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A study of convergence problem in functional norm

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Abstract. In this paper, we delve into the convergence challenges concerning with the functions $\tilde{g}$ and $\tilde{g}'$ representing as conjugate Fourier series and derived conjugate Fourier series respectively within generalized Hölder norm. Our approach involves utilizing the generalized Nörlund-Matrix ($N^{p,q}A$) means of the function $\tilde{g}$ of conjugate Fourier series and the function $\tilde{g}'$ of derived conjugate Fourier series in order to examine the convergence phenomenon of these functions. Furthermore, we conduct a comparative analysis of the convergence outcomes through applications.

1. Introduction

Numerous researchers, including [4, 5, 10–13, 23, 25] and others, have dedicated their studies to the utilization of Fourier series for function approximation. Their focus extends to different function spaces. These investigators, with the help of many references, have incorporated summability methods as essential tools to advance their investigations.

Present paper explores the challenges related to the convergence of conjugate function $\tilde{g}$ and conjugate derived function $\tilde{g}'$ associated with conjugate Fourier series and derived conjugate Fourier series respectively in generalized Hölder norm, where g is a 2π periodic function associated with Fourier series. This exploration is conducted by making use of generalized Nörlund-Matrix ($N^{p,q}A$) operator.

The structure of the paper is outlined as follows: In the second section, we present key definitions central to our research work. In Section 3, we establish the auxiliary results that are utilized in proving our main results. The fourth section focuses on establishing the convergence results of the functions $\tilde{g}$ and $\tilde{g}'$ within generalized Hölder space. In this section, we also discuss some applications of our convergence results. Section 5 serves as the conclusion, summarizing our key insights.

2. Preliminaries

2.1. *Conjugate Fourier series and its derived Series*

The conjugate Fourier series (C.F.S) i.e. conjugate series of Fourier series, is given by

$$\tilde{g}(t) := \sum_{\gamma=1}^{\infty}(a_\gamma \sin \gamma t - b_\gamma \cos \gamma t). \tag{1}$$

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The partial sums of (1) for the γ^{th} term are denoted as $\tilde{s}_\gamma(\tilde{g};t)$ and are defined by

$$\tilde{s}_\gamma(\tilde{g};t) - \tilde{g}(t) = \frac{1}{2\pi}\int_0^\pi \psi_{(t)}(\omega)\frac{\cos(\gamma+\frac{1}{2})\omega}{\sin\frac{\omega}{2}}d\omega,$$

where

$$\psi_{(t)}(\omega) = \tilde{g}(t+\omega) - \tilde{g}(t-\omega)$$

and

$$\tilde{g}(t) = -\frac{1}{2\pi}\int_0^\pi \gamma_{(t)}(\omega)\cot\left(\frac{\omega}{2}\right)d\omega[1]. \tag{2}$$

The first derived series of (1) i.e. derived conjugate Fourier series (D.C.F.S), is given by

$$\tilde{g}'(t) := \sum_{\gamma=1}^{\infty}(a_\gamma\cos\gamma t + b_\gamma\sin\gamma t). \tag{3}$$

The γ^{th} partial sum of the derived conjugate Fourier series (D.C.F.S.), denoted as $\tilde{s}'(\tilde{g}';t)$, is defined as follows,

$$\begin{aligned}&\tilde{s}_\gamma'(t) - \tilde{g}'(t)\\ &= -\frac{2}{\pi}\int_0^\pi \frac{\psi_{(t)}(\omega)}{4\sin\frac{\omega}{2}}\left(\gamma+\frac{1}{2}\right)\sin\left(\gamma+\frac{1}{2}\right)\omega d\omega - \frac{1}{\pi}\int_0^\pi \frac{\psi_{(t)}(\omega)}{4\sin\frac{\omega}{2}}\frac{\cos(\gamma+\frac{1}{2})\omega}{\tan\frac{\omega}{2}}d\omega[14],\end{aligned} \tag{4}$$

where $\tilde{g}'$ is the conjugate derived function of 2π-periodic function g, which is expressed as

$$\tilde{g}'(t) = \frac{1}{4\pi}\int_0^\pi \psi_{(t)}(\omega)\csc^2\left(\frac{\omega}{2}\right)d\omega.$$

Note 1. *A detailed work on Fourier series and its allied series can be found in [1].*

2.1.1. Generalized Nörlund-Matrix ($N^{p,q}A$) product operator

Generalized Nörlund-Matrix $N^{p,q}A$ product means is given by

$$\sigma_\gamma^{N^{p,q}A} = \frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}s_l, \tag{5}$$

where $N^{p,q}$ and A stands for generalized Nörlund transform and matrix transform respectively. When the limit of $\sigma_\gamma^{N^{p,q}A}$ approaches s as γ tends to infinity, it is asserted that the Fourier series is summable to s through the product $N^{p,q}A$.
The regularity of $N^{p,q}$ and A methods implies the regularity of product $N^{p,q}A$ method.

Note 2. *The readers may refer [3] for more details of matrix (A) and generalized Nörlund ($N^{p,q}$) transforms.*

Remark 2.1. *$N^{p,q}A$ product operator is reduced to*

(i) $N^{p,q}C^\beta$ *operator if* $a_{\gamma,\mu} = \frac{\binom{\gamma-\mu+\beta-1}{\beta-1}}{\binom{\gamma+\beta}{\beta}}$.

(ii) $N^{p,q}C^1$ *operator if* $a_{\gamma,\mu} = \frac{1}{\gamma+1}$.

(iii) $N^{p,q}H^p$ *operator if* $a_{\gamma,\mu} = \frac{1}{\log^{p-1}(\gamma+1)\prod_{m=0}^{p-1}\log^m(\mu+1)}$.

(iv) *$N^{p,q}H$ operator if* $a_{\gamma,\mu} = \frac{1}{(\gamma-\mu+1)\log\gamma}$.

(v) *$N^{p,q}N^{p}$ operator if* $a_{\gamma,\mu} = \frac{p_{\gamma-\mu}}{P_\gamma}$, *where* $P_\gamma = \sum_{\mu=0}^{\infty} p_\mu$.

(vi) *$N^{p}A$ operator if* $q_\gamma = 1$, *for all* γ.

(vii) *$C^{\beta}A$ operator if* $p_\gamma = \binom{\gamma+\beta-1}{\beta-1}, \beta > 0$ *and* $q_\gamma = 1$, *for all* γ.

2.1.2. Generalized Hölder space

Consider an arbitrary function $\rho : [0, 2\pi] \to \mathbb{R}$ with the conditions $\rho(\omega) > 0$ for $0 < \omega \leq 2\pi$, and $\lim_{\omega\to 0^+} \rho(\omega) = \rho(0) = 0$.
The function class $H_\nu^{(\rho)}$ is now characterized by the following definition

$$H_\nu^{(\rho)} := \left\{ g \in L^\nu[0, 2\pi] : \sup_{\omega\neq 0} \frac{\|g(\cdot, +\omega) - g(\cdot)\|_\nu}{\rho(\omega)} < \infty, \nu \geq 1 \right\} [1].$$

The norm associated with the function class $H_\nu^{(\rho)}$ can be formulated as

$$\|g\|_\nu^{(\rho)} = \|g\|_\nu + \sup_{\omega\neq 0} \frac{\|g(\cdot, +\omega) - g(\cdot)\|_\nu}{\rho(\omega)}; \nu \geq 1.$$

It is observed that $\rho(\omega)$ and $\tau(\omega)$ represent the moduli of smoothness, satisfying the condition that $\frac{\rho(\omega)}{\tau(\omega)}$ is positive, non-decreasing, and

$$\|g\|_{(\nu)}^{(\tau)} \leq \max\left(1, \frac{\rho(2\pi)}{\tau(2\pi)}\right) \|g\|_\nu^{(\rho)} < \infty.$$

It is evident that $H_\nu^{(\rho)}$ is a complete normed linear space. Furthermore, our observation reveals that

$$H_\nu^{(\rho)} \subset H_\nu^{(\tau)} \subset L^\nu\,[0, 2\pi].$$

2.2. Degree of convergence

The degree of convergence of a summation method to a given function g is a measure that how fast σ_γ converges to g, which is given by

$$\|g - \sigma_\gamma\| = O\left(\frac{1}{\varphi_\gamma}\right) [2],$$

where σ_γ is a trigonometric polynomial of degree γ and $\varphi_\gamma \to \infty$ as $\gamma \to \infty$.

2.3. Notations

$$T_\gamma^{N^{p,q}A}(\omega) = \frac{1}{2\pi}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\frac{\cos\left(l+\frac{1}{2}\right)\omega}{\sin(\frac{\omega}{2})};$$

$$\tilde{H}_1' = -\frac{\mu}{2\pi}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\frac{\sin(l+\frac{1}{2})\omega}{\sin\frac{\omega}{2}};$$

$$\tilde{H}_2' = -\frac{1}{4\pi}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\frac{\cos(l\omega)}{\sin^2\frac{\omega}{2}}.$$

3. Preliminary Results

The proof of our main theorems necessitates the use of the following auxiliary results.

Lemma 3.1. *If $\{p_\gamma\}$ and $\{q_\gamma\}$ are monotonic decreasing and monotonic increasing sequences respectively, then*

$$(\gamma+1)p_\gamma q_0 = O(R_\gamma). \tag{6}$$

Proof. The proof of this Lemma is straight forward. □

Lemma 3.2. $|T_\gamma^{N^{p,q}A}(\omega)| = O\left(\frac{1}{\omega}\right)$, *for* $0 < \omega \le \frac{1}{\gamma+1}$.

Proof. When $0 < \omega \le \frac{1}{\gamma+1}$, it is ensured that $\sin\left(\frac{\omega}{2}\right) \ge \frac{\omega}{\pi}$, $|\cos\left(l+\frac{1}{2}\right)\omega| \le 1$.

$$\begin{aligned}
\left|T_\gamma^{N^{p,q}A}(\omega)\right| &\le \frac{1}{2\pi}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\frac{|\cos\left(l+\frac{1}{2}\right)\omega|}{|\sin(\frac{\omega}{2})|}\\
&\le \frac{1}{2\pi}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\frac{1}{\frac{\omega}{\pi}}\\
&\le \frac{1}{2\omega}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\\
&\le \frac{1}{2\omega}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu\\
&= O\left(\frac{1}{\omega}\right).
\end{aligned}$$

□

Lemma 3.3. $|T_\gamma^{N^{p,q}A}(\omega)| = O\left(\frac{1}{\omega^2(\gamma+1)}\right)$, *for* $\frac{1}{\gamma+1} < \omega \le \pi$.

Proof. when $\frac{1}{\gamma+1} < \omega \le \pi$, it is ensured that $\sin(\frac{\omega}{2}) \ge \frac{\omega}{\pi}$.

$$\begin{aligned}
\left|T_\gamma^{N^{p,q}A}(\omega)\right| &\le \frac{1}{2\pi}\frac{1}{R_\gamma}\left|\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\frac{\cos\left(l+\frac{1}{2}\right)\omega}{|\sin(\frac{\omega}{2})|}\right|\\
&\le \frac{1}{2\pi}\frac{1}{R_\gamma}\left|\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\frac{\cos\left(l+\frac{1}{2}\right)\omega}{\frac{\omega}{\pi}}\right|\\
&\le \frac{1}{2\omega}\frac{1}{R_\gamma}\left|\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\cos\left(l+\frac{1}{2}\right)\omega\right|.
\end{aligned}$$

Using Abel's lemma we have,

$$\begin{aligned}
\left|T_\gamma^{N^{p,q}A}(\omega)\right| &\le \frac{1}{2\omega}\frac{1}{R_\gamma}\left[\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \left|\sum_{l=0}^{\mu-1}(a_{\mu,l}-a_{\mu-1,l+1})\sum_{v=0}^{l}\cos\left(v+\frac{1}{2}\right)\omega\right| + a_{\mu,\mu}\sum_{l=0}^{\mu}\cos\left(l+\frac{1}{2}\right)\omega\right]\\
&\le \frac{1}{2\omega}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \left|\sum_{l=0}^{\mu-1}\Delta a_{\mu,l}\sum_{v=0}^{l}\cos\left(v+\frac{1}{2}\right)\omega\right| + a_{\mu,\mu}\left|\sum_{l=0}^{\mu}\cos\left(l+\frac{1}{2}\right)\omega\right|
\end{aligned}$$

$$\leq \frac{1}{2\omega}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma}p_{\gamma-\mu}q_\mu\left[\sum_{l=0}^{\mu-1}|\Delta a_{\mu,l}|+a_{\mu,\mu}\right]\max_{0\leq\upsilon\leq m}\left|\sum_{\upsilon=0}^{m}\cos\left(\upsilon+\frac{1}{2}\right)\omega\right|$$
$$\leq \frac{1}{2\omega}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma}p_{\gamma-\mu}q_\mu\left[O\left(\frac{1}{\mu+1}\right)+O\left(\frac{1}{\mu+1}\right)\right]\cdot\frac{1}{\omega}$$
$$\leq \frac{1}{\omega^2}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma}p_{\gamma-\mu}q_\mu\left(\frac{1}{\mu+1}\right).$$

Further using Abel's lemma, we get

$$\left|T_\gamma^{N^{p,q}A}(\omega)\right| \leq \frac{1}{\omega^2}\frac{1}{R_\gamma}\left|\sum_{\mu=0}^{\gamma-1}(p_{\gamma-\mu}q_\mu-p_{\gamma-\mu-1}q_{\mu+1})\sum_{\upsilon=0}^{\mu}\frac{1}{\mu+1}+p_0q_\gamma\sum_{\mu=0}^{\gamma}\frac{1}{\mu+1}\right|$$
$$\leq \frac{1}{\omega^2}\frac{1}{R_\gamma}\left[\sum_{\mu=0}^{\gamma-1}|(p_{\gamma-\mu}q_\mu-p_{\gamma-\mu-1}q_{\mu+1})|+|p_0q_\gamma|\right]\max_{0\leq\upsilon\leq m}\left|\sum_{\upsilon=0}^{m}\frac{1}{\upsilon+1}\right|$$

Using lemma 3.1, we have

$$\left|T_\gamma^{N^{p,q}A}(\omega)\right| \leq \frac{1}{\omega^2}\frac{1}{R_\gamma}p_\gamma q_0$$
$$= O\left(\frac{1}{\omega^2(\gamma+1)}\right)$$

□

Lemma 3.4. *[13] Let $g\in H_\nu^{(\rho)}$, then for $0<\omega\leq\pi$*

(i) $\|\psi(\cdot,\omega)\|_\nu = O(\rho(\omega))$;

(ii) $\|\psi(\cdot+x,\omega)-\psi(\cdot,\omega)\|_\nu = \begin{cases}O(\rho(\omega)),\\ O(\rho(x));\end{cases}$

(iii) *If $\rho(\omega)$ and $\tau(\omega)$ are modulus of smoothness, then*

$$\|\psi(\cdot+x,\omega)-\psi(\cdot,\omega)\|_\nu = O(\tau(|x|)\left(\tfrac{\rho(\omega)}{\tau(\omega)}\right).$$

Lemma 3.5.

(i) *[22, Lemma 2]* $|\tilde{H}_1'| = O(\gamma+1)$, *for* $0<\omega\leq\frac{1}{\gamma+1}$;

(ii) *[22, Lemma 3]* $|\tilde{H}_1'| = O\left(\frac{1}{\omega^2(\gamma+1)}\right)$, *for* $\frac{1}{\gamma+1}<\omega\leq\pi$.

Lemma 3.6. $|\tilde{H}_2'| = O\left(\frac{1}{\omega^2}\right)$, *for* $0<\omega\leq\frac{1}{\gamma+1}$.

Proof. For $0<\omega\leq\frac{1}{\gamma+1}$, $\sin(\frac{\omega}{2})\geq\frac{\omega}{\pi}$, $|\cos(l\omega)|\leq 1$.

$$|\tilde{H}_2'| \leq \frac{1}{4\pi}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma}p_{\gamma-\mu}q_\mu\sum_{l=0}^{\mu}a_{\mu,l}\frac{|\cos(l\omega)|}{|\sin^2(\frac{\omega}{2})|}$$
$$\leq \frac{1}{4\pi}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma}p_{\gamma-\mu}q_\mu\sum_{l=0}^{\mu}a_{\mu,l}\frac{1}{\frac{\omega^2}{\pi^2}}$$

$$\leq \frac{\pi}{4\omega^2}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}$$
$$\leq \frac{\pi}{4\omega^2}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu$$
$$= O\Big(\frac{1}{\omega^2}\Big).$$

□

Lemma 3.7. $|\tilde{H}_2'| = O\left(\frac{1}{\omega^3(\gamma+1)}\right)$, *for* $\frac{1}{\gamma+1} < \omega \leq \pi$.

Proof. When $\frac{1}{\gamma+1} < \omega \leq \pi$, it ensures that $\sin(\frac{\omega}{2}) \geq \frac{\omega}{\pi}$.

$$|\tilde{H}_2'| \leq \frac{1}{4\pi}\frac{1}{R_\gamma}\left|\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\frac{\cos(l\omega)}{|\sin^2(\frac{\omega}{2})|}\right|$$
$$\leq \frac{1}{4\pi}\frac{1}{R_\gamma}\left|\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\frac{\cos(l\omega)}{\frac{\omega^2}{\pi^2}}\right|$$
$$\leq \frac{\pi}{4\omega^2}\frac{1}{R_\gamma}\left|\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \sum_{l=0}^{\mu} a_{\mu,l}\cos(l\omega)\right|.$$

Using Abel's lemma we have,

$$|\tilde{H}_2'| \leq \frac{\pi}{4\omega^2}\frac{1}{R_\gamma}\left[\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \left|\sum_{l=0}^{\mu-1}(a_{\mu,l}-a_{\mu-1,l+1})\sum_{\upsilon=0}^{l}\cos(\upsilon\omega)\right| + a_{\mu,\mu}\sum_{l=0}^{\mu}\cos(l\omega)\right]$$
$$\leq \frac{\pi}{4\omega^2}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \left|\sum_{l=0}^{\mu-1}\Delta a_{\mu,l}\sum_{\upsilon=0}^{l}\cos(\upsilon\omega)\right| + a_{\mu,\mu}\left|\sum_{l=0}^{\mu}\cos(l\omega)\right|$$
$$\leq \frac{\pi}{4\omega^2}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \left[\sum_{l=0}^{\mu-1}|\Delta a_{\mu,l}| + a_{\mu,\mu}\right]\max_{0\leq\upsilon\leq m}\left|\sum_{\upsilon=0}^{m}\cos(\upsilon\omega)\right|$$
$$\leq \frac{\pi}{4\omega^2}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \left[O\Big(\frac{1}{\mu+1}\Big)+O\Big(\frac{1}{\mu+1}\Big)\right]\cdot\frac{1}{\omega}$$
$$\leq \frac{\pi}{4\omega^3}\frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_\mu \Big(\frac{1}{\mu+1}\Big).$$

Further using Abel's lemma, we get

$$|\tilde{H}_2'| \leq \frac{\pi}{4\omega^3}\frac{1}{R_\gamma}\left|\sum_{\mu=0}^{\gamma-1}(p_{\gamma-\mu}q_\mu - p_{\gamma-\mu-1}q_{\mu+1})\sum_{\upsilon=0}^{\mu}\frac{1}{\mu+1} + p_0q_\gamma\sum_{\mu=0}^{\gamma}\frac{1}{\mu+1}\right|$$
$$\leq \frac{\pi}{4\omega^3}\frac{1}{R_\gamma}\left[\sum_{\mu=0}^{\gamma-1}|(p_{\gamma-\mu}q_\mu - p_{\gamma-\mu-1}q_{\mu+1})| + |p_0q_\gamma|\right]\max_{0\leq\upsilon\leq m}\left|\sum_{\upsilon=0}^{m}\frac{1}{\upsilon+1}\right|$$

Using lemma 3.1, we have

$$|\tilde{H}_2^{'}| \leq \frac{\pi}{4\omega^3}\frac{1}{R_\gamma}p_\gamma q_0 = O\left(\frac{1}{\omega^3(\gamma+1)}\right)$$

□

4. Main Theorems

4.1. Analysis of convergence of the function of conjugate Fourier series

Now, we proceed to establish our fundamental theorem.

Theorem 4.1. *If g is a function of period 2π and Lebesgue integrable, then the degree of convergence of $\tilde{g}$ of conjugate Fourier series in generalized Hölder class using generalized Nörlund-Matrix ($N^{p,q}A$) operator, is given by*

$$\|\tilde{M}_{\gamma,\mu}(\cdot)\|_\nu^{(\tau)} = O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{1}{\omega^2}\Big)d\omega\Big)\Big(\log\Big(\frac{1}{\gamma+1}\Big)+1\Big).$$

Proof. We write

$$\tilde{s}_\gamma(\tilde{g};t)-\tilde{g}(t) = \frac{1}{2\pi}\int_0^\pi \psi_{(t)}(\omega)\frac{\cos(\gamma+\frac{1}{2})\omega}{\sin\frac{\omega}{2}}d\omega.$$

Now,

$$\begin{aligned}\tilde{M}_{\gamma,\mu}(t) &= \sigma_\gamma^{N^{p,q}A} - \tilde{g}(t)\\ &= \frac{1}{R_\gamma}\sum_{\mu=0}^{\gamma}p_{\gamma-\mu}q_\mu\sum_{i=0}^{\mu}a_{\mu,i}(\tilde{s}_\gamma(\tilde{g};t)-\tilde{g}(t))\\ &= \frac{1}{2\pi}\frac{1}{R_\gamma}\int_0^\pi\psi_t(\omega)\sum_{\mu=0}^{\gamma}p_{\gamma-\mu}q_\mu\sum_{l=0}^{\mu}a_{\mu,l}\frac{\cos\left(l+\frac{1}{2}\right)\omega}{\sin(\frac{\omega}{2})}d\omega\\ &= \int_0^\pi\psi_t(\omega)T_\gamma^{N^{p,q}A}(\omega)d\omega. \end{aligned} \tag{7}$$

So,

$$\tilde{M}_{\gamma,\mu}(t+x)-\tilde{M}_{\gamma,\mu}(t) = \int_0^\pi(\psi(t+x,\omega)-\psi(t,\omega))T_\gamma^{N^{p,q}A}(\omega)d\omega.$$

Using generalized Minkowski's inequality [1],

$$\begin{aligned}\|\tilde{M}_{\gamma,\mu}(\cdot+x)-\tilde{M}_{\gamma,\mu}(\cdot)\|_\nu &\leq \int_0^\pi\|\psi(\cdot+x,\omega)-\psi(\cdot,\omega)\|_\nu T_\gamma^{N^{p,q}A}(\omega)d\omega\\ &= \int_0^{\frac{1}{\gamma+1}}\|\psi(\cdot+x,\omega)-\psi(\cdot,\omega)\|_\nu T_\gamma^{N^{p,q}A}(\omega)d\omega\\ &\quad+\int_{\frac{1}{\gamma+1}}^{\pi}\|\psi(\cdot+x,\omega)-\psi(\cdot,\omega)\|_\nu T_\gamma^{N^{p,q}A}(\omega)d\omega\end{aligned}$$

$$= U + V. \tag{8}$$

Using Lemmas 3.2 and 3.4 (iii), we get

$$\begin{aligned}
|U| &= \int_0^{\frac{1}{\gamma+1}} \|\psi(\cdot + x, \omega) - \psi(\cdot, \omega)\|_\nu T_\gamma^{N^{p,q}A}(\omega) d\omega \\
&\leq \left[\tau(|x|) \int_0^{\frac{1}{\gamma+1}} \frac{\rho(\omega)}{\tau(\omega)} \frac{1}{\omega} d\omega\right] \\
&\leq \left[\tau(|x|) \frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})} \lim_{\epsilon \to 0} \int_\epsilon^{\frac{1}{\gamma+1}} \frac{1}{\omega} d\omega\right] \\
&= O\left[\tau(|x|) \frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})} \log\left(\frac{1}{\gamma+1}\right)\right].
\end{aligned} \tag{9}$$

Using Lemmas 3.3 and 3.4 (iii), we obtain

$$\begin{aligned}
|V| &= \int_{\frac{1}{\gamma+1}}^{\pi} \|\psi(\cdot + x, \omega) - \psi(\cdot, \omega)\|_\nu T_\gamma^{N^{p,q}A}(\omega) d\omega \\
&= \left[\tau(|x|) \int_{\frac{1}{\gamma+1}}^{\pi} \frac{\rho(\omega)}{\tau(\omega)} \left(\frac{1}{(\gamma+1)\omega^2}\right) d\omega\right] \\
&= O\left(\frac{1}{\gamma+1} \tau(|x|) \int_{\frac{1}{\gamma+1}}^{\pi} \frac{\rho(\omega)}{\tau(\omega)} \left(\frac{1}{\omega^2}\right) d\omega\right).
\end{aligned} \tag{10}$$

From (8), (9) and (10) we get,

$$\sup_{x \neq 0} \frac{\|\tilde{M}_{\gamma,\mu}(\cdot + x) - \tilde{M}_{\gamma,\mu}(\cdot)\|_\nu}{\tau(|x|)} = O\left[\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})} \log\left(\frac{1}{\gamma+1}\right)\right] + O\left(\frac{1}{\gamma+1} \int_{\frac{1}{\gamma+1}}^{\pi} \frac{\rho(\omega)}{\tau(\omega)} \left(\frac{1}{\omega^2}\right) d\omega\right). \tag{11}$$

By applying generalized Minkowski's inequality [1] again and utilizing Lemma 3.4 (i), we obtain

$$\begin{aligned}
\|\tilde{M}_{\gamma,\mu}(\cdot)\|_\nu &= \|\sigma_\gamma^{N^{p,q}A} - \tilde{g}(t)\|_\nu \\
&\leq \left(\int_0^{\frac{1}{\gamma+1}} + \int_{\frac{1}{\gamma+1}}^{\pi}\right) \|\psi(\cdot, \omega)\|_\nu T_\gamma^{N^{p,q}A}(\omega) d\omega \\
&= O\left(\int_0^{\frac{1}{\gamma+1}} \frac{1}{\omega} \rho(\omega) d\omega\right) + O\left(\int_{\frac{1}{\gamma+1}}^{\pi} \left(\frac{1}{\gamma+1} \left(\frac{1}{\omega^2}\right) \rho(\omega)\right) d\omega\right) \\
&= O\left(\rho\left(\frac{1}{\gamma+1}\right) \log\left(\frac{1}{\gamma+1}\right)\right) + O\left(\frac{1}{\gamma+1} \int_{\frac{1}{\gamma+1}}^{\pi} \rho(\omega) \left(\frac{1}{\omega^2}\right) d\omega\right).
\end{aligned} \tag{12}$$

It is well known that

$$\|\tilde{M}_{\gamma,\mu}(\cdot)\|_\nu^{(\tau)} = \|\tilde{M}_{\gamma,\mu}(\cdot)\|_\nu + \sup_{x \neq 0} \frac{\|\tilde{M}_{\gamma,\mu}(\cdot + x) - \tilde{M}_{\gamma,\mu}(\cdot)\|_\nu}{\tau(|x|)} \tag{13}$$

Using (11), (12) and (13) we get,

$$\|\tilde{M}_{\gamma,\mu}(\cdot)\|_\nu^{(\tau)} = O\left(\rho\left(\frac{1}{\gamma+1}\right) \log\left(\frac{1}{\gamma+1}\right)\right) + O\left(\frac{1}{\gamma+1} \int_{\frac{1}{\gamma+1}}^{\pi} \rho(\omega) \left(\frac{1}{\omega^2}\right) d\omega\right)$$

$$+ O\Big(\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\log\Big(\frac{1}{\gamma+1}\Big)\Big) + O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{1}{\omega^2}\Big)d\omega\Big). \tag{14}$$

Due to monotonicity of $\tau(\omega)$, it follows that

$$\begin{aligned}\rho(\omega) &= \frac{\rho(\omega)}{\tau(\omega)}\tau(\omega)\\ &\leq \frac{\rho(\pi)}{\tau(\pi)}\tau(\pi).\end{aligned}$$

For $0 < \omega \leq \pi$ we get,

$$\|\tilde{M}_{\gamma,\mu}(\cdot)\|_{\nu}^{(\tau)} = O\Big(\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\log\Big(\frac{1}{\gamma+1}\Big)\Big) + O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{1}{\omega^2}\Big)d\omega\Big). \tag{15}$$

Now we have,

$$\begin{aligned}\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{1}{\omega^2}\Big)d\omega &\geq \frac{1}{\gamma+1}\cdot\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big[-\frac{1}{\omega}\Big]_{\frac{1}{\gamma+1}}^{\pi}\\ &= O\Big(\frac{1}{\gamma+1}\cdot\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}(\gamma+1)\Big)\\ &= O\Big(\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big).\end{aligned}$$

Thus,

$$O\Big(\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big) = O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{1}{\omega^2}\Big)d\omega\Big). \tag{16}$$

Now, from (15) and (16)

$$\|\tilde{M}_{\gamma,\mu}(\cdot)\|_{\nu}^{(\tau)} = O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{1}{\omega^2}\Big)d\omega\Big)\Big(\log\Big(\frac{1}{\gamma+1}\Big)+1\Big). \tag{17}$$

□

Corollary 4.2. *Let $\tilde{g} \in H_{(\alpha,\beta),\nu}; \nu \geq 1$ and assume that $\rho(\omega) = \omega^{\alpha}, \tau(\omega) = \omega^{\beta}$ and $0 \leq \beta < \alpha \leq 1$, then*

$$\|\tilde{M}_{\gamma,\mu}(\cdot)\|_{\nu}^{(\tau)} = \begin{cases} O\Big((\gamma+1)^{\alpha-\beta-2}\,(1-\log(\gamma+1))\Big), & \text{if } 0 \leq \beta < \alpha < 1,\\ O\Big[\frac{1}{\gamma+1}\,(\ln(\pi(\gamma+1)))\,(1-\ln(\gamma+1))\Big], & \text{if } \beta = 0, \alpha = 1.\end{cases}$$

4.1.1. Application

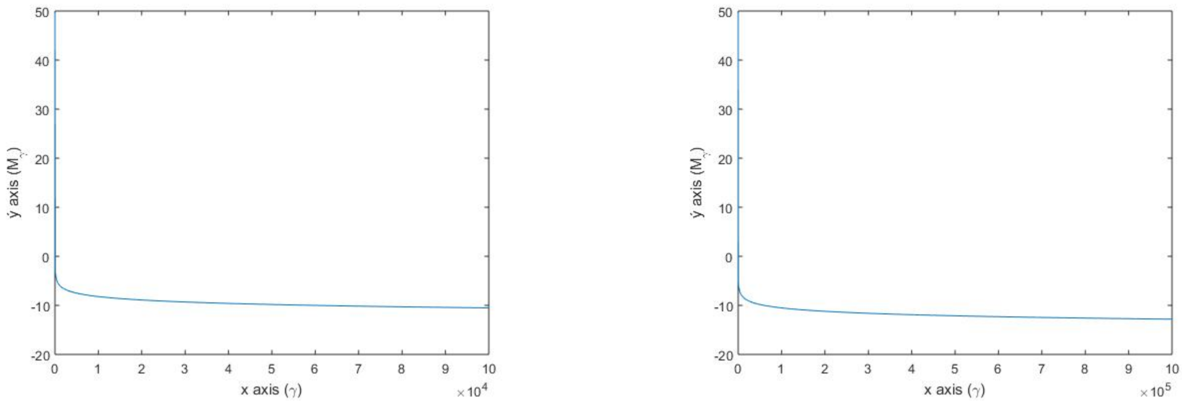
Let us consider $\frac{\rho(\omega)}{\tau(\omega)} = e^{\omega}\omega^3$. Then, from (17), we have

$$\|\tilde{M}_{\gamma,\mu}(\cdot)\|_{\nu}^{(\tau)} = \|\sigma_{\gamma}^{N^{p,q}A}(\tilde{g};t) - \tilde{g}(t)\|_{\nu} = O\Big[\frac{1}{\gamma+1}\Big(e^{\pi}(\pi-1) + \frac{\gamma e^{\frac{1}{\gamma+1}}}{\gamma+1}\Big)(1-\ln(\gamma+1))\Big].$$

Presently, we depict the graphs of $\tilde{M}_{\gamma,\mu}(\cdot)$ for different values of γ:

γ	$\Vert\tilde{M}_{\gamma,\mu}(t)\Vert_\nu = O\Big[\frac{1}{\gamma+1}\Big(e^{\pi}(\pi-1)+\frac{\gamma e^{\frac{1}{\gamma+1}}}{\gamma+1}\Big)(1-\ln(\gamma+1))\Big]$
1000	0.29843601
10000	0.04150614
100000	0.00531507
1000000	0.00064793
.	.
∞	0

Table 1: Values of $\Vert\tilde{M}_{\gamma,\mu}(\cdot)\Vert$ for different γ.



(a) For γ=100000 (b) For γ=1000000

Figure 1: Graphs of $\Vert\tilde{M}'_{\gamma,\mu}(\cdot)\Vert$ for different γ.

4.2. *Analysis of convergence of the function of derived conjugate Fourier series*

Now, we proceed to establish another fundamental theorem.

Theorem 4.3. *If g is a function of period 2π and Lebesgue integrable, then the degree of convergence of $\tilde{g}'$ of derived conjugate Fourier series in generalized Hölder class using generalized Nörlund-Matrix ($N^{p,q}A$) operator, is given by*

$$\Vert\tilde{M}'_{\gamma,\mu}(\cdot)\Vert_\nu^{(\tau)} = O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big).$$

Proof. According to [15], the integral representation of $\tilde{s}'_\gamma(\tilde{g}'\,;t)$ is provided as follows:

$$\begin{aligned}\tilde{s}'_\gamma(\tilde{g}'\,;t) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(k)\frac{\partial}{\partial k}\Big(\sum_{r=1}^{\mu}\sin r(k-t)\Big)dk\\ &= -\frac{1}{\pi}\int_0^{\pi}\frac{d}{d\omega}\Big(\frac{\cos(\frac{\omega}{2})-\cos(\mu+\frac{1}{2})\omega}{2\sin(\frac{\omega}{2})}\Big)(g(t+\omega)+g(t-\omega))d\omega\\ &= -\frac{1}{\pi}\int_0^{\pi}\frac{d}{d\omega}\Big(\frac{\cot(\frac{\omega}{2})}{2}\Big)\psi_t(\omega)d\omega+\frac{1}{\pi}\int_0^{\pi}\frac{d}{d\omega}\Big(\frac{\cos(\mu+\frac{1}{2})\omega}{2\sin(\frac{\omega}{2})}\Big)\psi_t(\omega)d\omega\\ &= \frac{1}{4\pi}\int_0^{\pi}\csc^2(\frac{\omega}{2})\psi_t(\omega)d\omega-\frac{2}{\pi}\int_0^{\pi}\frac{\psi_t(\omega)}{4\sin(\frac{\omega}{2})}\Big(\mu+\frac{1}{2}\Big)\sin\Big(\mu+\frac{1}{2}\Big)\omega d\omega\end{aligned}$$

$$- \frac{1}{\pi} \int_0^{\pi} \frac{\psi_t(\omega)}{4\sin(\frac{\omega}{2})} \frac{\cos(\mu + \frac{1}{2})\omega}{\tan(\frac{\omega}{2})} d\omega.$$

$$\begin{aligned}\tilde{s}_{\gamma}^{'}(\tilde{g}^{'};t) - \frac{1}{4\pi}\int_0^{\pi} \csc^2\left(\frac{\omega}{2}\right)\psi_t(\omega)d\omega = &-\frac{2}{\pi}\left(\mu + \frac{1}{2}\right)\int_0^{\pi} \frac{\psi_{(t)}(\omega)}{4\sin\frac{\omega}{2}} \sin\left(\mu + \frac{1}{2}\right)\omega d\omega \\ &- \frac{1}{\pi}\int_0^{\pi} \frac{\psi_{(t)}(\omega)}{4\sin\frac{\omega}{2}} \frac{\cos(\mu + \frac{1}{2})\omega\cos(\frac{\omega}{2})}{\sin\frac{\omega}{2}} d\omega\end{aligned}$$

$$\tilde{s}_{\gamma}^{'}(\tilde{g}^{'};t) - \tilde{g}^{'}(t) = -\frac{2\mu}{\pi}\int_0^{\pi} \frac{\psi_t(\omega)}{4\sin\frac{\omega}{2}} \sin\left(\mu + \frac{1}{2}\right)\omega d\omega - \frac{1}{\pi}\int_0^{\pi} \frac{\psi_t(\omega)}{4\sin\frac{\omega}{2}} \frac{\cos(\mu\omega)}{\sin\frac{\omega}{2}} d\omega.$$

Now,

$$\begin{aligned}\tilde{M}_{\gamma,\mu}^{'}(t) &= \sigma_{\gamma}^{N^{pq}A}(\tilde{g}^{'};t) - \tilde{g}^{'}(t) \\ &= \frac{1}{R_{\gamma}}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_{\mu}\sum_{l=0}^{\mu} a_{\mu,l}(\tilde{s}_{\gamma}^{'}(\tilde{g}^{'};t) - \tilde{g}^{'}(t)) \\ &= -\frac{2\mu}{\pi}\frac{1}{R_{\gamma}}\int_0^{\pi} \frac{\psi_t(\omega)}{4\sin\frac{\omega}{2}}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_{\mu}\sum_{l=0}^{\mu} a_{\mu,l}\sin(l + \frac{1}{2})\omega d\omega \\ &\quad - \frac{1}{\pi}\frac{1}{R_{\gamma}}\int_0^{\pi} \frac{\psi_t(\omega)}{4\sin\frac{\omega}{2}}\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_{\mu}\sum_{l=0}^{\mu} a_{\mu,l}\frac{\cos(l\omega)}{\sin\frac{\omega}{2}} d\omega \\ &= -\frac{\mu}{2\pi}\frac{1}{R_{\gamma}}\int_0^{\pi} \psi_t(\omega)\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_{\mu}\sum_{l=0}^{\mu} a_{\mu,l}\frac{\sin(l + \frac{1}{2})\omega}{\sin\frac{\omega}{2}} d\omega \\ &\quad - \frac{1}{4\pi}\frac{1}{R_{\gamma}}\int_0^{\pi} \psi_t(\omega)\sum_{\mu=0}^{\gamma} p_{\gamma-\mu}q_{\mu}\sum_{l=0}^{\mu} a_{\mu,l}\frac{\cos(l\omega)}{\sin^2\frac{\omega}{2}} d\omega \\ &= \int_0^{\pi}\left(\psi_t(\omega)\tilde{H}_1^{'}d\omega\right) + \int_0^{\pi}\left(\psi_t(\omega)\tilde{H}_2^{'}d\omega\right) \\ &= \int_0^{\pi} \psi_t(\omega)\left(\tilde{H}_1^{'} + \tilde{H}_2^{'}\right)d\omega. \end{aligned} \tag{18}$$

So,

$$\tilde{M}_{\gamma,\mu}^{'}(t+x) - \tilde{M}_{\gamma,\mu}^{'}(t) = \int_0^{\pi} (\psi(t+x,\omega) - \psi(t,\omega))(\tilde{H}_1' + \tilde{H}_2')d\omega.$$

Using generalized Minkowski's inequality [1], we have

$$\begin{aligned}\|\tilde{M}_{\gamma,\mu}^{'}(\cdot + x) - \tilde{M}_{\gamma,\mu}^{'}(\cdot)\|_{\nu} &\leq \int_0^{\pi} \|\psi(\cdot + x,\omega) - \psi(\cdot,\omega)\|_{\nu}(\tilde{H}_1' + \tilde{H}_2')d\omega \\ &= \int_0^{\frac{1}{\gamma+1}} \|\psi(\cdot + x,\omega) - \psi(\cdot,\omega)\|_{\nu}(\tilde{H}_1' + \tilde{H}_2')d\omega \\ &\quad + \int_{\frac{1}{\gamma+1}}^{\pi} \|\psi(\cdot + x,\omega) - \psi(\cdot,\omega)\|_{\nu}(\tilde{H}_1' + \tilde{H}_2')d\omega \\ &= Y + Z. \end{aligned} \tag{19}$$

Using Lemmas 3.4(iii), 3.5(i) and 3.6, we get

$$\begin{aligned}
|Y| &= \int_0^{\frac{1}{\gamma+1}} \|\psi(\cdot+x,\omega)-\psi(\cdot,\omega)\|_\nu(\tilde{H}_1'+\tilde{H}_2')d\omega\\
&\le \Big[\tau(|x|)\int_0^{\frac{1}{\gamma+1}} \frac{\rho(\omega)}{\tau(\omega)}\Big(\gamma+1+\frac{1}{\omega^2}\Big)d\omega\Big]\\
&= \Big[\tau(|x|)\lim_{\epsilon\to 0}\int_\epsilon^{\frac{1}{\gamma+1}} \frac{\rho(\omega)}{\tau(\omega)}\Big(\gamma+1+\frac{1}{\omega^2}\Big)d\omega\Big]\\
&= \Big[\tau(|x|)\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\lim_{\epsilon\to 0}\Big(\gamma\omega+\omega-\frac{1}{\omega}\Big)_\epsilon^{\frac{1}{\gamma+1}}\Big]\\
&= \Big[\tau(|x|)\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big((\gamma+1)\Big(\frac{1}{\gamma+1}\Big)-(\gamma+1)\Big)\Big]\\
&= O\Big(\gamma\ \tau(|x|)\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big). \qquad (20)
\end{aligned}$$

Using Lemmas 3.4(iii), 3.5(ii) and 3.7, we get

$$\begin{aligned}
|Z| &= \int_0^{\frac{1}{\gamma+1}} \|\psi(\cdot+x,\omega)-\psi(\cdot,\omega)\|_\nu(\tilde{H}_1'+\tilde{H}_2')d\omega\\
&= \Big[\tau(|x|)\int_{\frac{1}{\gamma+1}}^{\pi} \frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{1}{(\gamma+1)\omega^2}+\frac{1}{(\gamma+1)\omega^3}\Big)d\omega\Big]\\
&= O\Big(\frac{1}{\gamma+1}\tau(|x|)\int_{\frac{1}{\gamma+1}}^{\pi} \frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big). \qquad (21)
\end{aligned}$$

From (19), (20) and (21) we get,

$$\sup_{x\neq 0}\frac{\|\tilde{M}_{\gamma,\mu}'(\cdot+x)-\tilde{M}_{\gamma,\mu}'(\cdot)\|_\nu}{\tau(|x|)} = O\Big(\gamma\ \frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big)+O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big). \qquad (22)$$

Applying generalized Minkowski's inequality [1] once again and using Lemma 3.4(i), we get

$$\begin{aligned}
\|\tilde{M}_{\gamma,\mu}'(\cdot)\|_\nu &= \|\sigma_\gamma^{N^{p,q}A}(\tilde{g}';t)-\tilde{g}'(t)\|_\nu\\
&\le \Big(\int_0^{\frac{1}{\gamma+1}}+\int_{\frac{1}{\gamma+1}}^{\pi}\Big)\|\psi(\cdot,l)\|_\nu(\tilde{H}_1{}'+\tilde{H}_2{}')d\omega\\
&= O\Big(\int_0^{\frac{1}{\gamma+1}}\Big((\gamma+1)+\frac{1}{\omega^2}\Big)\rho(\omega)d\omega\Big)+O\Big(\int_{\frac{1}{\gamma+1}}^{\pi}\Big(\frac{1}{\gamma+1}\Big(\frac{\omega+1}{\omega^3}\Big)\rho(\omega)\Big)d\omega\Big)\\
&= O\Big[\gamma\ \rho\Big(\frac{1}{\gamma+1}\Big)\Big]+O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\rho(\omega)\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big). \qquad (23)
\end{aligned}$$

We know that

$$\|\tilde{M}_{\gamma,\mu}'(\cdot)\|_\nu^{(\tau)} = \|\tilde{M}_{\gamma,\mu}'(\cdot)\|_\nu+\sup_{x\neq 0}\frac{\|\tilde{M}_{\gamma,\mu}'(\cdot+x)-\tilde{M}_{\gamma,\mu}'(\cdot)\|_\nu}{\tau(|x|)} \qquad (24)$$

Using (22), (23) and (24) we get,

$$\begin{aligned}\|\tilde{M}'_{\gamma,\mu}(\cdot)\|_{\nu}^{(\tau)} &= O\Big[\gamma\,\rho\Big(\frac{1}{\gamma+1}\Big)\Big] + O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\rho(\omega)\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big)\\ &\quad + O\Big[\gamma\,\tau(\frac{1}{\gamma+1})\Big] + O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big). \qquad (25)\end{aligned}$$

Due to monotonicity of $\tau(\omega)$, it follows that

$$\begin{aligned}\rho(\omega) &= \frac{\rho(\omega)}{\tau(\omega)}\tau(\omega)\\ &\le \frac{\rho(\pi)}{\tau(\pi)}\tau(\pi).\end{aligned}$$

For $0 < \omega \le \pi$, we get

$$\|\tilde{M}'_{\gamma,\mu}(\cdot)\|_{\nu}^{(\tau)} = O\Big(\gamma\,\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big) + O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big). \qquad (26)$$

Now,

$$\begin{aligned}\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega &\ge \frac{1}{\gamma+1}\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\int_{\frac{1}{\gamma+1}}^{\pi}\Big(\frac{1}{\omega^2}+\frac{1}{\omega^3}\Big)d\omega\\ &= \frac{1}{\gamma+1}\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big[-\frac{1}{\omega}-\frac{1}{2\omega^2}\Big]_{\frac{1}{\gamma+1}}^{\pi}\\ &= \Big(\frac{\gamma+3}{2}\cdot\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big)\\ &\ge O\Big(\frac{\gamma}{2}\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big).\end{aligned}$$

Thus,

$$O\Big(\frac{\rho(\frac{1}{\gamma+1})}{\tau(\frac{1}{\gamma+1})}\Big) = O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big). \qquad (27)$$

Now, from (26) and (27), we get

$$\begin{aligned}\|\tilde{M}'_{\gamma,\mu}(\cdot)\|_{\nu}^{(\tau)} &= O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big) + O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big)\\ &= O\Big(\frac{1}{\gamma+1}\int_{\frac{1}{\gamma+1}}^{\pi}\frac{\rho(\omega)}{\tau(\omega)}\Big(\frac{\omega+1}{\omega^3}\Big)d\omega\Big). \qquad (28)\end{aligned}$$

□

Corollary 4.4. *Let $\tilde{g}' \in H_{(\alpha,\beta),\nu};\nu \ge 1$ and assume that $\rho(\omega)=\omega^{\alpha}, \tau(\omega)=\omega^{\beta}$ and $0 \le \beta < \alpha \le 1$, then*

$$\|\tilde{M}'_{\gamma,\mu}(\cdot)\|_{\nu}^{(\tau)} = \begin{cases} O\Big((\gamma+1)^{\alpha-\beta-2}, & \text{if } 0 \le \beta < \alpha < 1,\\ O\Big(\frac{1}{\gamma+1}(\ln(\pi(\gamma+1)) + (\gamma+1))\Big), & \text{if } \beta = 0, \alpha = 1.\end{cases}$$

4.2.1. Application

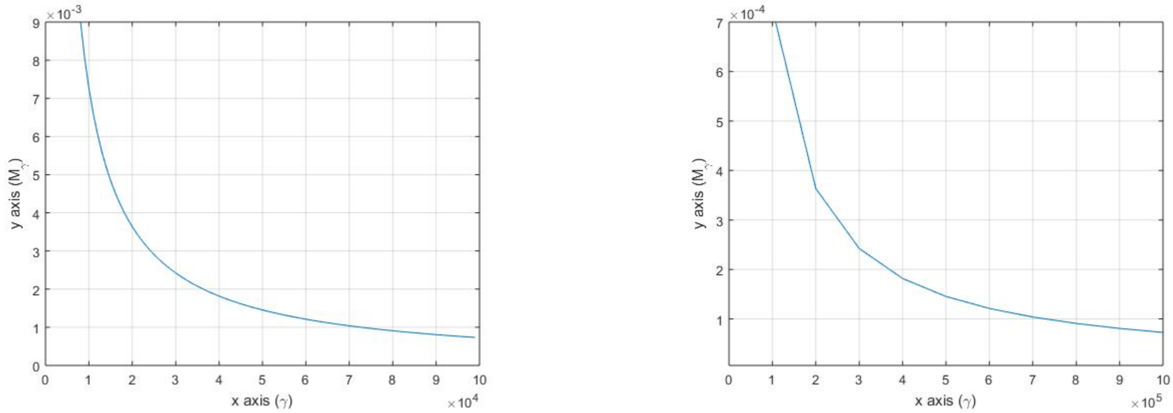
Let us consider $\frac{\rho(\omega)}{\tau(\omega)} = e^{\omega}\omega^3$. Then, from (28), we have

$$\|\tilde{M}'_{\gamma,\mu}(\cdot)\|_{\nu}^{(\tau)} = \|\sigma_{\gamma}^{N^{p,q}A}(\tilde{g}';t) - \tilde{g}'(t)\|_{\nu} = O\left(\frac{1}{\gamma+1}\left(\pi e^{\pi} - \frac{e^{\frac{1}{\gamma+1}}}{\gamma+1}\right)\right).$$

γ	$\|\tilde{M}'_{\gamma,\mu}(t)\|_{\nu} = O\left(\frac{1}{\gamma+1}\left(\pi e^{\pi} - \frac{e^{\frac{1}{\gamma+1}}}{\gamma+1}\right)\right)$
10000	0.00726
100000	0.00072
1000000	0.00007
.	.
∞	0

Table 2: Values of $\|\tilde{M}_{\gamma,\mu}(\cdot)\|$ for different γ.

Presently, we depict the graphs of $\tilde{M}'_{\gamma,\mu}(\cdot)$ for different values of γ:



(a) For γ=100000 (b) For γ=1000000

Figure 2: Graphs of $\|\tilde{M}'_{\gamma,\mu}(\cdot)\|$ for different γ.

5. Conclusion

After examining the convergence results in all the applications, it is clear that both the norms provide excellent approximations. We see that as the value of γ increases, the convergence rate of both $\tilde{g}$ and $\tilde{g}'$ intensifies. After a thorough analysis of the convergence results, it is unequivocally evident that both norms excel in offering remarkably best approximations which approaches to zero.

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