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Du-subsets of ordered Banach spaces and their best approximation properties with applications

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Abstract. The main goal of this paper is to investigate the best approximation of sets that do not correspond to proximal sets. We call these new sets "Du-sets". "D" and "u" refer to the relationship that these sets can have with downward and upward. The closed Du-sets are not necessarily convex, upward, downward, or star-shaped, but are nevertheless proximal. We will also achieve some useful and practical results.

1. Introduction

Mathematicians have developed a well-developed theory of best approximation based on elements of convex and reverse convex sets (complements of convex sets). Despite this, convexity can be a very restrictive assumption, so it is vital to conduct the study of the best approximation by looking at sets that aren't necessarily convex. A theoretical solution to the problem of best approximating downward subsets of the spaces $\mathbb{R}^I$ has been developed by Martinez-Legaz, Rubinov, and Singer in [4]. These properties were studied by Mohebi and other researchers for the Banach lattice space (for example see [8]). In [3], authors defined the new set of sets called I_m-quasi-upward sets and discussed how these sets are connected to upwards or downwards sets.

In this article we develop the theory of best approximation by subsets of ordered Banach spaces which are not necessarily convex, nor is it necessarily upward or downward, nor is it necessarily star-shaped, but it is proximal. We use the results obtained in downward and upward sets as a tool for finding the best approximation of a point to this new set.

The structure of the paper is as follows: In section 2, we recall the main definitions and some results on best approximation by elements of downward and upward sets. In section 3, we define the new sets that we call Du-sets and discuss the connection between Du-sets with downward hull and upward hull sets. In particular, we present the characterizations of best approximation by Du-sets in terms of separation from outside points.

2. Preliminaries

This section presents the first hypotheses and results.

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Definition 2.1. *Let X be a normed vector space. For a non-empty subset W of X and $x \in X$, define*

$$d(x, W) = \inf_{w \in W} \|x - w\|.$$

An element $w_0 \in W$ is called a best approximation for $x \in X$ if

$$\|x - w_0\| = d(x, W).$$

The set of all best approximation to x from W will be denoted by $P_W(x)$. In other words, $P_W(x) = \{w \in W : \|x - w\| = d(x, W)\}$.

It is well-known that $P_W(x)$ is a closed and bounded subset of X. If $x \notin X$, then $P_W(x)$ is located in the boundary of W.
Let X be a vector space. Assume that X is equipped with a convex and pointed cone $K \subset X$. (The latter means that $K \cap (-K) = \{0\}$.) Assume that the algebraic interior $intK$ of K is non-empty, that is, there exists an element $\mathbf{1} \in K$ such that for each $x \in X$ there exists $\varepsilon > 0$ with the property $\mathbf{1} + \alpha x \in K$ for all α with $|\alpha| < \varepsilon$. The cone K generates the order relation $\leq$ on X. By definition $x \leq y$ if and only if $y - x \in K$. We say that y is greater than x and write $y > x$ if $y - x \in K \backslash \{0\}$.
let $\mathbf{1} \in intK$. Using $\mathbf{1}$ we can define the following function:

$$P(x) = \inf\{\lambda \in \mathbb{R} : x \leq \lambda \mathbf{1}\}, \qquad (x \in X). \tag{1}$$

Since $\mathbf{1} \in intK$, it follows that P is finite (for more details, see [6]). It is easy to check that P is increasing, and

$$x \leq P(x)\mathbf{1} \qquad (x \in X). \tag{2}$$

Now, consider the function

$$\|x\| := \max(P(x), P(-x)), \quad x \in X. \tag{3}$$

It is easy to establish that $\|.\|$ is a norm on X. It follows from (2) that

$$x \leq \|x\|\mathbf{1}, \qquad -x \leq \|x\|\mathbf{1}, \qquad (x \in X). \tag{4}$$

Then, by (2) and (3) we have

$$B(t, r) := \{x \in X : \|x - t\| \leq r\} = \{x \in X : t - r\mathbf{1} \leq x \leq t + r\mathbf{1}\}. \tag{5}$$

We assume that X is equipped with the norm $\|.\|$ and the cone K is closed in the normed space X.
To continue, we shall use the functions φ^+ and φ_- defined on X by:

$$\varphi^+(x, y) = -P(-(x + y)) = \sup\{\lambda \in \mathbb{R} : \lambda \mathbf{1} \leq x + y\} \qquad x, y \in X, \tag{6}$$

and

$$\varphi_-(x, y) = P(x + y) = \inf\{\lambda \in \mathbb{R} : x + y \leq \lambda \mathbf{1}\} \qquad x, y \in X. \tag{7}$$

Since $\mathbf{1} \in intK$, it follows that the set $\{\lambda \in \mathbb{R} : \lambda \mathbf{1} \leq x + y\}$ is non-empty and bounded from above (by the number $\|x + y\|$). This set is closed. The following properties of φ^+ are evident from its definition:

$$-\infty < \varphi^+(x, y) \leq \|x + y\| \qquad for\ each\ x, y \in X, \tag{8}$$

$$\varphi^+(x, y)\mathbf{1} \leq x + y \qquad for\ all\ x, y \in X, \tag{9}$$

$$\varphi^+(x, y) = \varphi^+(y, x) \qquad for\ all\ x, y \in X, \tag{10}$$

$$\varphi^+(x,-x) = \sup\{\lambda \in \mathbb{R} : \lambda\mathbf{1} \le x - x = 0\} = 0 \qquad for\ all\ x \in X \tag{11}$$

Clearly

$$\varphi^+(x,y) = -\varphi_-(-x,-y) \quad and \quad \varphi_-(x,y) = -\varphi^+(-x,-y), \tag{12}$$

so all properties of φ_- can be derived from the corresponding properties of φ^+. Recall that a function $f : X \to \mathbb{R}$ is called topical if it is increasing ($x \ge y \Rightarrow f(x) \ge f(y)$) and plus-homogeneous ($f(x+\alpha\mathbf{1}) = f(x)+\alpha$ for all $x \in X$ and all $\alpha \in \mathbb{R}$).

Lemma 2.2. *[7] The function $\varphi^+(.,y)$ is topical.*

Proposition 2.3. *[7] The function $\varphi^+(.,y)$ is Lipschitz continuous.*

Recall that a set $D \subset X$ is called downward if $(w \in D, x \le w) \Rightarrow x \in D$, also, a set $U \subset X$ is called upward if $(u \in U, x \ge u) \Rightarrow x \in U$.

Theorem 2.4 (Theorem 3.1, [7]). *For a subset D of X, the following statements are equivalent:*
(1) D is downward, (resp. is upward).
(2) For each $x \in X\backslash D$ we have

$$\varphi^+(w,-x) < 0 \qquad (w \in D).$$

(resp. $\varphi_-(w,-x) > 0 \qquad (w \in D)$).
(3) For each $x \in X\backslash D$ there exists $l \in X$ such that

$$\varphi^+(w,l) < 0 \le \varphi^+(x,l) \qquad (w \in D).$$

(resp. $\varphi_-(x,l) < 0 \le \varphi_-(w,l) \qquad (w \in D)$).

Proposition 2.5. *[6] Let D be a downward subset of X, (resp. $U \subset X$ be an upward set) and $x \in X$. Then the following assertions are true:*
(1) If $x \in D$ (resp. $x \in U$), then $x - \varepsilon\mathbf{1} \in intD$ (resp. $x + \varepsilon\mathbf{1} \in intU$) for all $\varepsilon > 0$.
(2) We have

$$intD = \{x \in X : \ x + \varepsilon\mathbf{1} \in D\ for\ some\ \varepsilon > 0\}$$

(resp. $intU = \{x \in X : \ x - \varepsilon\mathbf{1} \in U\ for\ some\ \varepsilon > 0\}$)

Theorem 2.6. *[5] Let φ be the function defined by (6). Then for a function $f : X \to \mathbb{R}$ the following assertion are equivalent:*
(1) f is a topical function.
(2) For each $y \in X$ there exists $l_y \in X$ such that

$$\varphi_{l_y}(x) \le f(x) \qquad \forall x \in X, \quad and \quad \varphi_{l_y}(y) = f(y).$$

(3) f is X_φ-convex, where $X_\varphi = \{\varphi_l := \varphi(.,l) : \ l \in X\}$.

3. Best approximations of Du-sets with their separation properties

In this paper, we consider X to be an ordered Banach space. Our first step is to introduce the new Du-set, and then we will describe its best approximation. For any subset W of X, we denote by $intW$, clW, bdW, bd^uW, bd_lW, W_* and W^* the interior, closure, boundary, upper boundary (the boundary of W_*), lower boundary (the boundary of W^*), downward hull and upward hull of W, respectively. In this paper, we assume that W, W_*, W^* are closed sets. Let W be a subset of X and $a, b \in X$ such that $a < b$. Set $[a,b]_W := \{x \in W \mid a \le x \le b\}$.

Definition 3.1. *A set $W \subset X$ is called a Du-set if*
(i) $[a,b]_X = [a,b]_W$, for each $a, b \in W$ where $a < b$.
(ii) $[a,b]_W \ne \emptyset$, for $a \in W_*$ and $b \in W^*$ where $a < b$.

In the above definition, every downward or upward subset of X is a Du-set. Also a set $W \subset X$ is a Du-set if for each $y_1, y_2 \in W$ and each y where $y_1 \leq y \leq y_2$, we get $y \in W$. Let $I = \{1, ..., n\}$ and $\mathbb{R} = (-\infty, +\infty)$, the real line. Denote by $\mathbb{R}^I$ the space of all vectors $(x_i)_i \in I$, endowed with the max-norm and the coordinatewise order relation. We shall use the following notation:
If $x, y \in \mathbb{R}^I$, then $x \geq y \Leftrightarrow x_i \geq y_i$ for all $i \in I$.
If $x, y \in \mathbb{R}^I$, then $x \gg y \Leftrightarrow x_i > y_i$ for all $i \in I$.
$\mathbb{R}^I_+ = \{x = (x_i)_i \in I \in \mathbb{R}^I : \ x_i \geq 0 \ for \ all \ i \in I\}$.
$\mathbb{R}^I_{++} = \{x = (x_i)_i \in I \in \mathbb{R}^I : \ x_i > 0 \ for \ all \ i \in I\}$.
$\mathbf{1} = (1, ..., 1)$.

Three examples of Du-sets are shown below. The Du-sets shown in figures 1 and 2 are neither convex nor star-shaped.

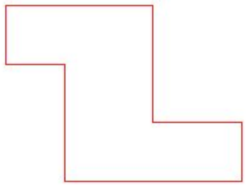

Figure 1: $W \subseteq \mathbb{R}^2$

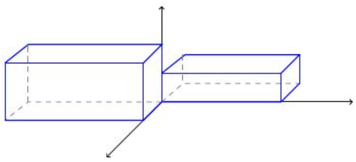

Figure 2: $W \subseteq \mathbb{R}^3$

In figure 3, w_2 is an upper boundary point, w_1, w_4 are lower boundary points, and w_3 is an interior point.

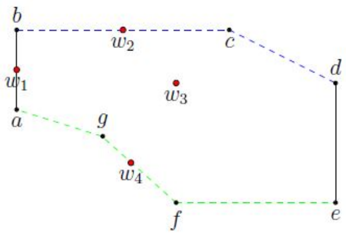

Figure 3: $W \subseteq \mathbb{R}^2$

Proposition 3.2. *Let W be a Du-subset of X then $W = W_* \cap W^*$.*

Proof. Clearly $W \subset W_* \cap W^*$. We show that $W_* \cap W^* \subset W$. Let $x \in W_* \cap W^*$, we have $a, b \in W$ such that $a \leq x \leq b$. So, $x \in [a, b]_X$. Thus $x \in [a, b]_W$, since W is a Du-set. Therefore, $x \in W$. □

Corollary 3.3. *Each Du-set is the intersection of a downward and an upward set.*

Proof. Because of Proposition 3.2, this is a trivial matter. □

With an example, we demonstrate that the inverse of the proposition 3.2 is not necessarily true.

Example 3.4. *Let* $\geq$ *be an order defined in the space* $\mathbb{R}^2$ *by* $x \geq y \Leftrightarrow x_i \geq y_i$ *for all* $i = 1, 2$. *Then if* $W_1 = \{w \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 1 \leq x_2 \leq 2\}$, $W_2 = \{w \in \mathbb{R}^2 : 3 \leq x_1 \leq 4,\ -1 \leq x_2 \leq 0\}$ *and* $W = W_1 \cup W_2$. *We have* $W = W_* \cap W^*$, *but for* $a = (2, 0) \in W_*$ *and* $b = (2, 1) \in W^*$, *we obtained* $[a, b]_W = \emptyset$.

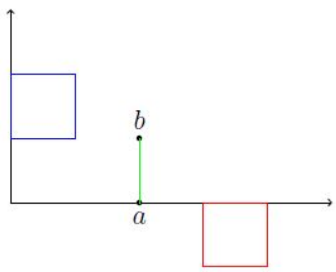

Figure 4: $W \subseteq \mathbb{R}^2$

Proposition 3.5. *Let* W *be a Du-subset of* X *and* $x \in X$. *Then*

$$intW = \{x \in W : x \pm \varepsilon\mathbf{1} \in W \ for\ some\ \varepsilon > 0\}. \tag{13}$$

Proof. Let $x \in intW$. Then there exists $\varepsilon > 0$ such that the closed ball $B(x, \varepsilon) \subset W$. So, by (5) $x \pm \varepsilon\mathbf{1} \in W$. Conversely, suppose that $x \in W$ and for a given $\varepsilon > 0$

$$x \pm \varepsilon\mathbf{1} \in W.$$

As W is a Du-set by Proposition 3.2, $W = W_* \cap W^*$. Since, $x + \varepsilon\mathbf{1} \in W_*$ and W_* is a downward set, by proposition 2.5, we have

$$x = x + \varepsilon\mathbf{1} - \varepsilon\mathbf{1} \in intW_*.$$

Similarly, since $x - \varepsilon\mathbf{1} \in W^*$ and W^* is an upward set, by proposition 2.5,

$$x = x - \varepsilon\mathbf{1} + \varepsilon\mathbf{1} \in intW^*.$$

Therefore, $x \in intW$. □

The proof of the following corollary is trivial.

Corollary 3.6. *Let* W *be a Du-subset of* X *and* $w \in \bar{W}$. *Then the following assertions are true:*
(1) $w \in bd^u W \ iff \ w + \lambda\mathbf{1} \notin W \quad \forall\lambda > 0$,
(2) $w \in bd_l W \ iff \ w - \lambda\mathbf{1} \notin W \quad \forall\lambda > 0$.

Lemma 3.7. *Let* W *be a Du-subset of* X, *and* φ^+, φ_- *be the function defined by (6) and (7). Then the following assertions are true:*
(1)If $y \in bd^u W$ *then* $\varphi^+(w, -y) \leq 0$ *for all* $w \in W$.
(2)If $y \in bd_l W$ *then* $\varphi_-(w, -y) \geq 0$ *for all* $w \in W$.

Proof. (1) Assume that $y \in bd^u W$ and there exists $w_0 \in W$ such that $\varphi^+(w_0, -y) > 0$. Then we have $\sup\{\lambda \in \mathbb{R};\ \lambda\mathbf{1} \le w_0 - y\} > 0$. So there exists $\lambda_0 > 0$ such that $\lambda_0\mathbf{1} \le w_0 - y$. This means that $y < \lambda_0\mathbf{1} + y \le w_0$. As $y \in bd^u W$ and $w_0 \in W$, this make a contradiction by part (1) of Corollary 3.6, which completes the proof of this part.
(2) Assume $y \in bd_l W$ and there exists $w_0 \in W$ s.t. $\varphi_-(w_0, -y) < 0$. Then $\inf\{\lambda \in \mathbb{R};\ w_0 - y \le \lambda\mathbf{1}\} < 0$. So there exists $\lambda_0' < 0$ such that $w_0 - y \le \lambda_0'\mathbf{1}$. This means that $\lambda_0'\mathbf{1} + y \ge w_0$. Therefore, $w_0 \le y + \lambda_0'\mathbf{1} \le y$. Since W is a Du-set, we obtained $y + \lambda_0'\mathbf{1} \in W$. Which contradicts part (2) of Corollary 3.3. This completes the proof. □

Lemma 3.8. *Let W be a closed Du-subset of X, $w_0 \in W$ and $l = -w_0$. Then if φ^+, φ_- be the functions defined by (6) and (7), the following assertions are true:*
(1)If $w_0 \in bd^u W$ then $\varphi^+(w, l) \le 0 = \varphi^+(w_0, l)$ for all $w \in W$.
(2)If $w_0 \in bd_l W$ then $\varphi_-(w_0, l) = 0 \le \varphi_-(w, l)$ for all $w \in W$.

Proof. (1) Since $w_0 \in bd^u W$, it follows, by Lemma 3.7 that

$$\varphi^+(w, l) = \varphi^+(w, -w_0) \le 0, \quad \forall w \in W.$$

Also, we have

$$\varphi^+(w_0, l) = \varphi^+(w_0, -w_0) = \sup\{\lambda \in \mathbb{R};\ \lambda\mathbf{1} \le w_0 - w_0\} = 0.$$

(2) In a similar fashion, we observe

$$\varphi_-(w_0, l) = 0 \le \varphi_-(w, l) \quad for\ all\ w \in W.$$

□

Theorem 3.9. *Let W be a closed Du-set, $x \in X$ and $r = dist(x, W)$. Then*
(i) W is proximal and $dist(x, W) = \max\{dist(x, W_*), dist(x, W^*)\}$.
(ii) $P_W(x) = P_{W_*}(P_W(x)) = P_{W^*}(P_W(x))$.

Proof. Since $W \subseteq W_*$ and $W \subseteq W^*$, we obtained

$$dist(x, W) \ge \max\{dist(x, W_*), dist(x, W^*)\}. \tag{14}$$

Let $r' = \max\{dist(x, W_*), dist(x, W^*)\}$, we have $x - r'\mathbf{1} \in W_*$, $x + r'\mathbf{1} \in W^*$ and $x - r'\mathbf{1} \le x + r'\mathbf{1}$. As W is a Du-set, there exists $w \in W$ such that $x - r'\mathbf{1} \le w \le x + r'\mathbf{1}$. So, $\|x - w\| \le r'$. Since $r = dist(x, W)$, we obtained $r \le r'$. By (14) and $r \le r'$, we have $dist(x, W) = \max\{dist(x, W_*), dist(x, W^*)\}$.
(ii) There is no significance to it since $W \subseteq W_*$ and $W \subseteq W^*$. □

Now let W be a subset of X and $x \in X$. We set $\Lambda_W(x) := \inf\{\lambda \in \mathbb{R};\ x \in W + \lambda\mathbf{1}\}$. So, by Proposition[3.1, [12]], If W be a downward set, then Λ_W is a topical function, and if W be a closed downward set, then $W = \{x \in X;\ \Lambda_W(x) \le 0\}$.

Theorem 3.10. *Let $W \subseteq X$ be a Du-set. Then there exist two topical functions f, g such that $W = \{x \in X;\ f(x) \le 0 \le g(x)\}$.*

Proof. By Proposition 3.2 we have $W = W_* \cap W^*$. Set $f(x) = \Lambda_{W_*}(x)$ and $g(x) = -\Lambda_{-W^*}(-x)$. Therefore, $W = \{x \in X;\ f(x) \le 0 \le g(x)\}$. □

Theorem 3.11. *Let $W \subseteq X$ be a Du-set, $x \in X$ and $w_0 \in W$, $r = \|x - w_0\|$. Then the following assertions are equivalent:*
(i) $w_0 \in P_W(x)$.
(ii) There exists a topical function f such that

$$f(w) \le 0 \le f(y) \qquad \forall w \in W,\ \forall y \in B(x, r) \tag{15}$$

Proof. $(i) \Rightarrow (ii)$. Let $B := B(x, r)$ and $w_0 \in P_W(x)$.
Case 1. $dist(x, W) = dist(x, W_*) = dist(x, W^*)$. Then $B^\circ \cap (W_* \cup W^*) = \emptyset$.
As $B^\circ \cap W_* = \emptyset$, by Lemma 2.2 and part (3) of Theorem 2.4, for each $y \in B^\circ$, there exists $l_y \in X$ such that

$$\varphi^+(w, l_y) < 0 < \varphi^+(y, l_y) \qquad (\forall w \in W). \tag{16}$$

Also, (16) is true for each $y \in (bdB \backslash bdW_*)$. For $y \in bdB \cap bdW_*$, by part (1) of Lemma 3.8, we obtained

$$\varphi^+(w, -y) \leq 0 \leq \varphi^+(y, -y) \qquad (\forall w \in W),$$

in this case, we set $l_y = -y$. Consider $f := \sup_{y \in B} \varphi^+(., l_y)$.
Case 2. $dist(x, W) = dist(x, W_*)$. Then $B^\circ \cap W_* = \emptyset$, so by part (3) of Theorem 2.4, for all $y \in B^\circ$ there exists $l_y \in X$ such that

$$\varphi^+(w, l_y) < 0 < \varphi^+(y, l_y) \qquad (\forall w \in W).$$

Also (16) is true for $y \in (bdB \backslash bdW_*)$, since $B^\circ \cap W_* = \emptyset$. If $y \in bdB \cap bdW_*$ by part (1) of Lemma 3.8 we have

$$\varphi^+(w, -y) \leq 0 \leq \varphi^+(y, -y) \qquad (\forall w \in W),$$

in this case, we set $l_y = -y$. Set $f := \sup_{y \in B} \varphi^+(., l_y)$.
Case 3. $dist(x, W) = dist(x, W^*)$. Then $B^\circ \cap W^* = \emptyset$, so by part (3) of Theorem 2.4 we have

$$\varphi_-(y, l_y) < 0 < \varphi_-(w, l_y) \qquad (\forall w \in W).$$

So,

$$-\varphi_-(w, l_y) < 0 < -\varphi_-(y, l_y) \qquad (\forall w \in W). \tag{17}$$

Also (17) is true for each $y \in bdB$ such that $y \notin W^*$ by part (3) of Theorem 2.4, since $B^\circ \cap W^* = \emptyset$. If $y \in bdB$, $y \in W^*$ then by part (2) of Lemma 3.8

$$\varphi_-(y, -y) \leq 0 \leq \varphi_-(w, -y) \qquad (\forall w \in W)$$

in this case, set $l_y = -y$. Consider $f := \sup_{y \in B} -\varphi_-(., l_y)$. Therefore for cases 1,2,3, we have (15).
Now we show that f /cong∞. Suppose there exists $x_0 \in X$ such that $f(x_0) = \infty$. Then, by definition of f ($f(x) = \sup_{y \in B} \varphi^+(x, y)$) we have $\varphi^+(x_0, y) \geq M$, for some $y \in B$ and all $M \geq 0$. In contradiction with boundary φ^+, this statement is not true.
$(ii) \Rightarrow (i)$. As f is topical, by Theorem 2.6, for $x - r\mathbf{1}$ there exists $l_0 \in X$ such that

$$\varphi_+(w, l_0) \leq f(w), \quad f(x - r\mathbf{1}) = \varphi_+(x - r\mathbf{1}, l_0) \quad (\forall w \in W). \tag{18}$$

By (15) and (18), we obtained

$$\varphi^+(w, l_0) \leq f(w) \leq 0 \leq f(x - r\mathbf{1}) = \varphi^+(x - r\mathbf{1}, l_0) \qquad (\forall w \in W).$$

So,

$$\varphi^+(w, l_0) \leq 0 \leq \varphi^+(x - r\mathbf{1}, l_0) = \varphi^+(x, l_0) - r,$$

since φ^+ is plus-homogeneous. Then, $\varphi^+(x, l_0) \geq r$. By (9), we have

$$r\mathbf{1} \leq \varphi^+(x, l_0)\mathbf{1} \leq x + l_0. \tag{19}$$

Let $w \in W$ be arbitrary, $t_w := \varphi^+(w, -x)\mathbf{1} + x$, we obtained $t_w \leq w$ since $\varphi^+(w, -x)\mathbf{1} \leq w - x$. So,

$$\varphi^+(t_w, l_0) \leq \varphi^+(w, l_0) \leq 0 \tag{20}$$

since φ^+ is increasing. By (19) and (20) we have

$$\varphi^+(t_w, -x) \le \varphi^+(t_w, l_0 - r\mathbf{1}) = \varphi^+(t_w, l_0) - r \le 0 - r = -r,$$

since φ^+ is topical. Therefore,

$$\begin{aligned} -r \ge \varphi^+(t_w, -x) &= \varphi^+(\varphi^+(w, -x)\mathbf{1} + x, -x) \\ &= \varphi^+(w, -x) + \varphi^+(x, -x) \\ &= \varphi^+(w, -x) \end{aligned}$$

since φ^+ is plus-homogeneous and $\varphi^+(x, -x) = 0$. So, $\varphi^+(w, -x) \le -r < 0$. Also,

$$r \le |\varphi^+(w, -x)| = |\varphi^+(w, -x) - \varphi^+(x, -x)| < \|x - w\|.$$

Then,

$$\|x - w_0\| = r \le \|x - w\|, \qquad (\forall w \in W),$$

which completes the proof. □

Now let $x \in X$ and set $\hat{x} := \{x + \lambda\mathbf{1} : \quad \lambda \in \mathbb{R}\}$.

Proposition 3.12. *Let $x \in X$ such that $\hat{x} \cap W \neq \emptyset$. Then the following assertions are true:*
(1) $x \in W^* \cup W_*$.
(2) If $x \in W_*$ then $x - r\mathbf{1} \in P_W(x)$.
(3) If $x \in W^*$ then $x + r\mathbf{1} \in P_W(x)$.

Proof. (1). As $\hat{x} \cap W \neq \emptyset$, then there exists $\lambda_0 \in \mathbb{R}$ such that $w_0 = x + \lambda_0\mathbf{1} \in W$. So, $x < w_0$ or $x > w_0$. Therefore $x \in W_*$ or $x \in W^*$ i.e. $x \in W^* \cup W_*$.
(2). Let $w_0 \in W$ and $r = \|w_0 - x\|$, since W is proximal. We have

$$x \le w_0 + r\mathbf{1}. \tag{21}$$

As we see in the proof of part (1) there exists $\lambda_0 \in \mathbb{R}$ such that $x + \lambda_0\mathbf{1} \in W$. We show that $\lambda_0 < 0$. Let $\lambda_0 > 0$. Since $x \in W_*$, there exists w_1 such that $w_1 < x$. So, $w_1 < x < x + \lambda_0\mathbf{1}$. Thus $x \in W$. In other words, this is a contradiction. Since $\|x - (x + \lambda_0\mathbf{1})\| = |\lambda_0| = -\lambda_0 \ge r$, then $\lambda_0 < -r$. By (21) we obtained $w = x + \lambda_0\mathbf{1} \le x - r\mathbf{1} \le w_0$. So, $x - r\mathbf{1} \in W$. Therefore, $x - r\mathbf{1} \in P_W(x)$.
(3). It is similar to what was stated in (2). □

Corollary 3.13. *Let W be a closed Du-subset of X, $\hat{x} \cap W \neq \emptyset$ and $x \in W^*$. Then*

$$d(t, W) = \min\{\lambda \ge 0 : \ t - \lambda\mathbf{1} \in W\}.$$

Proof. Let $A = \{\lambda \ge 0; \ t - \lambda\mathbf{1} \in W\}$. If $t \in W$ then $t - 0\mathbf{1} = t \in W$ and so $\min A = 0 = d(t, W)$. Suppose that $t \notin W$, then $r := d(t, W) > 0$. Let $\lambda > 0$ be arbitrary such that $t - \lambda\mathbf{1} \in W$. Thus we have

$$\lambda = \|\lambda\mathbf{1}\| = \|t - (t - \lambda\mathbf{1})\| \ge d(t, W) = r.$$

Since, by Proposition 3.12, $t - r\mathbf{1} \in W$ it follows that $r \in A$. Hence, $\min A = r$, which completes the proof. □

4. Applying Optimization Theory to Non-convex Du-sets

Convexity plays a critical role in optimization theory due to the inherent separation properties of convex sets and the linear approximation properties of convex functions. The historical development of convex analysis was primarily motivated by its application to convex optimization problems. However, the importance of non-convex optimization has increased significantly in recent years. This change acknowledges the fact that not all optimization problems can be easily formulated in the context of convex sets.

Monotonicity analysis establishes itself as another valuable tool in the mathematical framework used to analyze systems in various fields, including economics and engineering. This framework demonstrated by its successful exploration in recent studies. Notably, Hansen et al. [2] further explored its application, while monotonic optimization was first introduced in Robinov et al. [13] and developed in detail in two seminal papers by Tuy et al. [[17], [18]].

This section uses the monotonic optimization framework to provide a valuable application for dealing with optimization problems defined on Du-sets. Notably, the fact that Du-sets may not exhibit convexity presents a significant challenge for traditional optimization techniques. We will demonstrate the efficacy of this approach by focusing on a specific example within the class of Du-sets.

A monotonic optimization problem is an optimization problem of the form

$$\max\{h(x) \mid f_i(x) \leq 0,\ i = 1, ..., j\ ,\quad g_i(x) \geq 0\ i = j+1, ..., m\ x \in [a, b], \} \tag{22}$$

where the objective function $h(x)$ and the constraint functions $f_i(x), g_i(x)$ are increasing functions on the box $[a, b] \in \mathbb{R}^n$.

By setting $f(x) = \max\{f_1(x), ..., f_j(x)\}$ and $g(x) = \min\{g_{j+1}, ..., g_m(x)\}$ Problem (22) is transformed into:

$$\max\{h(x) \mid f(x) \leq 0 \leq g(x),\ x \in [a, b]\},$$

where f, g, h are increasing functions.

Leveraging Theorem 3.10,we can now reformulate optimization problem (22) into the following equivalent form:

$$\max\{h(x) \mid\ x \in W\}$$

where W is a closed bounded Du-subset of $\mathbb{R}^n_+$.

Tuy was solved a monotonic optimization problem, over a normal set, (recall a set G is normal if $x \in G$ then $[0, x] \subset G$), by using the "separation property" of normal sets which is analogous to the separation property of convex sets. One of the previously obtained results was that, any convex set can be approximated as accurately as desired by a nested sequence of polyhedra. Tuy et al addressed this issue by proposing an algorithm for the outer approximation of polyblocks. , which is analogous to the algorithms for the outer approximation of polyhedra, but do not quite match them.

The basic idea of polyblock outer approximation algorithms is to construct a sequence of polyblocks ρ_i such that:

$$\rho_1 \supset \rho_2 \supset ... \supset \rho_k \supset W.$$

Recall a set $\rho \subset \mathbb{R}^n_+$ is called a polyblock if it is a union of a finite number of boxes $[0, z]$, where $z \in T$ and $|T| < +\infty$. The set T is called the vertex set of the polyblock.

Let W be a Du-subset of $\mathbb{R}^n_+$. By proposition 3.2, $W = W_* \cap W^*$. We have $0 \in W_*$ and $W_* \cap \mathbb{R}^n_+$ is a normal set.

Leveraging the aforementioned properties, the polyblock algorithm (see [15],[22]) is applicable to all Du-subsets of $\mathbb{R}^n_+$. In particular, it is well-suited for addressing problems involving non-convex Du-subsets.

We present a slight modification of this algorithm tailored for Du-sets:

Algorithm1: Polyblock Outer Approximation Algorithm

Input: An upper semicontinuous increasing function $f(.) : W \subset \mathbb{R}^n_+ \to \mathbb{R}$, where W is a compact Du-set.
Output: an $\varepsilon-$optimal solution
Initialization: Let the initial polyblock ρ_1 be box $[0, b]$ that encloses W. The vertex set $\tau_1 = \{b\}$. Let $\varepsilon \geq 0$ be a small positive number $CBV_0 = -\infty$, $k = 0$.
repeat
$k = k + 1$.
From τ_k, select $z_k \in argmax\{f(z) \mid z \in \tau_k\}$.
$\pi_W(z_k) = \lambda z_k,\ \lambda = \max\{\alpha > 0 \mid\ \alpha z_k \in W\}$.
if $\pi_W(z_k) = z_k$, i.e., $z_k \in W$ then
$\bar{x}_k = z_k$ and $CBV_k = f(z_k)$.
else

11: If $\pi_W(z_k) \in W$ and $f(\pi_W(z_k)) \geq CBV_{k-1}$, then let the curent best solution $\bar{x}_k = \pi_W(z_k)$ and $CBV_k = f(\pi_W(z_k))$. Otherwise, $\bar{x}_k = \bar{x}_{k-1}$ and $CBV_k = CBV_{k-1}$.
12: Let $x = \pi_W(z_k)$ and $\tau_{k+1} = (\tau_k \setminus \tau_*) \cup \{v^i = v + (x_i - \nu_i)e^i \mid v \in \tau_*,\ i \in \{1, ..., n\}\}$, where $\tau_* = \{v \in \tau_k \mid v > x\}$.
13 : end if
14 : **until** $|f(z_k) - CBV_k| \leq \varepsilon$.
15 :Let $x^* = \bar{x}_k$ and terminate the algorithm.

To illustrate how the Algorithm works, we give the following example:

Example 4.1. *Consider the problem*

$$\begin{aligned}
&\max\ x_1x_2 \\
&subject\quad x_1, x_2 \geq 0, \\
&\qquad\qquad x_2 \leq 4 \textit{ and } x_1 \leq 2, \\
&\qquad\qquad x_1 + x_2 \leq 4 \textit{ and } 2 < x_1, \\
&\qquad\qquad x_1 \leq 3.
\end{aligned}$$

In fact $\max\{x_1x_2 \mid (x_1, x_2) \in W\}$, *where* $W = \{(x_1, x_2) \mid 0 \leq x_1 \leq 2,\ 0 \leq x_2 \leq 4\} \cup \{2 < x_1 \leq 3,\ x_1 + x_2 \leq 4\}$ *is a Du-set. To solve this problem with* $\varepsilon = 0/001$, *Our proposed algorithm is Algorithm 1. We show the steps of the Algorithm in the following table:*

k	τ_{k+1}	z_k	$\pi_W(z_k)$	$f(z_k)$	CBV_k	$d = \lvert f(z_k) - CBV_k \rvert$
$k = 0$	$\{(5,5)\}$	$(5,5)$	$(2,2)$	25	4	21
$k = 1$	$\{(2,5),(2,2),(5,2)$	$(2,5)$	$(\frac{8}{5},4)$	10	$\frac{32}{5}$	3.6
$k = 2$	$\{(\frac{8}{5},5),(2,4),(2,2),(5,2)\}$	$(5,2)$	$(\frac{20}{7},\frac{8}{7})$	10	$\frac{32}{5}$	3.6
$k = 3$	$\{(\frac{8}{5},5),(2,4),(2,2),(5,\frac{8}{7}),(\frac{20}{7},2)\}$	$(2,4)$	$(2,4)$	8	8	0

Thus $x^* = (2,4)$ *is an optimal solution.*

References

[1] G. Birkhoff, *Lattice Theory*, Vol. 25, American Mathematical Society, Colloquium Publications, 1967.

[2] P. Hansen, B. Jaumard and S. H. Lu, *Some further results on monotonicity in globally optimal design* , Journal of Mechanics, Transmissions and Automation in Design. **111**(1989).

[3] M. Iranmanesh and F. Solimany, *I_m-quasi upward sets, with their best approximations* , Numerical Functional Analysis and Optimization. **31(11)**(2010), 1261–1271.

[4] J. E. Martinez-Legaz, A. M. Rubinov and I. Singer, *Downward sets and their separation and approximation properties*, J. Global Optimization. **23**(2002), 11-137.

[5] H. Mohebi, *Topical Functions and their Properties in a Class of Ordered Banach Spaces*, In book: Continuous Optimization, January 2005, 343–361.

[6] H. Mohebi and A. Rubinov, *Metric Projection Onto a Closed set: Necessary and Sufficient Conditions for the Global Minimum*, Journal of M athematics of Operations Research, **31**(2006), No.1, 124–132.

[7] H. Mohebi and A. Rubinov, *Best Approximation by Downward sets with Applacations* , Anal. Theory Appl. **22**(2006), no. 1, 20–40.

[8] H. Mohebi and E. Naraghirad, *Distance from a point a downward set in a Banach lattice*, Pacific Journal of Optimization. Vol. 4(2008), no. 3, 641–649.

[9] P. Papalambros and H. L. Li, *Notes on the operational utility of monotonicity in optimization*, ASME Journal of Mechanisms, Transmission and Automation in Design. **105**(1993), 174–180.

[10] P. Papalambros and D. J. Wilde, *Principles of Optimal Design- Modeling and Computation* , Cambridge University Press, NewYork., 1986.

[11] A. M. Rubinov, *Sublinear Operations and their Applacations*, Uspehi Mat. Nauk, (Russian Math. Surv.), Optimization, **32** (1977), 113–174.

[12] A. M. Rubinov and I. Singer, *Topical and Sub-Topical functions, Downward sets and Abstract Convexity*, Optimization, vol. 50 (2001), 307–351.

[13] A. M. Rubinov, H. Tuy and H. Mays, *Algorithm for a monotonic global optimization problem*, SIMS, University of Ballarant, Ausffalia, 1998.

[14] Z. Soltani, H. R. Goudarzi, *Best Approximation by Upward Sets* , Journal of mathematics and computer science, **14**(2015), 345–353.

[15] H. Tuy, *Convex Analysis and Global Optimization Nonconvex Optimization and Its Applications*, Kluwer Academic Publishers, 1998.

[16] H. Tuy, *Normal Sets, Ployblocks and Monotonic Optimization*, Vietnam Journal of Mathematics, Springer Verlag., **27:4**(1999), 277–300.

[17] H. Tuy, *Monotonic optimization: Problems and solution approaches*, Institute of Mathematics. Hanoi, 1999.

[18] H. Tuy, *Monotonic optimization: Problems and solution approaches*, SIAM J. Optim., **11** (2000a), 464–494.
[19] H. Tuy and T. L. Luc, *A new approach to optimization under monotonic constraints*, J. Glob. Optim., **18**(2000), 1–15.
[20] H. Tuy and W. Oettli, *On necessary and sufficient conditions for global optimization*, Matematicas Applicadas, **15**(1994), 39–41.
[21] D. J. Wilde, *A maximum activity principle for eliminating over-constrained optimization cases*, ASME Journalof Mechanisms, Transmissions and Automation in Design **10S**(1936), 312–319.
[22] Y. J. A. Zhang, L. Qian and J. Hang *Monotonic Optimization in Communication and Networking Systems*, Found. Trends in Netw., vol. 7, no. 1(2013), 1–75.