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# An optimal interpolation formula of Hermite type in the Sobolev space

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**Abstract.** This article delves into the construction of an optimal interpolation formula designed for approximating functions within the Hilbert space $L_2^{(2)}(0,1)$. This space encompasses functions that are square integrable with a second generalized derivative in the interval $[0,1]$. The interpolation formula takes the form of a linear combination of function values and their first-order derivative at equidistant nodes within the interval $[0,1]$. The coefficients are determined by minimizing the norm of the error functional in the dual space $L_2^{(2)*}(0,1)$. This error functional is defined as the disparity between the function and its approximation.

Key outcomes of the study include explicit expressions for the coefficients and the norm of the error functional. The optimization problem is methodically formulated and solved, resulting in a system of linear equations for the coefficients. Analytical solutions are achieved, yielding a clear expression for the optimal coefficients.

Furthermore, integrating the obtained optimal interpolation formula over the interval $[0,1]$, yields the Euler-Maclaurin quadrature formula. The application of these results is demonstrated in estimating the error of the interpolation formula for functions in $L_2^{(2)}(0,1)$.

## 1. Introduction

A numerical interpolation technique aims to approximate a function based on available data at distinct points. This data may encompass both function values and diverse derivatives. In the present context, emphasis is solely on local interpolation, achieved through the utilization of low-order polynomials. The term "local" signifies that data is exclusively derived from the vicinity of the point where the function value is being estimated.

The history of spline functions is rooted in the work of drafting technicians, who often had to draw a smoothly turning curve between points on a drawing (see, for example, [14]). This process is called wrapping, and it can be done with several special devices, such as a French curve made of plastic, which presents the drafter with several curves of various curvatures to choose from.

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Long wooden slats were also used, passed through the control points with the help of weights laid on the draftsman's table and attached to the slats. Weights were called ducks, and wooden planks were called splines as early as 1891. The elasticity of the wooden slats allowed them to bend only slightly as they passed through the given points. Essentially, the wood solved the differential equation and minimized the strain energy. The latter is known to be a simple function of curvature. The mathematical theory of these curves owes much to early explorers, especially Isaac Schoenberg in the 1940s and 1950s. Other vital names associated with the early development of the subject (i.e. before 1964) are Garrett Birkhoff, C. de Boor, J. H. Ahlberg, E. N. Nilson, H. Garabedian, R. S. Johnson, F. Landis, A. Whitney, J. L. Walsh, and J.K. Holladay. The first book to systematically expose the theory of splines was [2]. This book presented the theory of polynomial splines of odd degrees, and further, the theory of generalized splines was discussed.

It should be noted that when studying polynomial splines of odd degree, algebraic and variational approaches are possible: 1) the algebraic approach studies in detail the systems of linear equations defining splines and 2) the variational approach uses the internal properties of splines and the leading integral relations between functions from the classes $K^{(n)}(a,b)$ and $K^{(2n)}(a,b)$ and approximating splines (see Chapter 6 of [2]), where $K^{(m)}(a,b)$ is a class of functions $f(x)$ defined on the segment $[a,b]$, having an absolutely continuous $(n-1)$-st derivative and an $n$-th derivative belonging to the space $L_2(a,b)$. The first approach was mainly used to study cubic and bicubic splines. The approach based on intrinsic properties is used to study generalized splines.

The present work is devoted to constructing an optimal interpolation formula with derivative based on variational methods. First, we give the definition of a generalized spline from the book [2].

Let $L$ be a linear differential operator given by the formula

$$L \equiv a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_0(x),$$

where the functions $a_j(x)$ $(j = 0,1,\ldots,n)$ belong to $C^n[a,b]$ and $a_n(x) \neq 0$ on the segment $[a,b]$. Denote by $L^*$ the operator adjoint to $L$:

$$L^* \equiv (-1)^n \frac{d^n}{dx^n}\{a_n(x)\cdot\} + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}\{a_{n-1}(x)\cdot\} + \ldots + a_0(x).$$

**Definition 1.1.** *If the $\Delta:\ a = x_0 < x_1 < \ldots < x_N = b$ mesh is given on the segment $[a,b]$, then the generalized defect spline $k$ $(0 \leq k \leq n)$ with respect to $\Delta$ is a function $S_\Delta(x)$ from the class $K^{(2n-k)}(a,b)$ satisfying differential equation*

$$L^*LS_\Delta = 0 \tag{1}$$

*on each open interval $(x_{i-1},x_i)$ $(i = 1,2,\ldots,N)$. To say that the spline $S_\Delta(x)$ has order $2n$ when it is necessary to specify the order of the operator $L^*L$ that defines $S_\Delta(x)$.*

If $k = 0$ and the coefficients of the operator $L$ are sufficiently smooth, then the spline $S_\Delta(x)$ has continuous derivatives of all orders and satisfies the equation (1) everywhere on $[a,b]$; in this case, the continuity of the $(2n-1)$th derivative implies continuity of $2n$-th and all higher derivatives. Thus, for this important class of differential operators, the condition that the defect vanishes is equivalent to the continuity of all derivatives of the spline. For ordinary splines (defect 1), breaks in the $(2n-1)$-th derivative are allowed only at the grid nodes.

Next, we discuss some recent results on splines.

It should be noted the work [40] provided an overview of the results on the convergence of the interpolation process for polynomial splines and derivatives for 50 years until 2013.

Digital bitmaps often need to be rendered at higher and lower resolutions. Digital image resampling is an integral part of image processing. The most efficient and sufficiently accurate image resampling methods can create strong fluctuations near sharp color transitions. To improve this, [9] considered tension splines. The presented spline stretching procedure provides an elegant solution to image resampling by constructing a smooth approximation with a clear non-oscillatory discontinuity resolution. To demonstrate the effectiveness of the proposed algorithm, numerical results are given on real digital images.

In [15], a hyperbolic stretched spline was defined as a solution to a differential multipoint boundary value problem. A discrete hyperbolic stretched spline was obtained using difference analogues of differential operators; Its calculation does not require exponential functions, even if its continuous extension is still a spline of hyperbolic type. In that paper, the main computational aspects were considered, and the main features of this approach were shown.

Most papers discuss fourth-order tension splines applied to a convex (or monotonic) interpolation problem or a two-point boundary value problem for ordinary differential equations. Higher-order tension splines are described in several articles, but no appendices are given. A possible reason for this is the lack of an appropriate algorithm for their evaluation. In [8], the authors presented an explicit algorithm for evaluating splines with tension of arbitrary order. They paid special attention to the stable and accurate calculation of hyperbolic functions used in the algorithm.

In [29] presented a family of trigonometric stretch curves similar to cubic Bezier curves. Some properties of the proposed curves are discussed. The authors proposed an efficient interpolation method based on stretched trigonometric splines with various properties such as unit splitting, geometric invariance, convex hull property, etc. This new interpolation method is used to construct curves and surfaces. In addition, it is possible to locally correct the shape of constructed curves and surfaces by changing the tension parameter, which is included mainly because of its importance for object rendering. To illustrate the performance and practical value of this model, as well as its accuracy and efficiency, the authors presented several simulation examples.

In cubic interpolation splines theory, an algorithm requires only $O(n)$ arithmetic operations. In addition, smoothing cubic splines can be calculated using the Reinsch algorithm, which reduces their calculation to interpolating cubic splines and performs arithmetic operations in $O(n)$. [28] shows that many features of tuning a polynomial cubic spline carry over to a broader class of $L$ splines, where $L$ is a 4th-order linear differential operator with constant coefficients. The criteria are given so that the associated matrix $R$ is strictly diagonally dominant, implying a fast interpolation algorithm exists. At the end of the article, the authors gave an example of two $L$ splines interpolating data.

In the introductory part of these papers [6, 10, 16, 19, 20, 34], the problem of interpolation by classical methods, i.e., algebraic and trigonometric polynomials, was mentioned. The lack of interpolation by algebraic polynomials is noted. Then, the problem of interpolation by polynomial splines, free from this shortcoming, was given. At the same time, natural splines that give a minimum norm in Sobolev space and fundamental splines in various Hilbert spaces were discussed. Finally, the problem of constructing optimal interpolation formulas in Banach spaces was presented by Sobolev. These articles then discussed the construction of optimal interpolation formulas that are exact for both algebraic polynomials and trigonometric functions. The authors found an extremal function that allows us to obtain an upper bound for the error of the interpolation formula in a given Hilbert space. A system of linear equations for the (optimal) coefficients was obtained, which gives the smallest value of the error. The solution of this system gives the coefficients of the optimal interpolation formula.

In [4, 5, 21], using the Sobolev method, the first part of the optimal interpolation problems were solved, i.e., an explicit expression for the squared norm of the error functional was found, and a system of linear algebraic equations for the coefficients of the optimal interpolation formula was obtained. In the work [5], the problem of constructing optimal interpolation formulas in a Hilbert space was studied. Here, using the Sobolev method, an algorithm was given for solving a system of linear algebraic equations for the coefficients of optimal interpolation formulas.

In the paper [30] using S.L. Sobolev's method interpolation splines minimizing the semi-norm in a Hilbert space were constructed. Explicit formulas for coefficients of interpolation splines were obtained. The obtained interpolation spline was exact for polynomials of degree $m-2$ and $e^{-x}$. Also some numerical results were presented.

In the work [13], using S.L. Sobolev's method, interpolation $D^m$-splines that minimizes the expression $\int_0^1(\varphi^{(m)}(x))^2dx$ in the $L_2^{(m)}(0,1)$ space are constructed. Explicit formulas for the coefficients of the interpolation splines are obtained. The obtained interpolation spline is exact for polynomials of degree $m-1$. Some numerical experiments were presented. Moreover the connection between the obtained interpolation splines and the optimal quadrature formulas were shown.

Several works are devoted to the construction of optimal quadrature formulas based on the Sobolev method and their application. For instance, the calculation of Fourier coefficients (see, [11, 12]), numerical integration of highly oscillatory integrals (see, [22–24, 26, 32, 41]), in the papers [1, 33] the optimal quadrature formulas with derivatives for approximate solution of a singular integral equation of the first kind with Cauchy kernel, the optimal quadrature formulas for approximate integration fractional integrals [17, 18, 38] were constructed, etc.

Additionally, achieving a high order of approximation requires the incorporation of optimal interpolation formulas that include derivatives. Various researchers have undertaken the challenge of developing such optimal interpolation formulas, as discussed by Shadimetov, Hayotov, and Nuraliev(2019) [31].

Further, the remaining sections of this article are organized as follows. In the Section 2, definitions of some mathematical concepts that will be used in this article are given. In the Section 3, the problem of constructing an interpolation formula with derivative is posed. In the Section 4, the norm of the error functional of the optimal interpolation formula is found. In the fifth section, the minimum value of the norm of the error functional is obtained by finding the conditional extremum of a multivariable Lagrange function. In the sixth section, an algorithm of finding the coefficients of the interpolation formula is presented. Finally, in the Section 7, exactness of optimal interpolation formula is discussed and some numerical results are considered.

## 2. Mathematical preliminaries

To construct a discrete operator, we utilize the principles of generalized functions and definitions in [39].

**Discrete argument functions**

Below we use mainly the concept of discrete argument functions and operations on them. The theory of discrete argument functions was presented in [35, 36]. We provide a few definitions of functions with discrete arguments to ensure comprehensiveness.

**Definition 2.1.** *The function $\varphi[\beta]$ is a* function of discrete argument *if it is given on some set of integer values of $\beta$.*

**Definition 2.2.** The inner product *of two discrete argument functions $\varphi[\beta]$ and $\psi[\beta]$ is given by*

$$[\varphi[\beta], \psi[\beta]] = \sum_{\beta=-\infty}^{\infty} \varphi[\beta] \cdot \psi[\beta],$$

*if the series on the right-hand side converges absolutely.*

**Convolution**

The convolution of $\varphi$ and $\psi$ is represented as $\varphi(x) * \psi(x)$, which denotes the operator using the symbol $*$. To compute the convolution of arbitrary continuous functions $\varphi$ and $\psi$, follow the steps outlined below

$$\varphi(x) * \psi(x) = \int_{-\infty}^{+\infty} \varphi(x-y) \cdot \psi(y)\mathrm{d}y = \int_{-\infty}^{+\infty} \psi(x-y) \cdot \varphi(y)\mathrm{d}y.$$

**Definition 2.3.** *The convolution of two discrete functions $\varphi[\beta]$ and $\psi[\beta]$ is the inner product*

$$\varphi[\beta] * \psi[\beta] = \sum_{\gamma=-\infty}^{+\infty} \varphi[\gamma]\psi[\beta-\gamma] = \sum_{\gamma=-\infty}^{+\infty} \varphi[\beta-\gamma]\psi[\gamma].$$

**The Dirac delta-function**

The Dirac delta-function, also known as the unit impulse, is a generalized function defined over the real line. It is zero everywhere except at zero and the integral of the function over the entire real line is equal to one. The Dirac delta-function possesses the following properties

$$\delta(hx) = h^{-1}\delta(x),$$

$$\delta(x-a)f(x) = \delta(x-a)f(a),$$

$$\delta^{(\alpha)}(x) * f(x) = f^{(\alpha)}(x),$$

$$\phi_0(x) = \sum_{\beta=-\infty}^{+\infty} \delta(x-\beta), \ \sum_{\beta=-\infty}^{+\infty} e^{2\pi i x\beta} = \sum_{\beta=-\infty}^{+\infty} \delta(x-\beta).$$

## 3. Statement of the problem

### 3.1. *Optimal interpolation formula in the space $L_2^{(1)}(0,1)$*

Assuming that, given the table of the values $\varphi(x_\beta)$, $\beta = 0, 1, \ldots, N$ of functions $\varphi$ at points $x_\beta \in [0,1]$ ($x_\beta = h\beta$, $h = 1/N$). It is required approximate functions $\varphi$ by another more simple function $P_\varphi$, i.e.

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta(x)\varphi(x_\beta), \tag{2}$$

in the Sobolev space $L_2^{(1)}(0,1)$. The elements of this space are absolute continuous and square-integrable with first-order generalized derivative. Here $C_\beta(x)$ and $x_\beta(\in[0,1])$ are the coefficients and the nodes of the interpolation formula (2), respectively.

**Theorem 3.1.** *([3]) Coefficients of the optimal interpolation formula of the form* (2) *in the space* $L_2^{(1)}(0,1)$ *have the form*

$$C_\beta(x) = \frac{1}{2h}(|x-h(\beta-1)| + |x-h(\beta+1)| - 2|x-h\beta|), \ \ \beta = 0, 1, \ldots, N. \tag{3}$$

These optimal coefficients (3) can be rewritten in the following form

$$C_0(x) = \begin{cases} \frac{h-x}{h}, & 0 \le x \le h, \\ 0, & h < x \le 1, \end{cases} \tag{4}$$

$$C_\beta(x) = \begin{cases} \frac{x+h-h\beta}{h}, & h(\beta-1) < x \le h\beta, \\ \frac{h-x+h\beta}{h}, & h\beta < x \le h(\beta+1), \quad \beta = 1, 2, \ldots, N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

$$C_N(x) = \begin{cases} 0, & 0 \le x \le h(N-1), \\ \frac{h-1+x}{h}, & h(N-1) < x \le 1. \end{cases} \tag{6}$$

*3.2. The problem of constructing an interpolation formula with derivative*

In this work, we consider the problem of interpolating a function $\varphi(x)$ given by values of the function and its first derivative

$$\varphi(x_\beta),\ \varphi'(x_\beta),$$

at points $x_\beta$, $\beta = 0, 1, \dots, N$ $(0 = x_0 < x_1 < \dots < x_N = 1)$ in the space $L_2^{(2)}(0,1)$. Here, $L_2^{(2)}(0,1)$ is the Hilbert space of functions that, the fist-order generalized derivative is absolute continuous in the interval $[0,1]$ and a second generalized derivative belongs $L_2(0,1)$ space. The space is equipped with the norm

$$\|\varphi\|_{L_2^{(2)}} = \sqrt{\int_0^1 (\varphi''(x))^2 \mathrm{d}x}.$$

So, we consider the problem of interpolation of functions $\varphi(x)$ by a more straightforward function $P_\varphi(x)$ as follows

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta(x)\varphi(x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}(x)\varphi'(x_\beta). \tag{7}$$

Here $C_\beta(x)$ are the coefficients of the optimal interpolation formula (2).

The error associated with the approximate equality (7) takes the form of a difference expressed as

$$R_\varphi(x) = \varphi(x) - P_\varphi(x). \tag{8}$$

It should be noted that in this work, when we consider the approximation of the form (7), we impose the condition that the class of functions that transforms this approximate equality into an exact equality in $L_2^{(2)}(0,1)$ space should be the class of all linear functions. If we take $\varphi_1(x) = 1$ and $\varphi_2(x) = x$ as the basis functions for the space of all linear functions, the imposition of

$$R_{\varphi_1}(x) = \varphi_1(x) - P_{\varphi_1}(x) = 0,\ \ (R, x) = 0, \tag{9}$$
$$R_{\varphi_2}(x) = \varphi_2(x) - P_{\varphi_2}(x) = 0,\ \ (R, 1) = 0, \tag{10}$$

conditions on the error functional $R_\varphi(x)$ is enough for the approximation formula (7) to be exact for all linear functions.

Then in the space $L_2^{(2)}(0,1)$ at every fixed point $x = z$ of the interval $[0,1]$ the error (8) defines a linear continuous functional

$$R(x,z) = \delta(x-z) - \sum_{\beta=0}^{N} C_\beta(z)\cdot\delta(x-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}(z)\cdot\delta'(x-x_\beta), \tag{11}$$

and

$$\begin{aligned}
(R,\varphi) &= \int_{-\infty}^{\infty} R(x,z)\cdot\varphi(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\left(\delta(x-z) - \sum_{\beta=0}^{N} C_\beta(z)\cdot\delta(x-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}(z)\cdot\delta'(x-x_\beta)\right)\cdot\varphi(x)\mathrm{d}x \\
&= \varphi(z) - \sum_{\beta=0}^{N} C_\beta(z)\cdot\varphi(x_\beta) - \sum_{\beta=0}^{N} C_{\beta,1}(z)\cdot\varphi'(x_\beta).
\end{aligned}$$

In order to construct an optimal interpolation formula in the form of (7), it is imperative to compute the norm $|R|_{L_2^{(2)}}$ of the error functional (11). This necessity arises from the fact that, according to the Cauchy-Schwarz inequality, the estimation of the error (7) is expressed by the norm as follows:

$$\|(R,\varphi)\| \leq \|R\|_{L_2^{(2)*}} \cdot \|\varphi\|_{L_2^{(2)}}.$$

It is easy to see that the norm $\|R\|_{L_2^{(2)*}}$ depends on the coefficients $C_{\beta,1}(z)$. Then it should be found the smallest value of the norm $\|R\|_{L_2^{(2)*}}$ by the coefficient $C_{\beta,1}$. That is, it should be calculated the quantity

$$\inf_{C_{\beta,1}} \|R\|_{L_2^{(2)*}}. \tag{12}$$

The coefficients $\mathring{C}_{\beta,1}$ reaching the value (12) we call the optimal coefficients.
Thus, consequently

- we calculate the norm $\|R\|_{L_2^{(2)*}}$,
- we find $\mathring{C}_{\beta,1}$ which gives (12).

## 4. The error functional of the interpolation formula

To calculate $\|R\|_{L_2^{(2)*}}$, we use the extremal function $U_R(x)$ [35–37] satisfying the following equality

$$(R, U_R) = \|R\|_{L_2^{(2)*}} \cdot \|U_R\|_{L_2^{(2)}},$$

here,

$$U_R(x) = R(x) * G_2(x) + p_1 x + p_0,$$

where $p_0$, $p_1$ are unknown real coefficients and

$$G_2(x) = \frac{|x|^3}{12}.$$

Now we calculate the convolution

$$\begin{aligned} R(x) * G_2(x) &= \int_{-\infty}^{\infty} R(y)\cdot G_2(x-y)\mathrm{d}y \\ &= G_2(x-z) - \sum_{\beta=0}^{N} C_\beta(z) G_2(x-x_\beta) - \sum_{\beta=0}^{N} C_{\beta,1}(z) G_2'(x-x_\beta). \end{aligned}$$

We calculate the convolution in the above equation separately

$$\int_{-\infty}^{\infty}\left(\delta(y-z) - \sum_{\beta=0}^{N} C_\beta \delta(y-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\delta'(y-x_\beta)\right)\cdot \frac{|x-y|^3}{12}\mathrm{d}x$$

$$\begin{aligned} &= \int_{-\infty}^{\infty}\left(\delta(y-z) - \sum_{\beta=0}^{N} C_\beta \delta(y-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\delta'(y-x_\beta)\right)\cdot \frac{\mathrm{sgn}(x-y)(x-y)^3}{12}\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\left(\delta(y-z) - \sum_{\beta=0}^{N} C_\beta \delta(y-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\delta'(y-x_\beta)\right)\cdot \frac{\mathrm{sgn}(y-x)(y-x)^3}{12}\mathrm{d}x \\ &= G_2(x-z) - \sum_{\beta=0}^{N} C_\beta \frac{|x-x_\beta|^3}{12} - \sum_{\beta=0}^{N} C_{\beta,1}\frac{\mathrm{sgn}(x_\beta - x)(x_\beta - x)^2}{4}. \end{aligned}$$

So,

$$R(x) * G_2(x) = \frac{|x-z|^3}{12} - \sum_{\beta=0}^{N} C_\beta \frac{|x-x_\beta|^3}{12} + \sum_{\beta=0}^{N} C_{\beta,1} \frac{\operatorname{sgn}(x-x_\beta)(x-x_\beta)^2}{4}. \tag{13}$$

Then the extremal function has the following form

$$U_R(x) = \frac{|x-z|^3}{12} - \sum_{\beta=0}^{N} C_\beta \frac{|x-x_\beta|^3}{12} + \sum_{\beta=0}^{N} C_{\beta,1} \frac{\operatorname{sgn}(x-x_\beta)(x-x_\beta)^2}{4} + p_1 x + p_0.$$

Taking into account (9), (10) and above last expression, we have

$$\begin{aligned}(R, U_R) &= \int_{-\infty}^{\infty} R(x) \cdot U_R(x) \mathrm{d}x = \int_{-\infty}^{\infty} R(x) \cdot (R(x) * G_2(x) + p_1 x + p_0)\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} R(x) \cdot (R(x) * G_2(x))\, \mathrm{d}x + p_1 (R, x) + p_0 (R, 1) \\ &= \int_{-\infty}^{\infty} R(x) \cdot (R(x) * G_2(x))\, \mathrm{d}x. \end{aligned} \tag{14}$$

Using expression (13), from expression (14) we obtain

$$\begin{aligned}(R, U_R) = &\int_{-\infty}^{\infty} \left( \delta(x-z) - \sum_{\gamma=0}^{N} C_\gamma \cdot \delta(x-x_\gamma) + \sum_{\gamma=0}^{N} C_{\gamma,1} \cdot \delta'(x-x_\gamma) \right) \\ &\times \left( \frac{|x-z|^3}{12} - \sum_{\beta=0}^{N} C_\beta \frac{|x-x_\beta|^3}{12} + \sum_{\beta=0}^{N} C_{\beta,1} \frac{\operatorname{sgn}(x-x_\beta)(x-x_\beta)^2}{4} \right) \mathrm{d}x \\ &= \frac{|z-z|^3}{12} - \sum_{\beta=0}^{N} C_\beta \frac{|z-x_\beta|^3}{12} + \sum_{\beta=0}^{N} C_{\beta,1} \frac{\operatorname{sgn}(z-x_\beta)(z-x_\beta)^2}{4} \\ &- \sum_{\gamma=0}^{N} C_\gamma \frac{|x_\gamma - z|^3}{12} + \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma \frac{|x_\gamma - x_\beta|^3}{12} - \sum_{\gamma=0}^{N}\sum_{\beta=0}^{N} C_\gamma C_{\beta,1} \frac{\operatorname{sgn}(x_\gamma - x_\beta)(x_\gamma - x_\beta)^2}{4} \\ &- \sum_{\gamma=0}^{N} C_{\gamma,1} \frac{\operatorname{sgn}(x_\gamma - z)(x_\gamma - z)^2}{4} + \sum_{\gamma=0}^{N}\sum_{\beta=0}^{N} C_{\gamma,1} C_\beta \frac{\operatorname{sgn}(x_\gamma - x_\beta)(x_\gamma - x_\beta)^2}{4} \\ &- \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,1} C_{\gamma,1} \frac{\operatorname{sgn}(x_\gamma - x_\beta)(x_\gamma - x_\beta)}{2}. \end{aligned}$$

So, we have

$$\begin{aligned}(R, U_R) = &- \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,1} C_{\gamma,1} \frac{|x_\beta - x_\gamma|}{2} + 2 \sum_{\beta=0}^{N} C_{\beta,1} \sum_{\gamma=0}^{N} C_\gamma \frac{\operatorname{sgn}(x_\beta - x_\gamma)(x_\beta - x_\gamma)^2}{4} \\ &- 2 \sum_{\beta=0}^{N} C_{\beta,1} \frac{\operatorname{sgn}(x_\beta - z)(x_\beta - z)^2}{4} + \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma \frac{|x_\beta - x_\gamma|^3}{12} - 2 \sum_{\beta=0}^{N} C_\beta \frac{|x_\beta - z|^3}{12} \\ &= - \left( \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,1} C_{\gamma,1} \frac{|x_\beta - x_\gamma|}{2} - \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma \frac{|x_\beta - x_\gamma|^3}{12} + 2 \sum_{\beta=0}^{N} C_\beta \frac{|x_\beta - z|^3}{12} \right. \\ &\left. - 2 \sum_{\beta=0}^{N} C_{\beta,1} \left( \sum_{\gamma=0}^{N} C_\gamma \frac{\operatorname{sgn}(x_\beta - x_\gamma)(x_\beta - x_\gamma)^2}{4} - \frac{\operatorname{sgn}(x_\beta - z)(x_\beta - z)^2}{4} \right) \right). \end{aligned}$$

We also have the following equalities

$$\begin{aligned}(R,1) &= \int_{-\infty}^{\infty} R(x)1\mathrm{d}x = \int_{-\infty}^{\infty}\left(\delta(x-z) - \sum_{\beta=0}^{N} C_\beta\delta(x-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\delta'(x-x_\beta)\right)1\mathrm{d}x \\ &= 1 - \sum_{\beta=0}^{N} C_\beta - \sum_{\beta=0}^{N} C_{\beta,1}(1)' = 1 - \sum_{\beta=0}^{N} C_\beta = 0.\end{aligned}$$

Then

$$\sum_{\beta=0}^{N} C_\beta = 1,$$

$$\begin{aligned}(R,x) &= \int_{-\infty}^{\infty} R(x)x\mathrm{d}x = \int_{-\infty}^{\infty}\left(\delta(x-z) - \sum_{\beta=0}^{N} C_\beta\delta(x-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}\delta'(x-x_\beta)\right)x\mathrm{d}x \\ &= z - \sum_{\beta=0}^{N} C_\beta \cdot x_\beta - \sum_{\beta=0}^{N} C_{\beta,1} = 0.\end{aligned}$$

From the last equation we have the following expression

$$\sum_{\beta=0}^{N} C_{\beta,1} = z - \sum_{\beta=0}^{N} C_\beta \cdot x_\beta.$$

Therefore, we get the following expression for the norm of the error functional of the interpolation formula (7)

$$\begin{aligned}\|R\|^2_{L_2^{(2)*}} &= -\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,1}C_{\gamma,1}\frac{|x_\beta - x_\gamma|}{2} + \sum_{\beta=0}^{N} C_{\beta,1}\left(\sum_{\gamma=0}^{N} C_\gamma \frac{\mathrm{sgn}(x_\beta - x_\gamma)(x_\beta - x_\gamma)^2}{2}\right. \\ &\left. - \frac{\mathrm{sgn}(x_\beta - z)(x_\beta - z)^2}{2}\right) + \sum_{\beta=0}^{N} C_\beta\left(\sum_{\gamma=0}^{N} C_\gamma \frac{|x_\beta - x_\gamma|^3}{12} - \frac{|x_\beta - z|^3}{6}\right). \qquad (15)\end{aligned}$$

## 5. The minimum value of the norm of the error functional

In the following two conditions, it is necessary to find the smallest value of the expression (15) according to the coefficients $C_{\beta,1}$

$$\begin{aligned}&\sum_{\beta=0}^{N} C_\beta = 1, \\ &\sum_{\beta=0}^{N} C_{\beta,1} = z - \sum_{\beta=0}^{N} C_\beta \cdot x_\beta. \qquad (16)\end{aligned}$$

We find the minimum of expression (15) under condition (16). For this, we come to the problem of finding the conditional extremum of a multivariable Lagrange function. We construct the Lagrange function in the following form

$$\Lambda = \|R\|^2_{L_2^{(2)*}} + 2\lambda(R,x) = \|R\|^2_{L_2^{(2)*}} + 2\lambda\left(z - \sum_{\beta=0}^{N} C_\beta x_\beta - \sum_{\beta=0}^{N} C_{\beta,1}\right).$$

In that case, equating to 0 the partial derivatives of the function $\Lambda$ by $C_{\beta,1}$ and $\lambda$, we get the following system of the linear equations

$$\begin{aligned} \sum_{\gamma=0}^{N} C_{\gamma,1}\frac{|x_\beta - x_\gamma|}{2} + \lambda &= f(x_\beta), \;\; \beta = 0,1,\dots,N, \\ \sum_{\gamma=0}^{N} C_{\gamma,1} &= z - \sum_{\gamma=0}^{N} C_\gamma x_\gamma, \end{aligned}$$

here,

$$f(x_\beta) = \sum_{\gamma=0}^{N} C_\gamma \frac{\mathrm{sgn}(x_\beta - x_\gamma)(x_\beta - x_\gamma)^2}{4} - \frac{\mathrm{sgn}(x_\beta - z)(x_\beta - z)^2}{4}.$$

Also, considering $\sum_{\gamma=0}^{N} C_\gamma h\gamma = z$, we get $\sum_{\gamma=0}^{N} C_{\gamma,1} = 0$.

So, the last system takes the form

$$\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{|h\beta - h\gamma|}{2} + \lambda = f(h\beta, z), \;\; \beta = 0,1,\dots,N, \tag{17}$$

$$\sum_{\gamma=0}^{N} C_{\gamma,1} = 0, \tag{18}$$

where

$$f(h\beta, z) = \sum_{\gamma=0}^{N} C_\gamma \frac{\mathrm{sgn}(h\beta - h\gamma)(h\beta - h\gamma)^2}{4} - \frac{\mathrm{sgn}(h\beta - z)(h\beta - z)^2}{4}. \tag{19}$$

## 6. An algorithm of finding the coefficients of the interpolation formula

In order to find an analytical solution to the system (17)-(18), we need the discrete analogue [35]

$$D_1(h\beta) = \begin{cases} 0, & |\beta| \geq 2, \\ \frac{1}{h^2}, & |\beta| = 1, \\ -\frac{2}{h^2}, & \beta = 0 \end{cases} \tag{20}$$

of the differential operator $\frac{d^2}{dx^2}$. The discrete operator (20) has the following properties [35]

$$\begin{aligned} &D_1(h\beta) * 1 = 0, \\ &D_1(h\beta) * (h\beta) = 0, \\ &hD_1(h\beta) * \frac{|h\beta|}{2} = \delta_d(h\beta), \end{aligned} \tag{21}$$

where

$$\delta_d(h\beta) = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \neq 0. \end{cases}$$

We consider the left-hand side of the expression (17) as a new function

$$U_1(h\beta) = \sum_{\gamma=0}^{N} C_{\gamma,1} \cdot \frac{|h\beta - h\gamma|}{2} + \lambda. \tag{22}$$

Here, $C_{\gamma,1}$ is considered as a discrete function of the integer-valued argument. For $\gamma = -1, -2, \ldots$ and $\gamma = N+1, N+2, \ldots$ we define $C_{\gamma,1}$ as 0.

As a result, based on the definition of the convolution operation of functions with discrete arguments, from expression (22) we arrive at the following

$$U_1(h\beta) = C_{\beta,1} * \frac{|h\beta|}{2} + \lambda.$$

In that case, based on properties (21), we get the following

$$C_{\beta,1} = hD_1(h\beta) * U_1(h\beta). \tag{23}$$

In order to find the coefficients $C_{\beta,1}$ from relation (23), we must first determine the function $U_1(h\beta)$ at all integer values of $\beta$.

Based on (15), the equality

$$U_1(h\beta) = f(h\beta, z) \tag{24}$$

is valid for $\beta = 0, 1, \ldots, N$. Now we find representation of $U_1(h\beta)$ at $\beta < 0$ and $\beta > N$.

Let $\beta = -1, -2, \ldots$. Then from (22) we get the following

$$\begin{aligned} U_1(h\beta) &= \sum_{\gamma=0}^{N} C_{\gamma,1} \cdot \frac{-(h\beta - h\gamma)}{2} + \lambda = -\frac{h\beta}{2}\sum_{\gamma=0}^{N} C_{\gamma,1} + \sum_{\gamma=0}^{N} C_{\gamma,1} \cdot \frac{h\gamma}{2} + \lambda \\ &= \lambda + \frac{1}{2}\sum_{\gamma=0}^{N} C_{\gamma,1} \cdot (h\gamma) = \lambda^-. \end{aligned} \tag{25}$$

Similarly, for $\beta = N+1, N+2, \ldots$, we have

$$U_1(h\beta) = \lambda - \frac{1}{2}\sum_{\gamma=0}^{N} C_{\gamma,1} \cdot (h\gamma) = \lambda^+. \tag{26}$$

From (24)-(26), we get the following

$$U_1(h\beta) = \begin{cases} \lambda^-, & \beta = -1, -2, \ldots, \\ f(h\beta, z), & \beta = 0, 1, \ldots, N, \\ \lambda^+, & \beta = N+1, N+2, \ldots. \end{cases}$$

It is easy to show that

$$\lambda^- = f(0, z), \lambda^+ = f(1, z).$$

So,

$$U_1(h\beta) = \begin{cases} f(0, z), & \beta = -1, -2, \ldots, \\ f(h\beta, z), & \beta = 0, 1, \ldots, N, \\ f(1, z), & \beta = N+1, N+2, \ldots. \end{cases} \tag{27}$$

Using equation (27), we find coefficients $C_{\beta,1}$ based on equation (23). Then

$$C_{\beta,1} = hD_1(h\beta) * U_1(h\beta) = h\sum_{\gamma=-\infty}^{\infty} D_1(h\beta - h\gamma) \cdot U_1(h\gamma),$$

$$C_{\beta,1} = h\left[\sum_{\gamma=0}^{N} D_1(h\beta - h\gamma) f(h\gamma, z) + \sum_{\gamma=1}^{\infty} D_1(h\beta + h\gamma) f(0, z) + \sum_{\gamma=1}^{\infty} D_1(h(N+\gamma) - h\beta) f(1, z)\right].$$

From the above expression, for $\beta = 0$ we have the following

$$\begin{aligned} C_{0,1} &= h\left[\sum_{\gamma=0}^{N} D_1(h\gamma)\cdot f(h\gamma,z) + \sum_{\gamma=1}^{\infty} D_1(h\gamma)\cdot f(0,z) + \sum_{\gamma=1}^{\infty} D_1(h(N+\gamma))\cdot f(1,z)\right] \\ &= h\left[D_1(0)\cdot f(0,z) + D_1(h)\cdot f(h,z) + D_1(h)\cdot f(0,z)\right] \\ &= h\left[-\frac{2}{h^2}\cdot f(0,z) + \frac{1}{h^2}\cdot f(h,z) + \frac{1}{h^2}\cdot f(0,z)\right] = \frac{1}{h}[f(h,z) - f(0,z)]. \end{aligned}$$

Thus,

$$C_{0,1} = \frac{1}{h}(f(h,z) - f(0,z)). \tag{28}$$

Now, from (28) in cases where $\beta = 1, 2, \ldots, N-1$, we have the following

$$\begin{aligned} C_{\beta,1} &= h\sum_{\gamma=0}^{N} D_1(h\beta - h\gamma)\cdot f(h\gamma,z) \\ &= h\left[D_1(h)\cdot f(h(\beta-1),z) + D_1(0)\cdot f(h\beta,z) + D_1(h)\cdot f(h(\beta+1),z)\right] \\ &= \frac{1}{h}\left[f(h(\beta-1),z) - 2f(h\beta,z) + f(h(\beta+1),z)\right]. \end{aligned}$$

Then,

$$C_{\beta,1} = \frac{1}{h}\left[f(h(\beta-1),z) - 2f(h\beta,z) + f(h(\beta+1),z)\right], \quad \beta = 1, 2, \ldots, N-1. \tag{29}$$

Finally, from (28) for $\beta = N$ we get the following

$$\begin{aligned} C_{N,1} &= h\left[\sum_{\gamma=0}^{N} D_1(hN - h\gamma)\cdot f(h\gamma,z) + \sum_{\gamma=1}^{\infty} D_1(h(N+\gamma))\cdot f_1(0) + \sum_{\gamma=1}^{\infty} D_1(h\gamma)\cdot f(1,z)\right] \\ &= h\left[D_1(0)\cdot f(1,z) + D_1(h)\cdot f((N-1)h,z) + D_1(h)\cdot f(1,z)\right]. \end{aligned}$$

Thus we get the following

$$C_{N,1} = \frac{1}{h}[f(1-h,z) - f(1,z)]. \tag{30}$$

Taking into account equality (19), simplifying the expressions (28), (29), and (30) obtained for the coefficients, we get the following result.

**Theorem 6.1.** *Coefficients of the optimal interpolation formula of the form* (7) *in the space* $L_2^{(2)}(0,1)$ *have the form*

$$C_{0,1}(z) = \begin{cases} \frac{z(h-z)}{2h}, & 0 \le z \le h, \\ 0, & h < z \le 1, \end{cases}$$

$$C_{\beta,1}(z) = \begin{cases} \frac{(z-h\beta)^2 + h(z-h\beta)}{2h}, & h(\beta-1) < z \le h\beta, \\ \frac{-(z-h\beta)^2 + h(z-h\beta)}{2h}, & h\beta < z \le h(\beta+1), \\ 0, & \textit{otherwise}, \end{cases} \quad \beta = 1, 2, \ldots, N-1,$$

$$C_{N,1}(z) = \begin{cases} 0, & 0 \le z \le h(N-1), \\ \frac{(z-1)(z-1+h)}{2h}, & h(N-1) < z \le 1. \end{cases}$$

In this way, we have the optimal interpolation formula

$$P_\varphi(x) = \sum_{\beta=0}^{N}\left(C_\beta(x)\varphi(x_\beta) + C_{\beta,1}(x)\varphi'(x_\beta)\right)$$

in the space $L_2^{(2)}(0,1)$, which is exact for any linear function.

## 7. RESULTS AND DISCUSSION

*7.1. The exactness of the interpolation formula for the quadratic function*

Is the optimal interpolation formula

$$P_{\varphi}(z) \cong \sum_{\beta=0}^{N}\sum_{k=0}^{1} C_{\beta,k}(z)\cdot\varphi^{(k)}(x_\beta)$$

exact to the function $\varphi(x) = x^2$?

To answer this question we must determine the correctness of the following equality, let's analyze it step by step

$$P_{\varphi(z)}(z) = \sum_{\beta=0}^{N}\sum_{k=0}^{1} C_{\beta,k}(z)\cdot\varphi^{(k)}(x_\beta) = z^2. \tag{31}$$

We check that the equality (31) is appropriate for $h\beta \le z \le h\beta + h$ for the cases $\beta = 0, 1, ..., N-1$, separately. Initially, we consider $0 \le z \le h$, then

$$P_{z^2}(z) = \sum_{\beta=0}^{N} C_{\beta,0}(z)\cdot(h\beta)^2 + \sum_{\beta=0}^{N} C_{\beta,1}(z)\cdot 2(h\beta)$$

$$= C_{0,0}(z)\cdot 0^2 + C_{1,0}(z)\cdot h^2 + C_{0,1}(z)\cdot 2\cdot 0 + C_{1,1}(z)\cdot 2h = C_{1,0}(z)\cdot h^2 + C_{1,1}(z)2h.$$

Taking into account (4),(5),(6) and Theorem 6.1

$$C_{1,0}(z) = \begin{cases} \frac{z}{h}, & 0 \le z \le h, \\ \frac{2h-z}{h}, & h < z \le 2h, \\ 0, & 2h < z \le 1, \end{cases}$$

$$C_{1,1}(z) = \begin{cases} \frac{(z-h)^2+h(z-h)}{2h}, & 0 \le z \le h, \\ \frac{-(z-h)^2+h(z-h)}{2h}, & h < z \le 2h, \\ 0, & 2h < z \le 1, \end{cases}$$

and simplifying the following expression

$$P_{z^2}(z) = C_{1,0}(z)h^2 + C_{1,1}(z)2h = \frac{1}{h}\cdot z\cdot h^2 + \frac{1}{4h}\cdot(2(z-h)^2 + 2h(z-h))\cdot 2h = z^2.$$

So, when $0 \le z \le h$ we get $P_{z^2}(z) = z^2$.

Now we consider the case where $h(k-1) \le z \le hk,\ \ k = 2, \ldots, N-1$. In this case, four terms remain in the expression $P_{\varphi}(z)$

$$\begin{aligned} P_{\varphi}(z) &= \sum_{\beta=0}^{N} C_{\beta,0}(z)\cdot(h\beta)^2 + \sum_{\beta=0}^{N} C_{\beta,1}(z)\cdot 2h\beta \\ &= C_{k-1,0}(z)\cdot(h(k-1))^2 + C_{k,0}(z)\cdot(hk)^2 + C_{k-1,1}(z)\cdot 2(h(k-1)) + C_{k,1}(z)\cdot 2hk \\ &= \frac{1}{h}(hk-z)\cdot h^2(k-1)^2 + \frac{1}{h}(z-h(k-1))\cdot(hk)^2 + \frac{2h(k-1)}{4h}\cdot(-2(z-h(k-1))^2 + 2h(z-h(k-1))) \\ &\quad + \frac{2hk}{4h}\cdot(2(z-hk)^2 + 2h(z-hk)) = h(k-1)^2(hk-z) + hk^2(z-h(k-1)) \\ &\quad + (k-1)(-(z-h(k-1))^2 + h(z-h(k-1))) + k\cdot((z-hk)^2 + h(z-hk)) = z^2. \end{aligned}$$

Similarly, when $h(N-1) \le z \le 1$ it is easy to show that $P_{z^2}(z) = z^2$.

**Remark 7.1.** *The optimal interpolation formula* (2) *is exact to the function* $\varphi(x) = x^2$.

*7.2. Integration of the optimal interpolation formula* (7)

We find the form of the quadrature formula corresponding to the optimal interpolation formula (7). For this, we integrate the optimal interpolation formula (7) over $[0,1]$

$$\begin{aligned}\int_0^1 \varphi(z)\mathrm{d}z &\cong \int_0^1 \left(\sum_{\beta=0}^{N} C_\beta(z)\varphi(x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}(z)\varphi'(x_\beta)\right)\mathrm{d}z \\ &= \sum_{\beta=0}^{N}\int_0^1 C_\beta(z)\mathrm{d}z\cdot\varphi(x_\beta) + \sum_{\beta=0}^{N}\int_0^1 C_{\beta,1}(z)\mathrm{d}z\cdot\varphi'(x_\beta) \\ &= \sum_{\beta=0}^{N} A_\beta\varphi(x_\beta) + \sum_{\beta=0}^{N} A_{\beta,1}\varphi'(x_\beta). \end{aligned} \tag{32}$$

Taking into account (4),(5),(6) and Theorem 6.1, initially, we calculate for $\beta = 0$

$$\begin{aligned} &\varphi(x_0)\cdot\int_0^1 C_0(z)dz + \varphi'(x_0)\cdot\int_0^1 C_{0,1}(z)\mathrm{d}z \\ =\ & \varphi(x_0)\left(\int_0^h (1-\frac{z}{h})\mathrm{d}z + \int_h^1 0\mathrm{d}z\right) + \varphi'(x_0)\left(\int_0^h (\frac{z}{2}-\frac{z^2}{2h})\mathrm{d}z + \int_h^1 0\mathrm{d}z\right) \\ =\ & \varphi(x_0)\cdot\frac{h}{2} + \varphi'(x_0)\cdot\frac{h^2}{12}. \end{aligned}$$

So, we have $A_0 = \frac{h}{2}$; $A_{0,1} = \frac{h^2}{12}$.

Now, we consider the cases $1 \le \beta \le N-1$

$$\begin{aligned} &\varphi(x_\beta)\cdot\int_0^1 C_\beta(z)\mathrm{d}z + \varphi'(x_\beta)\cdot\int_0^1 C_{\beta,1}(z)\mathrm{d}z \\ &= \varphi(x_\beta)\left[\int_0^{h(\beta-1)} 0\mathrm{d}z + \int_{h(\beta+1)}^1 0\mathrm{d}z + \int_{h(\beta-1)}^{h\beta} C_\beta(z)\mathrm{d}z + \int_{h\beta}^{h(\beta+1)} C_\beta(z)\mathrm{d}z\right] \\ &+\varphi'(x_\beta)\left[\int_0^{h(\beta-1)} 0\mathrm{d}z + \int_{h(\beta+1)}^1 0\mathrm{d}z + \int_{h(\beta-1)}^{h\beta} C_{\beta,1}(z)\mathrm{d}z + \int_{h\beta}^{h(\beta+1)} C_{\beta,1}(z)\mathrm{d}z\right] \\ &= \varphi(x_\beta)\left[\left(\frac{z^2}{2h} + (1-\beta)z\right)\Bigg|_{h(\beta-1)}^{h\beta} + \left((1+\beta)z - \frac{z^2}{2h}\right)\Bigg|_{h\beta}^{h(\beta+1)}\right] \\ &+\frac{\varphi'(x_\beta)}{2h}\left[\left(\frac{(z-h\beta)^3}{3} + \frac{h(z-h\beta)^2}{2}\right)\Bigg|_{h(\beta-1)}^{h\beta} + \left(-\frac{(z-h\beta)^2}{3} + \frac{h(z-h\beta)^2}{2}\right)\Bigg|_{h\beta}^{h(\beta+1)}\right] \\ &= \varphi(x_\beta)\cdot h + \varphi(x_\beta)\cdot 0. \end{aligned}$$

Thus we get the following coefficients $A_\beta = h$, $A_{\beta,1} = 0$, $\beta = \overline{1,N-1}$.

As well as, we calculate the case $\beta = N$

$$\begin{aligned} &\varphi(x_N)\int_0^1 C_N(z)\mathrm{d}z + \varphi'(x_N)\int_0^1 C_{N,1}(z)\mathrm{d}z \\ &= \varphi(x_N)\int_{h(N-1)}^1 \frac{h-1+z}{h}\mathrm{d}z + \varphi'(x_N)\int_{h(N-1)}^1 \frac{(z-1)^2 + h(z-1)}{2h}\mathrm{d}z \\ &= \varphi(x_N)\cdot\frac{h}{2} + \varphi'(x_N)\cdot\left(\frac{-h^2}{12}\right), \end{aligned}$$

and we have $A_N = \frac{h}{2}$, $A_{N,1} = -\frac{h^2}{12}$.

Thus, the following is true.

**Theorem 7.2.** *Coefficients of quadrature formula* (32) *with the equal spaced nodes* $x_\beta = h\beta$ *have the following form*

$$A_\beta(z) = \begin{cases} \frac{h}{2}, & \beta = 0, \\ h, & 1 \leq \beta \leq< N-1, \\ \frac{h}{2}, & \beta = N, \end{cases}$$

$$A_{\beta,1}(z) = \begin{cases} \frac{h^2}{12}, & \beta = 0, \\ 0, & 1 \leq \beta \leq< N-1, \\ -\frac{h^2}{12}, & \beta = N, \end{cases}$$

We note here that the coefficients in Theorem 7.2 are the coefficients of the Euler-Maclaurin quadrature formula [7, 27]. Also, the Euler-Maclaurin-type optimal quadrature formulas were obtained in work [25].

**Remark 7.3.** *The quadrature formula* (32) *is exact to the functions* $1, x, andx^2$.

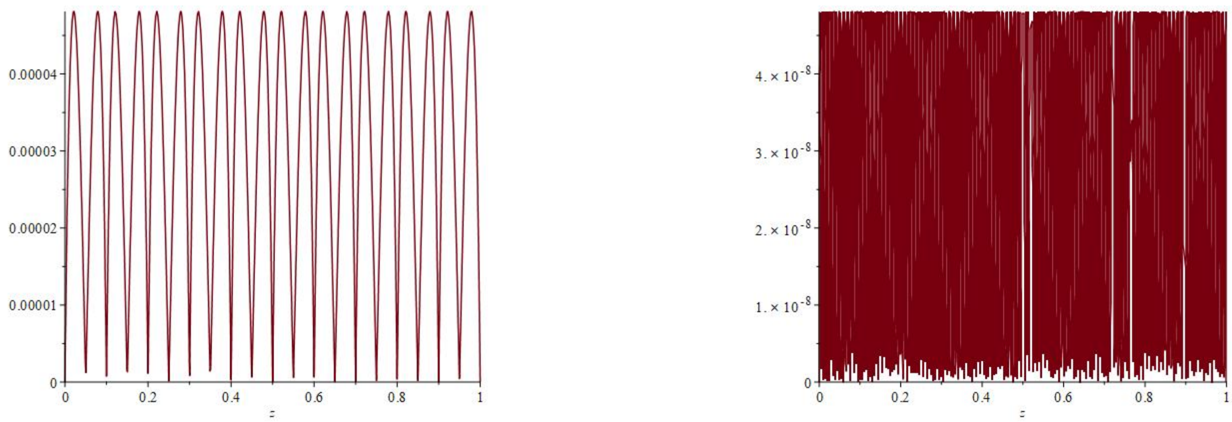
This remark can be easily demonstrated

$$\begin{aligned} \int_0^1 z^2 \mathrm{d}z &= A_0(h\cdot 0)^2 + A_1(h\cdot 1)^2 + \ldots + A_N(h\cdot N)^2 \\ &\quad + A_{0,1}(2h\cdot 0) + A_{1,1}(2h\cdot 1) + \ldots + A_{N,1}(2h\cdot N) \\ &= h\cdot h^2 + h\cdot 4h^2 + h\cdot 9h^2 + \ldots + \frac{h}{2}\cdot N^2\cdot h^2 + 2\cdot\left(-\frac{h^2}{12}\right) = \frac{1}{3}. \end{aligned}$$

*7.3. Numerical results*

We use the theoretical results obtained above to approximate several functions numerically. We also compare the numerical results with some results obtained in similar work.

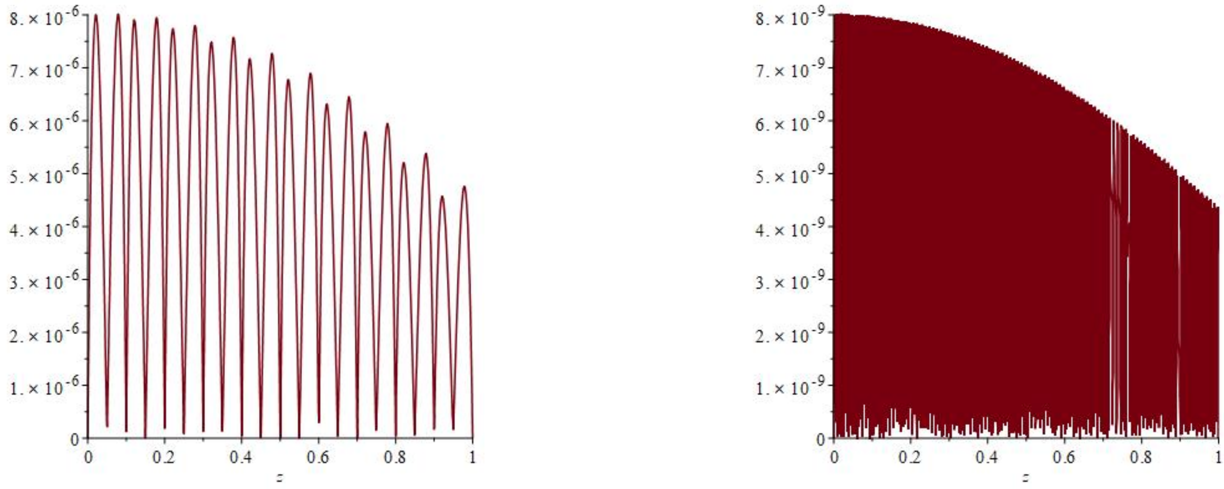
**Example 7.4.** *We analyze the approximation of the function* $\varphi(x) = x^3$ *using the optimal interpolation formula* (7) *in the interval* $[0, 1]$ *with a step size* $h = \frac{1}{N}$ *for both* $N = 10$ *and* $N = 100$.




(a) $N = 10$ (b) $N = 100$

Figure 1: The absolute error $|z^3 - P_{z^3}(z)|$ for $N = 10$ and $N = 100$.

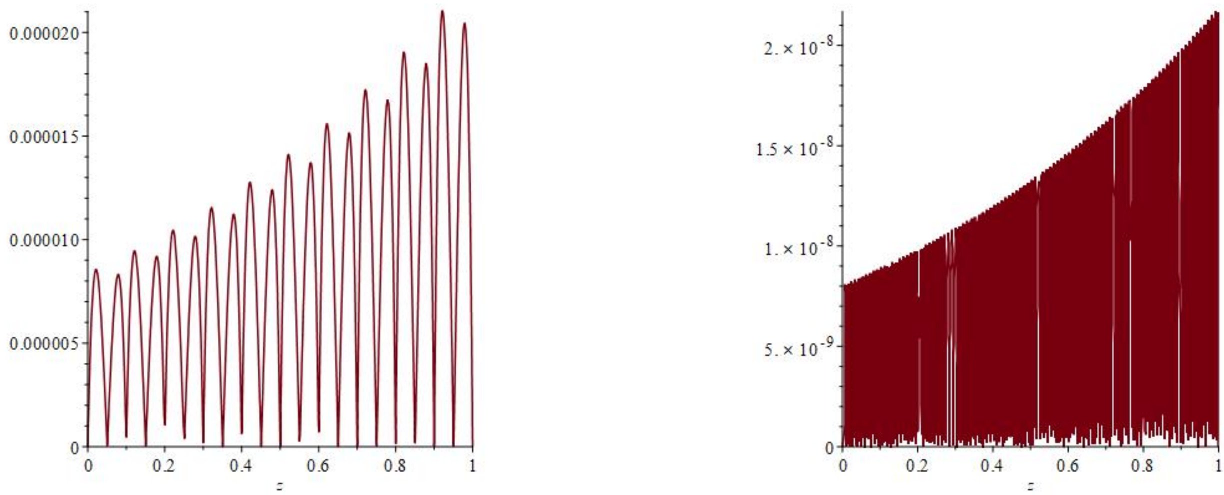
**Example 7.5.** *We analyze the approximation of the function* $\varphi(x) = \sin(x)$ *using the optimal interpolation formula* (7) *in the interval* $[0, 1]$ *with a step size* $h = \frac{1}{N}$ *for both* $N = 10$ *and* $N = 100$.




(a) $N = 10$ (b) $N = 100$

Figure 2: The absolute error $|\sin(z) - P_{\sin(z)}(z)|$ for $N = 10$ and $N = 100$.

**Example 7.6.** *We consider the approximation of the function $\varphi(x) = e^x$ using the optimal interpolation formula* (7) *in the interval* $[0, 1]$ *with a step size $h = \frac{1}{N}$ for both $N = 10$ and $N = 100$.*




(a) $N = 10$ (b) $N = 100$

Figure 3: The absolute error $|\exp(z) - P_{\exp(z)}(z)|$ for $N = 10$ and $N = 100$.

In conclusion, the order of approximation of the constructed optimal interpolation formula is $O(h^3)$. The obtain results can be applied to approximation boundary-value problems for ordinary differential equations using the Rayleigh-Ritz Method. Here, one can use coefficients of the optimal interpolation formula as basis functions.

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