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Commutators of maximal and potential operators on vanishing weighted Orlicz-Morrey spaces

Fatih Deringoz[a]

[a]*Department of Mathematics, Ahi Evran University, Kirsehir, Turkey*

Abstract. We study mapping properties of the commutators of maximal and potential operators in vanishing weighted Orlicz-Morrey spaces. We show that the vanishing properties defining that subspaces are preserved under the action of those operators.

1. Introduction

Morrey spaces $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ play an important role in the study of local behaviour and regularity properties of solutions to PDE (see e.g. [11–13]). It is well known that the Morrey spaces are non-separable if $\lambda > 0$. The lack of approximation tools for the entire Morrey space has motivated the introduction of appropriate subspaces like vanishing spaces. The definition of the vanishing Morrey spaces involves several vanishing conditions. Each condition generate a closed subspace of $\mathcal{M}^{p,\lambda}(\mathbb{R}^n)$. We use the notation of [1] and show these conditions as (V_0), (V_∞) and (V^*).

A natural step in the theory of functions spaces was to study Orlicz-Morrey spaces

$$\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$$

where the "Morrey-type measuring" of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one.

We refer to [2, 3, 8, 9] for the preservation of the vanishing property (V_0) and to [6, 7] for the preservation of the vanishing property (V_∞) of $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ by some classical operators and their commutators, respectively.

In this paper, we focus on the weighted versions of above results. More precisely, the purpose of this paper is to introduce vanishing weighted Orlicz-Morrey spaces $V_0\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ and $V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ and to show that vanishing properties (V_0) and (V_∞) are preserved under the action of the commutators of maximal operator and Riesz potential.

We use the following notation: $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and radius $r > 0$. The (Lebesgue) measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$ and χ_E denotes its characteristic function. $\varphi(B) \equiv \varphi(x,r)$ for a function φ defined on $\mathbb{R}^n \times (0,\infty)$ and $B \in \mathcal{B} := \{B(x,r) : x \in \mathbb{R}^n,\ r > 0\}$. We use C as a generic positive constant, i.e., a constant whose value may change with each appearance. The expression $A \lesssim B$ means that $A \leq CB$ for some independent constant $C > 0$, and $A \approx B$ means $A \lesssim B \lesssim A$.

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Email address: deringoz@hotmail.com (Fatih Deringoz)

2. Preliminaries

Even though the A_p class is well known, for completeness, we offer the definition of A_p weight functions.

Definition 2.1. *For, $1 < p < \infty$, a locally integrable function $w : \mathbb{R}^n \to [0, \infty)$ is said to be an A_p weight if*

$$\sup_{B\in\mathcal{B}} \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}} dx\right)^{\frac{p}{p'}} < \infty.$$

A locally integrable function $w : \mathbb{R}^n \to [0, \infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|}\int_B w(y)dy \le Cw(x), \qquad a.e.\ \ x \in B$$

for some constant $C > 0$. We define $A_\infty = \bigcup_{p\ge1} A_p$.

For any $w \in A_\infty$ and any Lebesgue measurable set E, we write $w(E) = \int_E w(x)dx$.
We recall the definition of Young function.

Definition 2.2. *A function $\Phi : [0, \infty] \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = \Phi(\infty) = \infty$.*

A Young function Φ is said to satisfy the Δ_2-condition, denoted also by $\Phi \in \Delta_2$, if

$$\Phi(2r) \le C\Phi(r), \qquad r > 0$$

for some $C > 0$.
A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2C}\Phi(Cr), \qquad r \ge 0$$

for some $C > 1$.
For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \qquad (\inf \emptyset = \infty).$$

A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $t \in [1, \infty)$ (resp. $t \in [0, 1]$) and $s \in [0, \infty)$,

$$\Phi(st) \le Ct^p\Phi(s).$$

Remark 2.3. *It is well known that if Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ if and only if $\Phi \in \Delta_2 \cap \nabla_2$.*

Definition 2.4 (weighted Orlicz Space). *For a Young function Φ and $w \in A_\infty$, the set*

$$L^\Phi_w(\mathbb{R}^n) \equiv \left\{ f - measurable : \int_{\mathbb{R}^n} \Phi(k|f(x)|)w(x)dx < \infty \text{ for some } k > 0 \right\}$$

is called the weighted Orlicz space. The local weighted Orlicz space $L^\Phi_{w,\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L^\Phi_w(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

Note that $L^{\Phi}_{w}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}_{w}(\mathbb{R}^n)} \equiv \|f\|_{L^{\Phi}_{w}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x)dx \leq 1\right\}$$

and we have

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L^{\Phi}_{w}}}\Big) w(x)dx \leq 1. \tag{1}$$

For $\Omega \subset \mathbb{R}^n$, let

$$\|f\|_{L^{\Phi}_{w}(\Omega)} := \|f\chi_{\Omega}\|_{L^{\Phi}_{w}}.$$

In [4], the weighted Orlicz–Morrey space $\mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ was introduced to unify weighted Orlicz spaces and generalized weighted Morrey spaces. The definition of $\mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ is as follows:

Definition 2.5. *Let φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $w \in A_{\infty}$ and Φ any Young function. Denote by $\mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ the generalized weighted Orlicz-Morrey space, the space of all functions $f \in L^{\Phi}_{w,\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)} \equiv \|f\|_{\mathcal{M}^{\Phi,\varphi}_{w}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{\Phi,\varphi,w}(f;x,r) < \infty,$$

where $\mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = \varphi(x,r)^{-1}\, \Phi^{-1}\Big(w(B(x,r))^{-1}\Big) \|f\|_{L^{\Phi}_{w}(B(x,r))}$.

For a Young function Φ and $w \in A_{\infty}$, we denote by $\mathcal{G}^{w}_{\Phi}$ the set of all functions $\varphi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ such that

$$\inf_{B\in\mathcal{B};\, r_B \leq r_{B_0}} \varphi(B) \gtrsim \varphi(B_0) \quad \text{for all } B_0 \in \mathcal{B}$$

and

$$\inf_{B\in\mathcal{B};\, r_B \geq r_{B_0}} \frac{\varphi(B)}{\Phi^{-1}\Big(w(B)^{-1}\Big)} \gtrsim \frac{\varphi(B_0)}{\Phi^{-1}\Big(w(B_0)^{-1}\Big)} \quad \text{for all } B_0 \in \mathcal{B},$$

where r_B and r_{B_0} denote the radius of the balls B and B_0, respectively.

It will be assumed that the functions φ are of the class $\mathcal{G}^{w}_{\Phi}$ in the sequel. We refer to [5, Section 5] for more information about this condition.

We consider the following subspaces of $\mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$:

Definition 2.6. *The vanishing weighted Orlicz-Morrey space at origin $V_0\mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in \mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = 0.$$

The vanishing weighted Orlicz-Morrey space at infinity $V_{\infty}\mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in \mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ such that

$$\lim_{r\to \infty} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = 0.$$

The vanishing subspace $V_{\infty}\mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ and $V_0\mathcal{M}^{\Phi,\varphi}_{w}(\mathbb{R}^n)$ are nontrivial if $\mathcal{G}^{w}_{\Phi}$ satisfies the additional conditions

$$\lim_{r\to \infty} \sup_{x\in\mathbb{R}^n} \frac{\Phi^{-1}\Big(w(B(x,r))^{-1}\Big)}{\varphi(x,r)} = 0$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi(x,r)} = 0,$$

respectively. Since then they contain bounded functions with compact support.

We recall that the space $BMO(\mathbb{R}^n) = \{b \in L^1_{\rm loc}(\mathbb{R}^n) : \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where $b_{B(x,r)} = \frac{1}{B(x,r)} \int_{B(x,r)} b(y)dy$.

Now, we define operators investigated in this paper.

The maximal commutator is defined by

$$M_b f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy.$$

Let us, also, define the commutator of the Riesz potential

$$[b, I_\alpha]f(x) := \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^{n-\alpha}} f(y)dy, \qquad 0 < \alpha < n.$$

3. Auxiliary Estimates

The following estimates play an essential role in the proof of our results.

Lemma 3.1. *[4, Lemma 5.6] Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$, $f \in L^{\Phi,\rm loc}_w(\mathbb{R}^n)$ and $B = B(x,r)$. Then*

$$\|M_b f\|_{L^\Phi_w(B)} \lesssim \frac{\|b\|_*}{\Phi^{-1}\big(w(B)^{-1}\big)} \sup_{t>r} \Big(1 + \ln\frac{t}{r}\Big) \Phi^{-1}\Big(w(B(x,t))^{-1}\Big) \|f\|_{L^\Phi_w(B(x,t))}. \tag{2}$$

Lemma 3.2. *[10, Theorem 5.9] Let $0 < \alpha < n$, $b \in BMO(\mathbb{R}^n)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$, and $\varphi(x,t)$ satisfies the condition*

$$r^\alpha \varphi(x,r) + \int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\varphi(x,t) t^\alpha \frac{dt}{t} \lesssim \varphi(x,r)^\beta \tag{3}$$

for some $\beta \in (0,1)$ and for every $x \in \mathbb{R}^n$ and $r > 0$. Then for the operator $[b, I_\alpha]$ we have the following pointwise estimate

$$|[b, I_\alpha]f(x)| \lesssim \|b\|_* (M_b f(x))^\beta \, \|f\|^{1-\beta}_{\mathcal{M}^{\Phi,\varphi}_w}. \tag{4}$$

We now give theorems that will guarantee the norm inequalities for the operators M_b and $[b, I_\alpha]$ in vanishing spaces.

Theorem 3.3. *[4, Theorem 1.2] Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$ and $\varphi \in \mathcal{G}^w_\Phi$ satisfies the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln\frac{t}{r}\Big)\varphi(x,t) \le C_0\, \varphi(x,r), \tag{5}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, where C_0 does not depend on x and r. Then the operator M_b is bounded from $M^{\Phi,\varphi}_w(\mathbb{R}^n)$ to $M^{\Phi,\varphi}_w(\mathbb{R}^n)$.

Theorem 3.4. *[10, Theorem 5.9] Let $0 < \alpha < n$, $b \in BMO(\mathbb{R}^n)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$ and $\varphi \in \mathcal{G}^w_\Phi$. Let $\beta \in (0,1)$ and define $\eta(x,t) \equiv \varphi(x,t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$. If conditions (3) and (5) hold, then $[b, I_\alpha]$ is bounded from $\mathcal{M}^{\Phi,\varphi}_w(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}_w(\mathbb{R}^n)$.*

4. Main Results

In this section, we show that the subspaces $V_0\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ and $V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ are invariant with respect to operators M_b and $[b, I_\alpha]$.

Theorem 4.1. *Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$. If $\varphi \in \mathcal{G}_\Phi^w$ satisfies the condition (5), then the maximal commutator M_b is bounded on $V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$.*

Proof. Since M_b is bounded in $\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ (cf. Theorem 3.3) we only have to show that it preserves the vanishing property (V_∞):

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = 0 \quad \Longrightarrow \quad \lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi,w}(M_b f;x,r) = 0.$$

If $f \in V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ then for any $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ such that

$$\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi,w}(f;x,t) < \epsilon \qquad \text{for every } t \ge R.$$

Using inequality (2), we get

$$\mathfrak{A}_{\Phi,\varphi,w}(M_b f;x,r) \lesssim \frac{\|b\|_*}{\varphi(x,r)}\sup_{r<t<\infty}\Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\Big(w(B(x,t))^{-1}\Big)\|f\|_{L_w^\Phi(B(x,t))} \lesssim \|b\|_*\epsilon$$

for any $x \in \mathbb{R}^n$ and every $r \ge R$ (with the implicit constants independent of x and r). Therefore

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^n}\mathfrak{A}_{\Phi,\varphi,w}(M_b f;x,r) = 0$$

and hence $M_b f \in V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$. □

Now we show that the vanishing property (V_∞) is also preserved by the operator $[b, I_\alpha]$.

Theorem 4.2. *Let $0 < \alpha < n$, $b \in BMO(\mathbb{R}^n)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$ and $\varphi \in \mathcal{G}_\Phi^w$. Let $\beta \in (0,1)$ and define $\eta(x,t) \equiv \varphi(x,t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$. If conditions (3) and (5) hold, then $[b, I_\alpha]$ is bounded from $V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $V_\infty\mathcal{M}_w^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof. The boundedness of the operator $[b, I_\alpha]$ in weighted Orlicz-Morrey spaces follows from Theorem 3.4. To show the preservation of vanishing property, we make use of the pointwise estimate (4).

Note that from (1) we get

$$\int_{B(x,r)}\Psi\left(\frac{(M_b f(z))^\beta}{\|M_b f\|^\beta_{L_w^\Phi(B(x,r))}}\right)w(z)dz = \int_{B(x,r)}\Phi\left(\frac{M_b f(z)}{\|M_b f\|_{L_w^\Phi(B(x,r))}}\right)w(z)dz \le 1.$$

Thus

$$\|(M_b f)^\beta\|_{L_w^\Psi(B(x,r))} \le \|M_b f\|^\beta_{L_w^\Phi(B(x,r))}. \tag{6}$$

By (4) and (6), we get

$$\mathfrak{A}_{\Psi,\eta,w}([b,I_\alpha]f;x,r) \lesssim \Big(\mathfrak{A}_{\Phi,\varphi,w}(M_b f;x,r)\Big)^\beta \|f\|^{1-\beta}_{\mathcal{M}_w^{\Phi,\varphi}} \tag{7}$$

for all $r > 0$ and $x \in \mathbb{R}^n$. If $f \in V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$, then $M_b f \in V_\infty\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ by Theorem 4.1. Consequently, we have $[b, I_\alpha]f \in V_\infty\mathcal{M}_w^{\Psi,\eta}(\mathbb{R}^n)$ taking into account estimate (7). □

Theorem 4.3. *Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$ and $\varphi \in \mathcal{G}_\Phi^w$ satisfies (5). Let also*

$$\lim_{r\to 0} \frac{\ln \frac{1}{r}}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} = 0 \tag{8}$$

and

$$m_\delta := \sup_{\delta<t<\infty} (1 + |\ln t|) \sup_{x\in\mathbb{R}^n} \varphi(x,t) < \infty \tag{9}$$

for every $\delta > 0$. Then the maximal commutator M_b is bounded on $V_0\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$.

Proof. The norm inequalities follow from Theorem 3.3, so we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(f;x,r) = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(M_b f;x,r) = 0. \tag{10}$$

We rewrite the inequality (2) in the form

$$\mathfrak{A}_{\Phi,\varphi,w}(M_b f;x,r) \le C \frac{\sup_{t>r} \left(1 + \ln \frac{t}{r}\right) \Phi^{-1}\left(w(B(x,t))^{-1}\right) \|f\|_{L^\Phi(B(x,t))}}{\varphi(x,r)}. \tag{11}$$

To show that $\sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(M_b f;x,r) < \varepsilon$ for small r, we split the right-hand side of (11):

$$\mathfrak{A}_{\Phi,\varphi,w}(M_b f;x,r) \le C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \tag{12}$$

where $\delta_0 > 0$ will be chosen as shown below (we may take $\delta_0 < 1$) and

$$I_{\delta_0}(x,r) := \frac{\sup_{r<t<\delta_0} \left(1 + \ln \frac{t}{r}\right) \Phi^{-1}\left(w(B(x,t))^{-1}\right) \|f\|_{L^\Phi(B(x,t))}}{\varphi(x,r)},$$

$$J_{\delta_0}(x,r) := \frac{\sup_{t>\delta_0} \left(1 + \ln \frac{t}{r}\right) \Phi^{-1}\left(w(B(x,t))^{-1}\right) \|f\|_{L^\Phi(B(x,t))}}{\varphi(x,r)}$$

and it is supposed that $r < \delta_0$. Now we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi,w}(f;x,t) < \frac{\varepsilon}{2CC_0}, \quad \text{for all} \quad 0 < t < \delta_0,$$

where C and C_0 are the constants from (12) and (5), which is possible since $f \in V\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$. Then

$$\Phi^{-1}\left(w(B(x,t))^{-1}\right) \|f\|_{L^\Phi(B(x,t))} < \frac{\varepsilon}{2CC_0} \varphi(x,t)$$

and we obtain the estimate of the first term uniform in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

For the second term, writing $1 + \ln \frac{t}{r} \le 1 + |\ln t| + \ln \frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r) \le \frac{m_{\delta_0} + \widetilde{m_{\delta_0}}}{\varphi(x,r)} \|f\|_{\mathcal{M}_w^{\Phi,\varphi}},$$

where m_{δ_0} is the constant from (9) with $\delta = \delta_0$ and $\widetilde{m_{\delta_0}}$ is a similar constant with omitted logarithmic factor. Then, by (8) it suffices to choose r small enough such that $\sup_{x\in\mathbb{R}^n} J_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, which completes the proof of (10). □

Similar to the proof of Theorem 4.2 we can show that the vanishing property (V_0) is also preserved by the operator $[b, I_\alpha]$. But now using Theorem 4.3 instead of Theorem 4.1 in the proof.

Theorem 4.4. *Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function which is of lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, $w \in A_{p_0}$ and $\varphi \in \mathcal{G}_\Phi^w$. Let $\beta \in (0,1)$ and define $\eta(x,t) \equiv \varphi(x,t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$. If conditions* (3), (5), (8) *and* (9) *hold, then $[b, I_\alpha]$ is bounded from $V_0\mathcal{M}_w^{\Phi,\varphi}(\mathbb{R}^n)$ to $V_0\mathcal{M}_w^{\Psi,\eta}(\mathbb{R}^n)$.*

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