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Milne-type inequalities for co-ordinated convex functions

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Abstract. In this research, our objective is to formulate a unique identity for Milne-type inequalities involving for functions of two variables having convexity on co-ordinates over $[\mu, \upsilon] \times [\omega, \varkappa]$. By employing this identity, we establish some new inequalities of the Milne-type for co-ordinated convex functions. Furthermore, the propose identity strengthens the theoretical basis of mathematical inequalities showcasing its significance in various fields.

1. Introduction and Preliminaries

Convexity is a fundamental mathematical concept originating from ancient Greek philosophy. It gained significant traction in the late 19th century with the introduction of convex functions by German mathematician Karl Hermann Amandus Schwarz [1]. Recently, convexity has applications in economics, engineering, computer science, and mathematics, particularly in optimization problems and inequalities [2, 3]. Extensive research indicates a strong relationship between convexity theory and integral inequalities, emphasizing their key functions in differential equations and applied mathematics. This relationship is essential because of the wide range of applications and significant impact of integral inequalities. The understanding of mathematical concepts is strengthened by investigating a variety of inequalities, such as Gronwall, Simpson's type, Chebyshev, Jensen, Holder, Milne, and Hermite-Hadamard inequality. It is suggested that those who are interested in learning more about these inequalities and their applications in real-world scenarios visit references [4–16].
The subsequent definitions will be extensively employed in this study.

Definition 1.1. *[17] A mapping $\Phi : \Delta \to \mathcal{R}$ is convex on the co-ordinates, if the following inequality holds:*

$$\Phi(t\mu + (1-t)\upsilon, t\omega + (1-t)\varkappa) \leq t\Phi(\mu, \omega) + (1-t)\Phi(\upsilon, \varkappa)$$

for all $(\mu, \omega), (\upsilon, \varkappa) \in \Delta$ and $t \in [0, 1]$.

A modification for convex functions on co-ordinates, which are also known as co-ordinated convex functions, was introduced by Dragomir [17, 18] as follows:

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Definition 1.2. *A function $\Phi : \Delta \subset \mathcal{R}^2 \to \mathcal{R}$ is convex on the coordinates on Δ. If the partial mappings*

$$\Phi_y : [\mu, \upsilon] \to \mathcal{R},\ \Phi_y(u) = \Phi(u, y)$$

and

$$\Phi_x : [\omega, \varkappa] \to \mathcal{R},\ \Phi_x(v) = \Phi(x, v)$$

are convex, where defined for all $y \in [\omega, \varkappa]$ and $x \in [\mu, \upsilon]$.

A formal definition for the co-ordinated convex functions stated as:

Definition 1.3. *[19] A mapping $\Phi : \Delta \to \mathcal{R}$ is said to be co-ordinated convex on Δ, for all $(\mu, \upsilon), (\omega, \varkappa) \in \Delta$ and $t, s \in [0, 1]$, then the following inequality holds:*

$$\Phi(t\mu + (1-t)\omega, s\upsilon + (1-s)\varkappa) \leq ts\Phi(\mu, \upsilon) + t(1-s)\Phi(\mu, \varkappa) + (1-t)s\Phi(\omega, \upsilon) + (1-t)(1-s)\Phi(\omega, \varkappa).$$

If Φ is a co-ordinated concave on Δ then the above inequality hold in reverse direction.

Theorem 1.4. *Suppose that $\Phi : \Delta \subset \mathcal{R}^2 \to \mathcal{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\Phi\left(\frac{\mu+\upsilon}{2}, \frac{\omega+\varkappa}{2}\right) \leq \frac{1}{(\upsilon-\mu)(\varkappa-\omega)} \int_{\mu}^{\upsilon} \int_{\omega}^{\varkappa} \Phi(x, y)dydx \leq \frac{\Phi(\mu, \omega) + \Phi(\upsilon, \omega) + \Phi(\mu, \varkappa) + \Phi(\upsilon, \varkappa)}{4}.$$

The inequalities mentioned above are precise.
The Milne-type inequality, a mathematical inequality focusing on estimating integrals, was established in the early twentieth century by British mathematician Edward Arthur Milne. This inequality, named after Milne, has acquired importance in mathematical inequalities due to its versatility and broad applications in optimisation theory, physics, and engineering [20–22].
The explanations provided by Dragomir and Agarwal about the mathematical analysis of errors related to the trapezoidal formula are highly significant and are covered in [23]. Kirmaci also used convex functions to define error bounds for the midpoint and trapezoidal formulas [24]. Budak et al. reported results for several function classes in [25], which investigates Milne-type inequalities for fractional integrals, presenting theoretical insights enriched by specific examples and graphical representations. Error bounds for Milne's formula in fractional and classical calculus have been obtained by Ali et al. [26], with specific applications to differentiable convex functions. Bakula and Pecaric have investigated Jensen's inequality for convex functions on coordinates within a rectangular plane [27]. Hezenci [28] introduce Hermite-Hadamard type inequalities for differentiable co-ordinated (s_1, s_2)-convex functions and provide additional inequalities that apply to Riemann-Liouville fractional integrals and k-Riemann-Liouville fractional integrals. Özdemir et al. [29] have identified Hadamard-type inequalities by investigating co-ordinated quasi-convexity. For extended s-convex functions, Xi et al. [30] gave some Hermite-Hadamard type integral inequalities on the co-ordinates in a rectangle. By employing Riemann-Liouville fractional integrals, in R^2 rectangle plane Sarikaya [31] present Hermite–Hadamard-type for co-ordinated convex functions, additionally proving an integral identity for fractional integrals. Erden and Sarikaya have proved novel inequalities of Hermite-Hadamard and Ostrowski types, tailored for convex functions defined on the co-ordinates within a rectangular region in the plane [32]. Farid et al. established the Fejer-Hadamard inequality for convex functions on coordinates within a plane's rectangular region. Additionally, they explore certain mappings associated with this inequality [33]. Latif and Dragomir [34] have formulated various novel inequalities applicable to two-variable differentiable coordinated convex and concave functions. These inequalities are particularly associated with the left side of the Hermite-Hadamard type inequality concerning co-ordinated convex functions in two variables. Kara et al. [35] have concluded novel additions of the Hermite-Hadamard-Fejér type inequality for the product of two interval-valued functions with coordinated convexity. By leveraging properties of exponentially convex m-and (α, m)-convex functions on the co-ordinates Aslan et al. [36] have reported novel classes of convexity, m-and (α, m)-exponentially convex functions on the co-ordinates.

Numerous research papers have explored generalizations and new formulations of inequalities, employing various types of convex functions. Various inequalities and results concerning co-ordinated convex functions, readers are encouraged to consult [37–47].

This study establishes and discusses a Milne-type inequality for coordinated convex functions. The main objective of this study, to prove a Milne-type inequality for convex functions on co-ordinates.

Main Results

To establish our main results, we need the following lemma.

Lemma 1.5. *Suppose that $\Phi : \Delta \subset \mathcal{R}^2 \to \mathcal{R}$ be a partial differentiable mapping on $\Delta = [\mu, \upsilon] \times [\omega, \varkappa]$. If $\frac{\partial^2 \Phi}{\partial t \partial s} \in L(\Delta)$, then the equality holds:*

$$\Omega(\mu, \upsilon; \omega, \varkappa) = (\upsilon - \mu)(\varkappa - \omega) \int_0^1 \int_0^1 \mathcal{P}(x,t)\mathcal{Q}(y,s) \frac{\partial^2 \Phi}{\partial t \partial s}(t\mu + (1-t)\upsilon, s\omega + (1-s)\varkappa) dtds$$

where

$$\begin{aligned}
\Omega(\mu, \upsilon; \omega, \varkappa) = & \frac{4\Phi(\mu,\omega) + 4\Phi(\upsilon,\omega) + 4\Phi(\mu,\varkappa) + 4\Phi(\upsilon,\varkappa)}{9} \\
& - \frac{2\Phi(\mu, \frac{\omega+\varkappa}{2}) + 2\Phi(\upsilon, \frac{\omega+\varkappa}{2}) - \Phi(\frac{\mu+\upsilon}{2}, \frac{\omega+\varkappa}{2}) + 2\Phi(\frac{\mu+\upsilon}{2}, \omega) + 2\Phi(\frac{\mu+\upsilon}{2}, \varkappa)}{9} \\
& - \frac{1}{3(\upsilon - \mu)} \int_\mu^\upsilon \left[2\Phi(x,\omega) - \Phi\left(x, \frac{\omega+\varkappa}{2}\right) + 2\Phi(x,\varkappa) \right] dx \\
& - \frac{1}{3(\varkappa - \omega)} \int_\omega^\varkappa \left[2\Phi(\mu,y) - \Phi\left(\frac{\mu+\upsilon}{2}, y\right) + 2\Phi(\upsilon,y) \right] dy \\
& + \frac{1}{(\upsilon-\mu)(\varkappa-\omega)} \int_\mu^\upsilon \int_\omega^\varkappa \Phi(x,y) dy dx,
\end{aligned}$$

$$\mathcal{P}(x,t) = \begin{cases} (t - \frac{2}{3}), & for \quad t \in [0, \frac{1}{2}) \\ (t - \frac{1}{3}), & for \quad t \in (\frac{1}{2}, 1] \end{cases}$$

and

$$\mathcal{Q}(y,s) = \begin{cases} (s - \frac{2}{3}), & for \quad s \in [0, \frac{1}{2}) \\ (s - \frac{1}{3}), & for \quad s \in (\frac{1}{2}, 1]. \end{cases}$$

Proof. By definition of $\mathcal{P}$, we can write

$$\begin{aligned}
& \int_0^1 \int_0^1 \mathcal{P}(x,t)\mathcal{Q}(y,s) \frac{\partial^2 \Phi}{\partial t \partial s}(t\mu + (1-t)\upsilon, s\omega + (1-s)\varkappa) dtds \\
& = \int_0^1 \mathcal{Q}(y,s) \left[\int_0^{\frac{1}{2}} \left(t - \frac{2}{3}\right) \frac{\partial^2 \Phi}{\partial t \partial s}(t\mu + (1-t)\upsilon, s\omega + (1-s)\varkappa) dt \right. \\
& \quad \left. + \int_{\frac{1}{2}}^1 \left(t - \frac{1}{3}\right) \frac{\partial^2 \Phi}{\partial t \partial s}(t\mu + (1-t)\upsilon, s\omega + (1-s)\varkappa) dt \right] ds.
\end{aligned}$$

By using integration by parts, we acquire

$$\int_0^1 \int_0^1 \mathcal{P}(x,t)\mathcal{Q}(y,s) \frac{\partial^2 \Phi}{\partial t \partial s}(t\mu + (1-t)\upsilon, s\omega + (1-s)\varkappa) dtds$$

$$
\begin{aligned}
&= \int_0^1 Q(y,s)\Bigg\{\left[\left(\frac{1}{\mu-\upsilon}\right)\left(t-\frac{2}{3}\right)\frac{\partial\Phi}{\partial s}(t\mu+(1-t)\upsilon, sc+(1-s)\varkappa)\right]_0^{\frac{1}{2}} \\
&\quad - \frac{1}{\mu-\upsilon}\int_0^{\frac{1}{2}} \frac{\partial\Phi}{\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)dt \\
&\quad + \left[\left(\frac{1}{\mu-\upsilon}\right)\left(t-\frac{1}{3}\right)\frac{\partial\Phi}{\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right]_{\frac{1}{2}}^1 - \frac{1}{\mu-\upsilon}\int_{\frac{1}{2}}^1 \frac{\partial\Phi}{\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\,dt\Bigg\}ds \\
&= \frac{1}{\upsilon-\mu}\Bigg[\frac{1}{6}\int_0^{\frac{1}{2}}\left(s-\frac{2}{3}\right)\frac{\partial\Phi}{\partial s}\left(\frac{\mu+\upsilon}{2}, s\omega+(1-s)\varkappa\right)ds + \frac{1}{6}\int_{\frac{1}{2}}^1\left(s-\frac{1}{3}\right)\frac{\partial\Phi}{\partial s}\left(\frac{\mu+\upsilon}{2}, s\omega+(1-s)\varkappa\right)ds \\
&\quad - \frac{2}{3}\int_0^{\frac{1}{2}}\left(s-\frac{2}{3}\right)\frac{\partial\Phi}{\partial s}(\upsilon, s\omega+(1-s)\varkappa)ds - \frac{2}{3}\int_{\frac{1}{2}}^1\left(s-\frac{1}{3}\right)\frac{\partial\Phi}{\partial s}(\upsilon, s\omega+(1-s)\varkappa)ds \\
&\quad + \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(s-\frac{2}{3}\right)\frac{\partial\Phi}{\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)dsdt \\
&\quad + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left(s-\frac{1}{3}\right)\frac{\partial\Phi}{\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)dsdt \\
&\quad - \frac{2}{3}\int_0^{\frac{1}{2}}\left(s-\frac{2}{3}\right)\frac{\partial\Phi}{\partial s}(\mu, s\omega+(1-s)\varkappa)ds - \frac{2}{3}\int_{\frac{1}{2}}^1\left(s-\frac{1}{3}\right)\frac{\partial\Phi}{\partial s}(\mu, s\omega+(1-s)\varkappa)ds \\
&\quad + \frac{1}{6}\int_0^{\frac{1}{2}}\left(s-\frac{2}{3}\right)\frac{\partial\Phi}{\partial s}\left(\frac{\mu+\upsilon}{2}, s\omega+(1-s)\varkappa\right)ds + \frac{1}{6}\int_{\frac{1}{2}}^1\left(s-\frac{1}{3}\right)\frac{\partial\Phi}{\partial s}\left(\frac{\mu+\upsilon}{2}, s\omega+(1-s)\varkappa\right)ds \\
&\quad + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left(s-\frac{2}{3}\right)\frac{\partial\Phi}{\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)dsdt \\
&\quad + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left(s-\frac{2}{3}\right)\frac{\partial\Phi}{\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)dsdt\Bigg].
\end{aligned}
$$

After computing these integral and using the change of the variable $x = t\mu + (1-t)\upsilon$ and $y = s\omega + (1-s)\varkappa$ for $(t,s) \in [0,1]$, then multiplying both sides with $(\upsilon-\mu)(\varkappa-\omega)$, we have the required result. □

Theorem 1.6. *Let assume that the conditions of Lemma 1.5 hold. If* $\left|\frac{\partial^2\Phi}{\partial t\partial s}\right|$ *is co-ordinated convex on* Δ, *then the following inequality holds:*

$$
|\Omega(\mu,\upsilon;\omega,\varkappa)| \le \frac{25(\upsilon-\mu)(\varkappa-\omega)}{576}\left[\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right| + \left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right| + \left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right| + \left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|\right].
$$

Proof. By taking the absolute value of Lemma 1.5, then it becomes

$$
|\Omega(\mu,\upsilon;\omega,\varkappa)| \le (\upsilon-\mu)(\varkappa-\omega)\int_0^1\int_0^1 |p(x,t)q(y,s)|\left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right|dtds. \tag{1}
$$

Since $\left|\frac{\partial^2\Phi}{\partial t\partial s}\right|$ is co-ordinated convex , then we have

$$
\begin{aligned}
&\left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right| \\
&\le ts\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right| + t(1-s)\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right| + (1-t)s\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right| + (1-t)(1-s)\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|.
\end{aligned}
$$

Utilizing the given fact that, we get

$$\int_0^1\int_0^1 |\mathcal{P}(x,t)\mathcal{Q}(y,s)|\left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right|dtds \tag{2}$$
$$\leq \frac{25\left[\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right|+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right|+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right|+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|\right]}{576}.$$

By using (2) in (1), then we attain required inequality. □

Theorem 1.7. *Let assume that the conditions of Lemma 1.5 is hold. If $\frac{\partial^2\Phi}{\partial t\partial s}$ is bounded, i.e,*

$$\left\|\frac{\partial^2\Phi}{\partial t\partial s}\right\|_\infty = \sup_{(x,y)\in(\mu,\upsilon)\times(\omega,\varkappa)}\left|\frac{\partial^2\Phi}{\partial t\partial s}(x,y)\right| < \infty$$

for all $(t,s)\in[0,1]$, then the following inequality holds:

$$|\Omega(\mu,\upsilon;\omega,\varkappa)| \leq \frac{25(\upsilon-\mu)(\varkappa-\omega)}{144}\left\|\frac{\partial^2\Phi}{\partial t\partial s}\right\|_\infty.$$

Proof. By using the Lemma 1.5 and utilizing the property of modulus, we acquire

$$|\Omega(\mu,\upsilon;\omega,\varkappa)| \leq (\upsilon-\mu)(\varkappa-\omega)\int_0^1\int_0^1 |\mathcal{P}(x,t)\mathcal{Q}(y,s)|\left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right|dtds.$$

Since $\left|\frac{\partial^2\Phi}{\partial t\partial s}\right|$ is bounded, we have

$$|\Omega(\mu,\upsilon;\omega,\varkappa)| \leq (\upsilon-\mu)(\varkappa-\omega)\left\|\frac{\partial^2\Phi}{\partial t\partial s}\right\|_\infty\int_0^1\int_0^1 |\mathcal{P}(x,t)\mathcal{Q}(y,s)|dtds. \tag{3}$$

After some calculation, we find

$$\int_0^1\int_0^1 |\mathcal{P}(x,t)\mathcal{Q}(y,s)|dtds = \frac{25}{144}. \tag{4}$$

By using (4) in (3), it yields

$$|\Omega(\mu,\upsilon;\omega,\varkappa)| \leq \frac{25(\upsilon-\mu)(\varkappa-\omega)}{144}\left\|\frac{\partial^2\Phi}{\partial t\partial s}\right\|_\infty.$$

Therefore, we have concluded the proof. □

Theorem 1.8. *Let assume that the conditions of Lemma 1.5 hold. If $\left|\frac{\partial^2\Phi}{\partial t\partial s}\right|^q$, $q>1$ is co-ordinated convex on Δ, then one has the inequality*

$$|\Omega(\mu,\upsilon;\omega,\varkappa)| \leq \frac{(\upsilon-\mu)(\varkappa-\omega)}{4}\left(\frac{(4^{p+1}-1)^2}{9^{p+1}(p+1)^2}\right)^{\frac{1}{p}}\left(\frac{\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right|^q+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right|^q+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right|^q+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|^q}{4}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Through the utilization of the familiar Hölder inequality in the context of a double integral, we have

$$
\begin{aligned}
&|\Omega(\mu,\upsilon;\omega,\varkappa)| \\
&\leq (\upsilon-\mu)(\varkappa-\omega)\left(\int_0^1\int_0^1 |\mathcal{P}(x,t)\mathcal{Q}(y,s)|^p dsdt\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 \left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right|^q dsdt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $\left|\frac{\partial^2\Phi}{\partial t\partial s}\right|^q$ is co-ordinated convex function on Δ, then we have

$$
\begin{aligned}
&\left(\int_0^1\int_0^1 \left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right|^q dsdt\right)^{\frac{1}{q}} \qquad (5)\\
&\leq \left(\int_0^1\int_0^1 \left\{ts\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right|^q + t(1-s)\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right|^q\right.\right.\\
&\quad\left.\left. +\ (1-t)s\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right|^q + (1-t)(1-s)\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|^q\right\}dsdt\right)^{\frac{1}{q}}\\
&= \left(\frac{\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right|^q + \left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right|^q + \left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right|^q + \left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}
$$

We also have

$$
\left(\int_0^1\int_0^1 |\mathcal{P}(x,t)\mathcal{Q}(y,s)|^p dsdt\right)^{\frac{1}{p}} = \left(\frac{(4^{p+1}-1)^2}{4^p 9^{p+1}(p+1)^2}\right)^{\frac{1}{p}}. \qquad (6)
$$

By combining (5) and (6), then we obtained required result. □

Theorem 1.9. *Let assume that the conditions of Lemma 1.5 hold. If* $\left|\frac{\partial^2\Phi}{\partial t\partial s}\right|^q$*,* $q \geq 1$ *is co-ordinated convex on* Δ*, then the following inequality holds:*

$$
|\Omega(\mu,\upsilon;\omega,\varkappa)| \leq (\upsilon-\mu)(\varkappa-\omega)\left(\frac{25}{144}\right)^{1-\frac{1}{q}}\left(\frac{25\left[\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right|^q + \left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right|^q + \left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right|^q + \left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|^q\right]}{576}\right)^{\frac{1}{q}}.
$$

Proof. From Lemma 1.5, we can write

$$
|\Omega(\mu,\upsilon;\omega,\varkappa)| \leq (\upsilon-\mu)(\varkappa-\omega)\int_0^1\int_0^1 |\mathcal{P}(x,t)\mathcal{Q}(y,s)|\left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right|dtds.
$$

By implementation of the Power Mean inequality tailored for double integrals, we can assert

$$
\begin{aligned}
|\Omega(\mu,\upsilon;\omega,\varkappa)| \leq& (\upsilon-\mu)(\varkappa-\omega)\left(\int_0^1\int_0^1 |\mathcal{P}(x,t)\mathcal{Q}(y,s)|dsdt\right)^{1-\frac{1}{q}} \qquad (7)\\
&\times\left(\int_0^1\int_0^1 |\mathcal{P}(x,t)\mathcal{Q}(y,s)|\left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right|^q dsdt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $\left|\frac{\partial^2\Phi}{\partial t\partial s}\right|^q$ is co-ordinated convex on Δ, then we have

$$
\left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right|^q
$$

$$\leq ts\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right|^q + t(1-s)\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right|^q + (1-t)s\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right|^q + (1-t)(1-s)\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|^q$$

and, thus we obtain

$$\left(\int_0^1\int_0^1 |\mathcal{P}(x,t)Q(y,s)|\left|\frac{\partial^2\Phi}{\partial t\partial s}(t\mu+(1-t)\upsilon, s\omega+(1-s)\varkappa)\right|^q dsdt\right)^{\frac{1}{q}} \tag{8}$$
$$\leq\left(\frac{25\left[\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right|^q+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right|^q+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right|^q+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|^q\right]}{576}\right)^{\frac{1}{q}}.$$

By using the fact that

$$\left(\int_0^1\int_0^1 |\mathcal{P}(x,t)Q(y,s)|dsdt\right)^{1-\frac{1}{q}} = \left(\frac{25}{144}\right)^{1-\frac{1}{q}} \tag{9}$$

Hence, by using (8) and (9) in (7), we get

$$|\Omega(\mu,\upsilon;\omega,\varkappa)| \leq (\upsilon-\mu)(\varkappa-\omega)\left(\frac{25}{144}\right)^{1-\frac{1}{q}}\left(\frac{25\left[\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\omega)\right|^q+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\mu,\varkappa)\right|^q+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\omega)\right|^q+\left|\frac{\partial^2\Phi}{\partial t\partial s}(\upsilon,\varkappa)\right|^q\right]}{576}\right)^{\frac{1}{q}}.$$

In consequence, we have completed the proof. □

2. Conclusion

Milne-type inequalities are widely recognized in mathematical analysis and optimization theory. In this investigation, we proposed a novel identity of Milne-type inequalities for functions of two variables, having convexity on co-ordinates over the domain $[\mu,\upsilon]\times[\omega,\varkappa]$. Leveraging this identity, we unveiled several results for Milne-type inequalities. This study is the first to derive Milne-type inequalities specifically for co-ordinated convex functions. In future research, authors may seek to explore the possibility of extending our results by investigating alternative classes of convex functions or exploring different types of fractional integral operators.

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