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Families of higher order q-Euler numbers and polynomials

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Abstract. In this paper, we define new type q-Euler numbers and polynomials with the help of p-adic q-integrals. Using the techniques of p-adic integral, the method of generating functions, and combinatorial techniques, some interesting sums and relations between them are calculated.

1. Introduction

The q-calculus plays an important role in number theory, combinatorics and other branches of mathematics. It was first examined by Euler [4]. There are still important works related to q-calculus.

Let p be an odd prime number. $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\ .\ |_p$ is normalized by $|\ p\ |_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|\ 1-q\ |_p < p^{\frac{-1}{p-1}}$. The q-extension (or q-analogue) of number x, denoted as $[x]_q$, is

$$[x]_q = \frac{1-q^x}{1-q}.$$

It is clear that $\lim_{q\to 1}[x]_q = x$. Let d be a fixed integer and

$$X = X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \ | \ x \equiv a \ (\mathrm{mod}\ dp^N)\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral was defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \int_X f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{x=0}^{dp^N-1} f(x)q^x$$

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for $| 1 - q |_p < 1$ (see [6, 13, 21, 22, 24, 25, 27–29]).

In [13], Kim gave the integral equations related to the p-adic q-integral. For example, for $n \in \mathbb{N}$,

$$\int_{\mathbb{Z}_p} x^n d\mu_q(x) = \frac{q-1}{\log q} B_{n,q},$$

where $B_{n,q}$ are q-Bernoulli numbers.

In [21], Kim defined the generalized q-Bernoulli numbers $B_{m,\chi}(q)$ as

$$B_{m,\chi}(q) = \int_X \chi(x)[x]^m d\mu_q(x) = \lim_{N\to\infty} \sum_{x=0}^{dp^N-1} [x]^m \chi(x) \frac{q^x}{dp^N}.$$

The author showed Carlitz's q-Bernoulli numbers as an integral by the q-analogue μ_q of the ordinary p-adic invariant measure.

In [9, 10], bosonic integral was considered from a more physical point of view to the bosonic limit $q \to 1$ as follows:

$$I_1(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x).$$

Furthermore, it can be considered the fermionic integral in contrast to the conventional "bosonic" . That is

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).$$

From here, it can be seen that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0),$$

where $f_1(x) = f(x+1)$. Moreover,

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{x=0}^{n-1} (-1)^{n-1-x} f(x),$$

where $f_n(x) = f(x+n)$ and $n \in \mathbb{Z}^+$ [8, 11]. For $| 1 - q |_p < 1$, it can be considered fermionic p-adic q-integral on $\mathbb{Z}_p$ which is the q-extension of $I_{-1}(f)$ as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[dp^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x.$$

From here, Kim et al. [9, 31] examined that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1}$$

As known, the higher order Euler polynomials are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^r(x) \frac{t^n}{n!}$$

for positive integer r. When $x = 0$, $E_n^r = E_n^r(0)$ are called Euler numbers of order r. In particular, when $r = 1$, $E_n(x) = E_n^1(x)$ are called the Euler polynomials. Also, in the case of $x = 0$ and $r = 1$, $E_n = E_n^1(0)$ are called

Euler numbers (see [8, 11, 15, 19, 23, 35, 42]). There are famous scientists working on Euler numbers and polynomials in several parts of mathematics. For example, in analysis, in statistics, in numerical analysis, in combinatorics, in number theory, and so on [1, 2, 6, 7, 14, 17, 19, 24, 26, 30, 37, 38].

Recently, different generalizations of Euler numbers are still defined in number theory [23, 26, 28, 36].

In [18, 19], Kim introduced the Euler polynomials of Nörlund type $E_n^{-r}(x)$ as follows:

$$\left(\frac{e^t+1}{2}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{-r}(x)\frac{t^n}{n!}.$$

The authors [23, 33] defined the q-Euler numbers as

$$E_{0,q} = 1, \;\; q(qE+1)^n + E_{n,q} = \begin{cases} [2]_q & \text{if } n = 0, \\ 0 & \text{if } n \neq 0, \end{cases} \tag{2}$$

with the usual convention of replacing E^n by $E_{n,q}$. These numbers are reduced to E_n when $q = 1$. From (2), it can be also derived

$$E_{n,q} = \frac{[2]_q}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{1+q^{l+1}}.$$

In [22], Kim defined the λ-Euler numbers, denoted by $E_n(\lambda)$, as

$$\int_{\mathbb{Z}_p} e^{tx}\lambda^x d\mu_{q=-1}(x) = \frac{2}{\lambda e^t+1} = \sum_{n=0}^{\infty} E_n(\lambda)\frac{t^n}{n!}.$$

In [25], Kim constructed p-adic q-Euler numbers and polynomials of higher order and defined new generating functions of multiple q-Euler numbers and polynomials. The author considered the extended higher order q-Euler numbers by

$$E_{m,q}^{(h,k)} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1+x_2+\ldots+x_r]_q^m q^{x_1(h-1)+x_2(h-2)+\ldots+x_r(h-r)}d\mu_{-q}(x_1)d\mu_{-q}(x_2)\ldots d\mu_{-q}(x_r).$$

The q-Euler polynomials, denoted as $E_{n,q}(x)$, are as follows [5, 12, 23, 31]:

$$E_{n,q}(x) = \int_{\mathbb{Z}_p}[x+y]_q^n d\mu_{-q}(y) = \frac{[2]_q}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{xl}}{1+q^{l+1}}.$$

Note that, in the case of $x = 0$, $E_{n,q}(0) = E_{n,q}$.

In [16], Kim showed the systemic study of some families of multiple q−Euler numbers and polynomials. For $n \in \mathbb{Z}^+$,

$$E_{n,q}^{(r)}(x) = 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m[m+x]_q^n,$$

where $E_{n,q}^{(r)}$ are the q-Euler polynomials of order $r \in \mathbb{N}$ [19, 20].

Kim [20] introduced the modified q-Euler numbers and polynomials. For any non-negative integer n, the modified q-Euler polynomials $\varepsilon_{n,q}(x)$ are defined by

$$\varepsilon_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-y}[x+y]_q^n d\mu_{-q}(y) = \frac{[2]_q}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{xl}}{1+q^l}.$$

For $x = 0$, $\varepsilon_{n,q}(0) = \varepsilon_{n,q}$ are called nth modified q-Euler numbers.

In [36], Rim and Jeong defined the modified q-Euler polynomials with weight α as follows:

$$\widetilde{\varepsilon}_{n,q}^{(\alpha)}(x)=\int_{\mathbb{Z}_p} q^{-y}[x+y]_{q^\alpha}^n d\mu_{-q}(y)=\frac{[2]_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{\alpha x l}}{1+q^{\alpha l}}$$

for $\alpha\in\mathbb{Q}$. For $x=0$, $\widetilde{\varepsilon}_{n,q}^{(\alpha)}(0)=\widetilde{\varepsilon}_{n,q}^{(\alpha)}$ are called nth modified q-Euler numbers with weight α. Also, these numbers hold

$$(q^\alpha\widetilde{\varepsilon}_q^{(\alpha)}+1)^n+\widetilde{\varepsilon}_{n,q}^{(\alpha)}=\begin{cases}[2]_q & \text{if } n=0,\\ 0 & \text{if } n\neq 0,\end{cases}$$

with the usual convention about replacing $(\widetilde{\varepsilon}_q^{(\alpha)})^n$ by $\widetilde{\varepsilon}_{n,q}^{(\alpha)}$.

In [38], Rim et al. gave another type of modified q-Euler polynomials as

$$\widetilde{\epsilon}_{n,q}(x)=\int_{\mathbb{Z}_p} q^{-y}(x+[y]_q)^n d\mu_{-q}(y)=\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}\binom{l}{k}\frac{[2]_q x^{n-l}}{(1-q)^l(1+q^k)},$$

where $\widetilde{\epsilon}_{n,q}(0)=\varepsilon_{n,q}$.

2. Higher Order q-Euler Numbers and Polynomials

In this section, firstly we define the q-Euler polynomials with order r of the second kind denoted as $E_{n,q}^r(x)$; the modified q-Euler polynomials with weight α and order r, denoted as $\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}(x)$; the modified q-Euler polynomials with weight α and order r of the second kind, denoted as $\widetilde{\epsilon}_{n,q}^{r,(\alpha)}(x)$. Then using the techniques of p-adic integral, the method of generating functions, and combinatorial techniques, we will give some combinatorial identities and sums of these numbers and polynomials. We also obtain some relations related to them.

Definition 2.1. *For non-negative integer n and positive integer r, the q-Euler polynomials with order r are defined by*

$$E_{n,q}^r(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r).$$

For $x=0$, $E_{n,q}^r(0)=E_{n,q}^r$ are called the q-Euler numbers with order r.

Definition 2.2. *For non-negative integer n, positive integer r, and rational number α, the modified q-Euler polynomials with weight α and order r are defined by*

$$\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_r}[x+x_1+\cdots+x_r]_{q^\alpha}^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r).$$

In the special case of $x=0$, the numbers $\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}(0)=\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}$ are called the modified q-Euler numbers with weight α and order r; for $\alpha=1$, $\widetilde{\varepsilon}_{n,q}^{r,(1)}(x)=\varepsilon_{n,q}^r(x)$ are called the modified q-Euler polynomials with order r and for $x=0$, $\alpha=1$, $\widetilde{\varepsilon}_{n,q}^{r,(1)}(0)=\varepsilon_{n,q}^r$ are called the modified q-Euler numbers with order r.

Definition 2.3. *For non-negative integer n, positive integer r, and rational number α, the modified q-Euler polynomials with weight α and order r of the second kind are defined by*

$$\widetilde{\epsilon}_{n,q}^{r,(\alpha)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_r}(x+[x_1+\cdots+x_r]_{q^\alpha})^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r).$$

For $\alpha=1$, $\widetilde{\epsilon}_{n,q}^{r,(1)}(x)=\widetilde{\epsilon}_{n,q}^r(x)$ is referred to as the modified q-Euler polynomials with order r.

Lemma 2.4. *For real number λ, then*

$$\int_{\mathbb{Z}_p} q^{\lambda y} d\mu_{-q}(y) = \frac{[2]_q}{1+q^{\lambda+1}}. \tag{3}$$

Proof. For $f(y) = q^{\lambda y}$ in (1), the proof is clear. □

Theorem 2.5. *For non-negative integer n and positive integer r, then*

$$E^r_{n,q}(x) = \frac{[2]^r_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{lx}}{(1+q^{l+1})^r}, \tag{4}$$

$$\varepsilon^r_{n,q}(x) = \frac{[2]^r_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{lx}}{(1+q^{l})^r}$$

and

$$\widetilde{\epsilon}^r_{n,q}(x) = [2]^r_q \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{n}{l}\binom{l}{k} \frac{x^{n-l}}{(1-q)^l\,(1+q^k)^r},$$

where $\widetilde{\epsilon}^r_{n,q}(0) = \widetilde{\varepsilon}^r_{n,q}$.

Proof. We will give the proof of (4). From the definition of q- Euler polynomials with order r and binomial theorem, we get

$$\begin{aligned}
E^r_{n,q}(x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x+x_1+\cdots+x_r]^n_q d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&= \frac{1}{(1-q)^n} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1-q^{x+x_1+\cdots+x_r})^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&= \frac{1}{(1-q)^n} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{l(x+x_1+\cdots+x_r)} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{lx}}{(1-q)^n} \int_{\mathbb{Z}_p} q^{lx_r} \cdots \int_{\mathbb{Z}_p} q^{lx_2} \int_{\mathbb{Z}_p} q^{lx_1} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r).
\end{aligned}$$

With the help of (3), we have

$$\begin{aligned}
E^r_{n,q}(x) &= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{lx}}{(1-q)^n} \frac{[2]_q}{1+q^{l+1}} \int_{\mathbb{Z}_p} q^{lx_r} \cdots \int_{\mathbb{Z}_p} q^{lx_2} d\mu_{-q}(x_2)\cdots d\mu_{-q}(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{lx}}{(1-q)^n} \left(\frac{[2]_q}{1+q^{l+1}}\right)^2 \int_{\mathbb{Z}_p} q^{lx_r} \cdots \int_{\mathbb{Z}_p} q^{lx_3} d\mu_{-q}(x_3)\cdots d\mu_{-q}(x_r) \\
&= \cdots \\
&= \frac{[2]^r_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{lx}}{(1+q^{l+1})^r},
\end{aligned}$$

as claimed. Other identities can be found in a similar way. So, the proof is complete. □

For example, when $x = 0$ in Theorem 2.5, it can be seen that

$$E^{r}_{n,q} = \frac{[2]^r_q}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{(1+q^{l+1})^r}$$

and

$$\varepsilon^{r}_{n,q} = \frac{[2]^r_q}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{(1+q^{l})^r}$$

immediately.

Theorem 2.6. *For non-negative integer n and positive integer r, then*

$$\widetilde{\varepsilon}^{r,(\alpha)}_{n,q}(x) = \frac{[2]^r_q}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l\, q^{\alpha l x}}{(1+q^{\alpha l})^r}, \tag{5}$$

and

$$\widetilde{\varepsilon}^{r,(\alpha)}_{n,q}(x) = [2]^r_q\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{n}{l}\binom{l}{k}\frac{x^{n-l}}{(1-q^\alpha)^l\,(1+q^{\alpha k})^r}$$

where $\widetilde{\varepsilon}^{r,(\alpha)}_{n,q}(0) = \widetilde{\varepsilon}^{r,(\alpha)}_{n,q}$.

Proof. These identities can be found similar to way proof of Theorem 2.5. □

Theorem 2.7. *For non-negative integers n and* $r > 1$, *then*

$$\widetilde{\varepsilon}^{r,(\alpha)}_{n,q}(x+1) + \widetilde{\varepsilon}^{r,(\alpha)}_{n,q}(x) = [2]_q\widetilde{\varepsilon}^{r-1,(\alpha)}_{n,q}(x) \tag{6}$$

and

$$qE^{r}_{n,q}(x+1) + E^{r}_{n,q}(x) = [2]_q E^{r-1}_{n,q}(x).$$

Proof. We will prove identity (6). Taking $f(x_1) = q^{-x_1-x_2-\cdots-x_r}[x+x_1+\cdots+x_r]^n_{q^\alpha}$ in (1), we write

$$q\int_{\mathbb{Z}_p} q^{-1-x_1-x_2-\cdots-x_r}[x+1+x_1+\cdots+x_r]^n_{q^\alpha}d\mu_{-q}(x_1) + \int_{\mathbb{Z}_p} q^{-x_1-x_2-\cdots-x_r}[x+x_1+\cdots+x_r]^n_{q^\alpha}d\mu_{-q}(x_1) = [2]_q q^{-x_2-\cdots-x_r}[x+x_2+\cdots+x_r]^n_{q^\alpha}$$

and apply p-adic integral of the above equality $r-1$ times with respect to $x_2, \cdots, x_r$, respectively, we have

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} & q^{-x_1-x_2-\cdots-x_r}[x+1+x_1+\cdots+x_r]^n_{q^\alpha}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\ &+\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-x_1-x_2-\cdots-x_r}[x+x_1+\cdots+x_r]^n_{q^\alpha}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\ &=[2]_q\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-x_2-\cdots-x_r}[x+x_2+\cdots+x_r]^n_{q^\alpha}d\mu_{-q}(x_2)\cdots d\mu_{-q}(x_r).\end{aligned}$$

From the definition of the modified q-Euler polynomials with weight α and order r, the identity is obtained. The proof of other identity is similar to the proof of (6). □

Theorem 2.8. *For non-negative integers n and $r > 1$, then*

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}(1) = [2]_q\widetilde{\epsilon}_{n,q}^{r-1,(\alpha)}(x) - \widetilde{\epsilon}_{n,q}^{r,(\alpha)}(x).$$

Proof. The proof is similar to the proof of Theorem 2.7. □

Theorem 2.9. *For non-negative integers n and $r \geq 1$, then*

$$q^x E_{n,q}^r(x) = \varepsilon_{n,q}^r(x) - (1-q)\varepsilon_{n+1,q}^r(x).$$

Proof. From definitions of the modified q-Euler polynomials with order r and q-Euler polynomials with order r, we have

$$\begin{aligned}(1-q)\varepsilon_{n+1,q}^r(x) =&(1-q)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_r}[x+x_1+\cdots+x_r]_q^{n+1}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_r}[x+\cdots+x_r]_q^n\left(1-q^{x+x_1+\cdots+x_r}\right)d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\ =&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-x_1-\cdots-x_r}[x+\cdots+x_r]_q^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\ &-q^x\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^n d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\ =&\varepsilon_{n,q}^r(x) - q^x E_{n,q}^r(x),\end{aligned}$$

as claimed. □

Theorem 2.10. *For non-negative integer n and positive integer r, we have*

$$E_{n,q}^r(x) = [2]_q^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m q^m[x+m]_q^n, \tag{7}$$

and

$$\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}(x) = [2]_q^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m[x+m]_{q^\alpha}^n. \tag{8}$$

Proof. We will give proof of (8). From (5) and binomial theorem, we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}(x)\frac{t^n}{n!} =&\sum_{n=0}^{\infty}\frac{[2]_q^r}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\frac{1}{(1+q^{\alpha l})^r}\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\frac{[2]_q^r}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m q^{\alpha l m}\frac{t^n}{n!}\\ =&[2]_q^r\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m\frac{1}{(1-q^\alpha)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l(x+m)}\frac{t^n}{n!}\\ =&[2]_q^r\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m\frac{(1-q^{\alpha(x+m)})^n}{(1-q^\alpha)^n}\frac{t^n}{n!}\\ =&[2]_q^r\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m[x+m]_{q^\alpha}^n\frac{t^n}{n!}.\end{aligned}$$

By equality of two exponential generating functions, we get (8). Similarly, the proof of (7) can be shown. □

Now, we define new type q-Euler polynomials as derive of $\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}(x)$ and $E_{n,q}^{r}(x)$. Let exponential generating functions of these polynomials be denoted as $F_q^{(r)}(t,x)$ and $G_q^{(r)}(t,x)$, respectively. Firstly, we examine

$$F_q^{(r)}(t,x)=\sum_{n=0}^{\infty}\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}(x)\frac{t^n}{n!}.$$

By (8), we can write

$$F_q^{(r)}(t,x)=[2]_q^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m e^{[x+m]_{q^\alpha}^n t}.$$

Also, from the above equality, we can consider the q-extension of $\widetilde{\varepsilon}_{n,q}^{r,(\alpha)}$ of Nörlund type, is defined by

$$F_q^{(-r)}(t,x)=\frac{1}{[2]_q^r}\sum_{m=0}^{r}\binom{r}{m}e^{[x+m]_{q^\alpha}^n t}=\sum_{n=0}^{\infty}\widetilde{\varepsilon}_{n,q}^{-r,(\alpha)}(x)\frac{t^n}{n!}.$$

Similarly, we can define the q-extension of $E_{n,q}^{r}(x)$ of Nörlund type by using (7) as

$$G_q^{(-r)}(t,x)=\frac{1}{[2]_q^r}\sum_{m=0}^{r}\binom{r}{m}q^m e^{[x+m]_q^n t}=\sum_{n=0}^{\infty}E_{n,q}^{-r}(x)\frac{t^n}{n!}.$$

Therefore, we obtain the following corollary:

Corollary 2.11. *For non-negative integer n and positive integer r, we have*

$$E_{n,q}^{-r}(x)=\frac{1}{[2]_q^r}\sum_{m=0}^{r}\binom{r}{m}q^m[x+m]_q^n \tag{9}$$

and

$$\widetilde{\varepsilon}_{n,q}^{-r,(\alpha)}(x)=\frac{1}{[2]_q^r}\sum_{m=0}^{r}\binom{r}{m}[x+m]_{q^\alpha}^n. \tag{10}$$

Proof. Taking $r\to -r$ in (7) and (8) and using the generalized binomial theorem, then the identities are obtained. □

Theorem 2.12. *For non-negative integer n, then*

$$\sum_{r=0}^{\infty}[2]_q^r E_{n,q}^{-r}(x)\frac{t^r}{r!}=\frac{1}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k q^{xk}e^{(1+q^{k+1})t},$$

and

$$\sum_{r=0}^{\infty}[2]_q^r\widetilde{\varepsilon}_{r,q}^{-r,(\alpha)}(x)\frac{t^r}{r!}=\frac{1}{(1-q^\alpha)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k q^{\alpha xk}e^{(1+q^{\alpha k})t}. \tag{11}$$

Proof. We will give proof of (11). From (10) and some combinatorial techniques, we can write

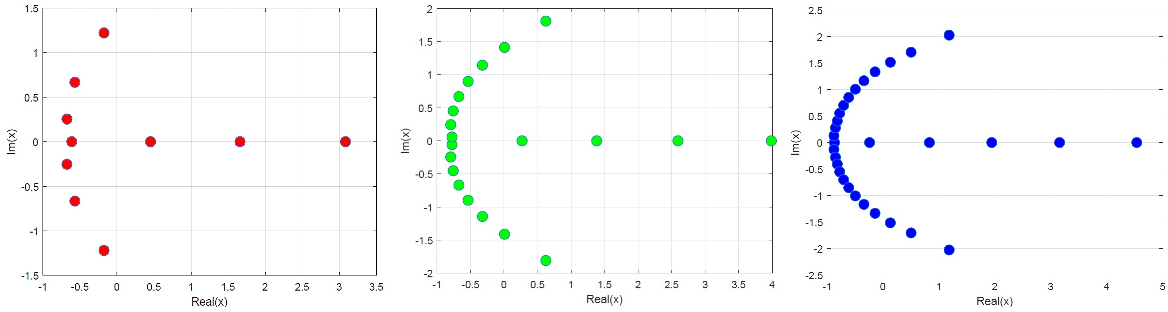
$$\begin{aligned}\sum_{r=0}^{\infty}[2]_q^r\widetilde{\varepsilon}_{r,q}^{-r,(\alpha)}(x)\frac{t^r}{r!}&=\sum_{r=0}^{\infty}\sum_{m=0}^{r}\binom{r}{m}[x+m]_{q^\alpha}^n\frac{t^r}{r!}=\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{r=0}^{\infty}[x+r]_{q^\alpha}^n\frac{t^r}{r!}\\&=e^t\frac{1}{(1-q^\alpha)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k q^{\alpha xk}e^{q^{\alpha k}t}\\&=\frac{1}{(1-q^\alpha)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k q^{\alpha xk}e^{(1+q^{\alpha k})t},\end{aligned}$$

as claimed. Other identity can be found similar way. □

3. Distribution and Stacks of Zeros of Higher Order Euler Polynomials

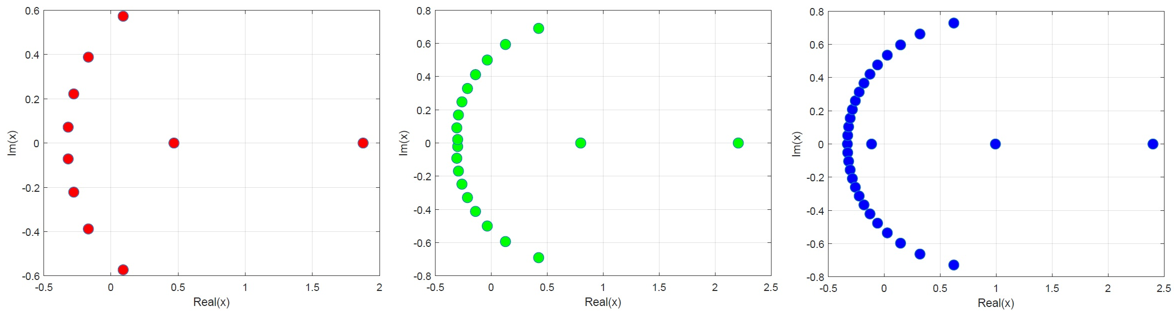
In this section, there are some works including interesting phenomenon of "scattering" of the zeros of Euler and q-Euler polynomials in complex plane [32, 34, 39–41].

We will show zeros of family of higher order q-Euler polynomials by using MATLAB2021b. We plot the zeros of the polynomials $E^{r}_{n,q}(x)$, $\widetilde{\varepsilon}^{r,(\alpha)}_{n,q}(x)$, and $\widetilde{\epsilon}^{r,(\alpha)}_{n,q}(x)$ in Figure 1, Figure 2, and Figure 3 for $n = 10, 20, 30$ and $x \in \mathbb{C}$, respectively. Also, stacks of zeros of these polynomials for $q = 1/2$ and $1 \leq n \leq 40$ from a 3D structure are presented in Figure 4. Lastly, we present the distribution of real zeros of them for $1 \leq n \leq 20$ in Figure 5.



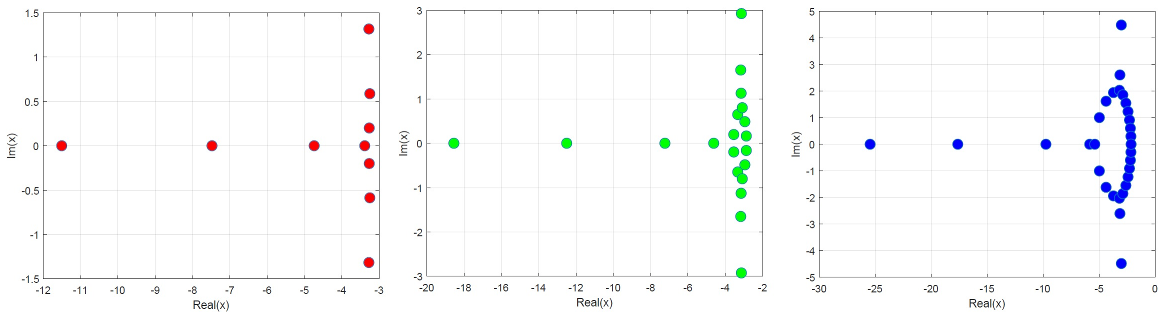
(a) $n = 10$ (b) $n = 20$ (c) $n = 30$

Figure 1: Zeros of $E^{3}_{n,1/2}(x)$



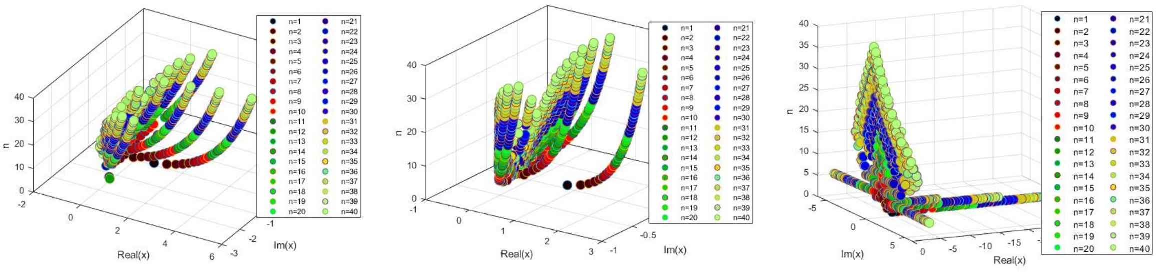
(a) $n = 10$ (b) $n = 20$ (c) $n = 30$

Figure 2: Zeros of $\widetilde{\varepsilon}^{3,(3)}_{n,1/2}(x)$



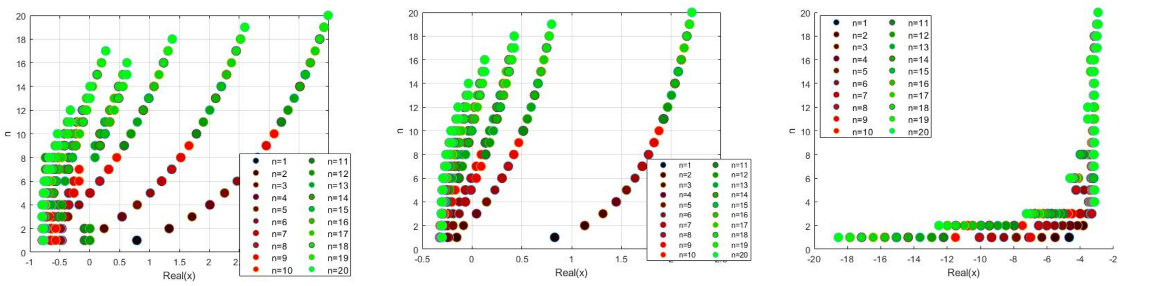
(a) $n = 10$ (b) $n = 20$ (c) $n = 30$

Figure 3: Zeros of $\widetilde{\epsilon}^{1,(1)}_{n,1/2}(x)$



(a) Stacks of zeros of $E^{3}_{n,1/2}(x)$ (b) Stacks of zeros of $\widetilde{\varepsilon}^{3,(3)}_{n,1/2}(x)$ (c) Stacks of zeros of $\widetilde{\varepsilon}^{1,(1)}_{n,1/2}(x)$

Figure 4: Stacks of polynomials



(a) Distribution of real zeros $E^{3}_{n,1/2}(x)$ (b) Distribution of real zeros $\widetilde{\varepsilon}^{3,(3)}_{n,1/2}(x)$ (c) Distribution of real zeros $\widetilde{\varepsilon}^{1,(1)}_{n,1/2}(x)$

Figure 5: Distribution of real zeros of polynomials

Table 1 shows numbers of complex and real zeros of $E^{r}_{n,q}(x)$ for $r = 3, q = 1/2$, and various values n. Table 2 shows approximate solutions satisfying the $E^{r}_{n,q}(x)$ for $r = 3, q = 1/2$, and various values n. A highly ordered structure of the complex roots of $E^{r}_{n,q}(x)$ is observed in Table 1. This numerical study is quite exciting in combinatorics.

Table 1: Numbers of complex and real zeros of $E^3_{n,1/2}(x)$.

Degree n	Real zeros	Complex zeros	Degree n	Real zeros	Complex zeros
1	1	0	21	5	16
2	2	0	22	6	16
3	3	0	23	5	18
4	2	2	24	6	18
5	3	2	25	5	20
6	4	2	26	6	20
7	3	4	27	5	22
8	4	4	28	6	22
9	3	6	29	5	24
10	4	6	30	6	24
11	5	6	31	5	26
12	4	8	32	6	26
13	5	8	33	5	28
14	4	10	34	6	28
15	5	10	35	5	30
16	4	12	36	6	30
17	5	12	37	7	30
18	4	14	38	6	32
19	5	14	39	5	34
20	4	16	40	6	34

Table 2: Approximate solutions of $E^3_{n,1/2}(x)$.

Degree n	Values x
1	0.7891032
2	-0.0840745, 1.3291870
3	-0.4605966, 0.2415933, 1.7115022
4	0.5417479, 2.0098701
5	-0.3632635, 0.7957111, 2.2553569
6	-0.6140164, -0.1781407, 1.0135564, 2.4642325
7	0.0056870, 1.2036925, 2.6461660
8	-0.6555541, 0.1699167, 1.3721568, 2.8074013
9	0.3181771, 1.5232866, 2.9522172
10	-0.6086962, 0.4531741, 1.6602661, 3.0836790
11	-0.7014964, -0.5312062, 0.5770020, 1.7854895, 3.2040610
12	-0.4146927, 0.6913118, 1.9007977, 3.3150983
13	-0.7653929, -0.3099667, 0.7974248, 2.0076346, 3.4181460
14	-0.2120978, 0.8964122, 2.1071520, 3.5142826
15	-0.7863637, -0.1203067, 0.9891529, 2.2002835, 3.6043813
16	-0.0338978, 1.0763758, 2.2877954, 3.6891590
17	-0.7911550, 0.0477139, 1.1586914, 2.3703244, 3.7692114
18	0.1250245, 1.2366155, 2.4484056, 3.8450393
19	-0.7800355, 0.1984580, 1.3105878, 2.5224921, 3.9170677
20	0.2683798, 1.3809860, 2.5929716, 3.9856602

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