}}. \tag{33}$$

That $t \geq t^*$. By taking into account, (32) and (33), we obtain

$$\frac{\psi(t)\psi''(t)}{(\psi'(t))^2} \geq \frac{9}{8} + \mu.$$

Hence

$$\psi(t)\psi''(t) - \left(\frac{9}{8} + \mu\right)(\psi'(t))^2 \geq 0.$$

Since $\psi(0) > 0$ and $\psi'(0) > 0$ thus, from Lemma 3.2 as a result, there exists a $0 < T_2 < +\infty$ so that $\lim_{t\to T_2} \psi(t) = \infty$, hence $\lim_{t\to T_2} \int_0^t \|v(s)\|^2_{H_0^2(\Omega)} ds = +\infty \Rightarrow \lim_{t\to T_2} \sup \|v(t)\|^2_{H_0^2(\Omega)} = +\infty$, therefore $v(x,t)$ discontinuing at some finite time T_2, $v(x,t)$ not exists for all time, and exists almost everywhere in $(0, T_2)$ here the upper bound of T_2 is specified by,

$$0 < T_2 \leq \frac{16\psi(0)}{(q_1+1)\psi'(0)} = \frac{16\beta}{(q_1+1)\|z(t_0)\|^2_{H_0^2(\Omega)}} < +\infty.$$

Thus, $z(x,t)$ discontinuing at some finite time $T_{3,}$ implies that $z(x,t)$ does not exist for every time, and cannot persist finite for every t, in this result in the nonexistence result stated in the theorem therefore the interval of existence for z is bounded, that contrary $T_{\max} = +\infty$. This concludes the proof. □

4. Upper bound of blow up time

In this part, we investigate the blow-up outcomes and approximate the upper bound of blow-up time. From (7) and (8), it is clear that

$$\begin{aligned} J(t) &\geq \frac{1}{2}\int_\Omega |\nabla z(x,t)|^2 dx + \frac{1}{2}\int_\Omega |\Delta z(x,t)|^2 dx \\ &\quad - \frac{1}{q_1+1}\max\left(\|z\|^{q_2+1}_{q(\cdot)+1}, \|z\|^{q_1+1}_{q(\cdot)+1}\right) \\ &\geq \frac{1}{2}\int_\Omega |\nabla z(x,t)|^2 dx + \frac{1}{2}\int_\Omega |\Delta z(x,t)|^2 dx \\ &\quad - \frac{1}{q_1+1}\max\left((K_1\|\Delta z\|_2)^{q_2+1}, (K_1\|\Delta z\|_2)^{q_1+1}\right) \\ &\geq \frac{1}{2}\gamma^2 - \frac{1}{q_1+1}\max\left(K_1^{q_2+1}\gamma^{q_2+1}, K_1^{q_1+1}\gamma^{q_1+1}\right) = g(\gamma) \quad \forall\gamma \geq 0, \end{aligned} \tag{34}$$

here $\gamma = \left(\int_\Omega |\Delta z(x,t)|^2 \, dx \right)^{\frac{1}{2}} = \|\Delta z\|_2$.

Assume that $K_1, \gamma_1, \gamma_0, J_1$ be positive constants such that

$$K_1 = \max(1, K^*), \quad \gamma_1 = K_1^{-\frac{q_1+1}{q_1-1}}, \quad \gamma_0 = \|\Delta z_0\|_2, \quad J_1 = \left(\frac{1}{2} - \frac{1}{q_1+1} \right) \gamma_1^2, \tag{35}$$

the subsequent outcome has been established.

Theorem 4.1. *Suppose that* $z_0 \in H_0^2(\Omega) \cap L^{q(\cdot)+1}(\Omega)$ *so that* $0 \le J(0) < \frac{2}{(q_1-1)\left(1-\left(\frac{\gamma_1}{\gamma_2}\right)^{q_1+1}\right)+2} J_1$ *and* $K_1^{-1} \ge \|\Delta z_0\|_2 > \gamma_1$ *hold. Under the given assumptions, the solution of problem (1) will blow-up in finite time T. Additionally, the upper estimate for the blow-up time T is given by* $\widehat{T}$

$$\widehat{T} = \frac{\|z_0\|_{H_0^2(\Omega)}}{(q_1-1)\left(1-\left(\frac{\gamma_1}{\gamma_2}\right)^{q_1+1}\right)\left(2J_1 - \left((q_1-1)\left(1-\left(\frac{\gamma_1}{\gamma_2}\right)^{q_1+1}\right)+2\right)J(0)\right)}, \tag{36}$$

here γ_1, K_1 *and* J_1 *are given in (35), and* γ_2 *to be given in (38).*

Lemma 4.2. *Assume that* $h : [0, +\infty) \to \mathbb{R}$ *be defined by*

$$h(\gamma) = \frac{1}{2}\gamma^2 - \frac{1}{q_1+1} K_1^{q_1+1} \gamma^{q_1+1}, \tag{37}$$

then h exhibits the subsequent properties:

i) h is increasing for $0 < \gamma \le \gamma_1$ *and decreasing for* $\gamma \ge \gamma_1$,

ii) $\lim_{\gamma \to +\infty} h(\gamma) = -\infty$ *and* $h(\gamma_1) = J_1$,

where γ_1 *and* J_1 *are given in (35).*

Proof. Assuming that $K_1 > 1$ and $q_1 > 1$, one can observe that $h(\gamma) = g(\gamma)$, for $0 < \gamma \le K_1^{-1}$, here g is defined in (34). Also, $h(\gamma)$ is differentiable and continuous in $[0, +\infty)$.

$$h'(\gamma) = \gamma \left(1 - K_1^{q_1+1} \gamma^{q_1+1}\right), \ 0 \le \gamma < K_1^{-1}.$$

Laterly (i) subsequent. Since $q_1 - 1 > 0$, we obtain $\lim_{\gamma \to +\infty} h(\gamma) = -\infty$. A simple calculation leads to $h(\gamma_1) = J_1$. Then (ii) holds valid. □

Lemma 4.3. *Given the assumptions of Theorem 4.1, there is a positive constant* $\gamma_2 > \gamma_1$ *so that*

$$\|\Delta z\|_2 \ge \gamma_2, \ t \ge 0, \tag{38}$$

$$\frac{1}{q(x)+1} \int_\Omega z(x,t)^{q(x)+1} \, dx \ge \frac{1}{q_1+1} K_1^{q_1+1} \gamma_2^{q_1+1}, \tag{39}$$

and

$$\frac{\gamma_2}{\gamma_1} \ge \left((q_1+1) \left(\frac{1}{2} - \frac{J(0)}{\gamma_1^2} \right) \right)^{\frac{1}{q_1-1}} > 1, \tag{40}$$

here γ_1, K_1 *and* J_1 *are given in (35).*

Proof. Since $J(0) < \frac{2}{(q_1-1)\left(1-\left(\frac{\gamma_1}{\gamma_2}\right)^{q_1+1}\right)+2} J_1 < J_1$, it subsequent from Lemma 4.2 that there is a positive constant $\gamma_2 > \gamma_1$, so that $J(0) = h(\gamma_2)$. By equation (34), we get $h(\gamma_0) = g(\gamma_0) \leq J(0) = h(\gamma_2)$, it subsequent from Lemma 4.2 (i) that $\gamma_0 \geq \gamma_2$, so (38) fulfills for $t = 0$. At the present we demonstrate (38) by contrast. Assume that $\|\Delta z(t^*)\|_2 < \gamma_2$ for certain $t^* > 0$. Due to the continuity of $\|\Delta z(\cdot, t)\|_2$ and $\gamma_2 > \gamma_1$, we can choose t^* so that $\gamma_2 > \|\Delta z(t^*)\|_2 > \gamma_1$, then it subsequent from (34) and (37) that

$$J(0) = h(\gamma_2) < h(\|\Delta z(t^*)\|_2) \leq J(t^*),$$

which contrast to Lemma 3.3. Thus, equation (38) follows.

By (2) and (8), we get

$$\begin{aligned} \int_\Omega \frac{1}{q(x)+1} z^{q(x)+1} dx &\geq \frac{1}{2}\int_\Omega |\nabla z|^2 dx + \frac{1}{2}\int_\Omega |\Delta z|^2 dx - J(0) \\ &\geq \frac{1}{2}\gamma_2^2 - h(\gamma_2) = \frac{1}{q_1+1} K_1^{q_1+1} \gamma_2^{q_1+1}, \end{aligned} \tag{41}$$

and (39) subsequent.

Since $J(0) < J_1$, by a simple calculation allows us to verify

$$\left((q_1+1)\left(\frac{1}{2} - \frac{J(0)}{\gamma_1^2}\right)\right)^{\frac{1}{q_1-1}} > 1,$$

then the second inequality in (40) satisfied, and we just need to specify the first inequality. Denote $\alpha = \frac{\gamma_2}{\gamma_1}$, then $\alpha > 1$ by the fact that $\gamma_2 > \gamma_1$. So, the result is derived from the fact that $J(0) = h(\gamma_2)$, $K_1 > 1$ and (35) that

$$\begin{aligned} J(0) &= h(\gamma_2) \\ &= h(\alpha\gamma_1) \\ &= \gamma_1^2\alpha^2\left(\frac{1}{2} - \frac{1}{q_1+1} K_1^{q_1+1} \alpha^{q_1-1}\gamma_1^{q_1-1}\right) \\ &= \gamma_1^2\alpha^2\left(\frac{1}{2} - \frac{1}{q_1+1}\alpha^{q_1-1}\right), \end{aligned}$$

this means that

$$\begin{aligned} 1 &< \left((q_1+1)\left(\frac{1}{2} - \frac{J(0)}{\gamma_1^2}\right)\right)^{\frac{1}{q_1-1}} \\ &\leq \left((q_1+1)\left(\frac{1}{2} - \frac{J(0)}{\gamma_1^2\alpha^2}\right)\right)^{\frac{1}{q_1-1}} \\ &\leq \alpha, \end{aligned}$$

thus, (40) holds.

Assume that

$$H(t) = J_1 - J(t) \text{ for } t \geq 0, \tag{42}$$

we present the following lemma. □

Lemma 4.4. *Given the assumptions of Theorem 4.1, the functional $H(t)$ defined in equation (42) exhibits the subsequent estimates:*

$$0 < H(0) \leq H(t) \leq \int_\Omega \frac{1}{q(x)+1} z^{q(x)+1} dx, \quad t \geq 0. \tag{43}$$

Proof. Lemma 3.3 indicates that $H(t)$ is a nondecreasing function with respect to t. Hence

$$H(t) \geq H(0) = J_1 - J(0) > 0, \quad t \geq 0. \tag{44}$$

Combining (8), (35), (38) and $\gamma_2 > \gamma_1$, we get

$$\begin{aligned} H(t) - \int_\Omega \frac{1}{q(x)+1} z^{q(x)+1} dx &= J_1 - \frac{1}{2} \int_\Omega |\Delta z|^2 dx \\ &\leq \left(\frac{1}{2} - \frac{1}{q(x)+1}\right)\gamma_1^2 - \frac{1}{2}\gamma_1^2 < 0. \end{aligned} \tag{45}$$

Equation (43) can be deduced from equations (44) and (45).

According to the three lemmas mentioned above, the proof of Theorem 4.1 is presented as follows:

Proof of Theorem 4.1: We establish the function

$$\psi(t) = \frac{1}{2} \int_\Omega z^2 dx + \frac{1}{2} \int_\Omega |\Delta z|^2 dx. \tag{46}$$

Subsequently, based on the definitions of $J(t)$ and $H(t)$, the derivative of $\psi(t)$ fulfills

$$\begin{aligned} \psi'(t) &= \int_\Omega z z_t dx + \int_\Omega \Delta z \Delta z_t dx \\ &= \int_\Omega z\left(z^{q(x)} + \Delta z - \Delta^2 z - \Delta^2 z_t\right) dx + \int_\Omega \Delta z \Delta z_t dx \\ &= -\int_\Omega |\nabla z|^2 dx - \int_\Omega |\Delta z|^2 dx + \int_\Omega z^{q(x)+1} dx \\ &= \left(-2J(t) - 2\int_\Omega \frac{1}{q(x)+1} z^{q(x)+1} dx\right) + \int_\Omega z^{q(x)+1} dx \\ &\geq -2(J_1 - H(t)) + \left(1 - \frac{2}{q_1+1}\right) \int_\Omega z^{q(x)+1} dx \\ &\geq -2J_1 + 2H(t) + \frac{q_1 - 1}{q_1 + 1} \int_\Omega z^{q(x)+1} dx. \end{aligned} \tag{47}$$

By (35) and (39), we see

$$\begin{aligned} 2J_1 &= \frac{q_1-1}{q_1+1}K_1^{-\frac{q_1+1}{q_1-1}} = \frac{q_1-1}{q_1+1}K_1^{q_1+1}\gamma_1^{q_1+1} \\ &= \frac{q_1-1}{q_1+1}\left(\frac{\gamma_1}{\gamma_2}\right)^{q_1+1}(K_1\gamma_2)^{q_1+1} \\ &\leq (q_1-1)\left(\frac{\gamma_1}{\gamma_2}\right)^{q_1+1}\int_\Omega \frac{1}{q(x)+1}z^{q(x)+1}dx \\ &\leq \frac{q_1-1}{q_1+1}\left(\frac{\gamma_1}{\gamma_2}\right)^{q_1+1}\int_\Omega z^{q(x)+1}dx. \end{aligned} \tag{48}$$

Then it results from Lemma 4.4, (47) and (48) it is evident that

$$\begin{aligned} \psi'(t) &\geq \lambda\int_\Omega z^{q(x)+1}dx. + 2H(t) \\ &\geq \lambda\int_\Omega \frac{q_1+1}{q(x)+1}z^{q(x)+1}dx - \frac{q_1+1}{2}\lambda\int_\Omega |\nabla z|^2 dx \\ &\quad -\frac{q_1+1}{2}\lambda\int_\Omega |\Delta z|^2 dx + 2H(t) \\ &= \varphi(t), \end{aligned} \tag{49}$$

here

$$\lambda = \frac{q_1-1}{q_1+1}\left(1-\left(\frac{\gamma_1}{\gamma_2}\right)^{q_1+1}\right) > 0.$$

Subsequently, through direct computation and references to equations (1), (9), (42), we obtain

$$\begin{aligned} \varphi'(t) &= \lambda(q_1+1)\left(\int_\Omega z^{q(x)}z_t dx - \int_\Omega \nabla z\nabla z_t dx - \int_\Omega \Delta z\Delta z_t dx\right) + 2H'(t) \\ &= (\lambda(q_1+1)+2)\int_\Omega \left(|z_t|^2 + |\Delta z_t|^2\right)dx. \end{aligned} \tag{50}$$

Applying Schwarz's inequality, we observe

$$\begin{aligned} \psi(t)\varphi'(t) &= \frac{(\lambda(q_1+1)+2)}{2}\left(\int_\Omega \left(|z|^2+|\Delta z|^2\right)dx.\right) \\ &\quad \times\left(\int_\Omega \left(|z_t|^2+|\Delta z_t|^2\right)dx.\right) \\ &\geq \frac{(\lambda(q_1+1)+2)}{2}\left(\int_\Omega zz_t dx + \int_\Omega \Delta z\Delta z_t dx\right)^2 \\ &= \frac{(\lambda(q_1+1)+2)}{2}\left(\psi'(t)\right)^2. \end{aligned} \tag{51}$$

By (49), (51), and the assumption $0 \leq J(0) < \frac{2}{(\lambda(q_1+1)+2)} J_1$ we know because $\varphi(0) = 2J_1 - (\lambda(q_1+1)+2) J(0) > 0$, that $\varphi(t) > 0$ for all $t \geq 0$. Thus, we get

$$\psi(t)\varphi'(t) \geq \frac{(\lambda(q_1+1)+2)}{2}\psi'(t)\varphi(t),$$

expressing this as

$$\frac{\varphi'(t)}{\varphi(t)} \geq \frac{(\lambda(q_1+1)+2)}{2}\frac{\psi'(t)}{\psi(t)}. \tag{52}$$

Integrating (52) from 0 to t and using (49), we get

$$\frac{\psi'(t)}{\psi^{\frac{(\lambda(q_1+1)+2)}{2}}(t)} \geq \frac{\varphi(0)}{\psi^{\frac{(\lambda(q_1+1)+2)}{2}}(0)}. \tag{53}$$

Integrating inequality (53) from 0 to t, observing that

$$\frac{1}{\psi^{\frac{\lambda(q_1+1)}{2}}(t)} \leq \frac{1}{\psi^{\frac{\lambda(q_1+1)}{2}}(0)} - \frac{\lambda(q_1+1)}{2}\frac{\varphi(0)}{\psi^{\frac{\lambda(q_1+1)}{2}+2}(0)}t. \tag{54}$$

Assume that

$$0 < T^* = \frac{2\psi(0)}{\lambda(q_1+1)\varphi(0)},$$

later $\psi(t)$ blows up at time T^* in $H_0^2(\Omega)$- norm. Thus, z finite time $T \leq T^*$, that is to say, z blows up at a time $T < \infty$. The inequality above implies that $T \leq T^* \leq \widehat{T}$, where $\widehat{T}$, is concluded by (36). □

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