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The results on solutions to a fuzzy problem

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Abstract. In this paper, we investigate the solutions of a fuzzy problem with fuzzy coefficient using the fuzzy Laplace transform method. The problem has four different solutions under the generalized Hukuhara differentiability. We examine whether the solutions are valid fuzzy functions. We give the results for the problem. Also, we explain the problem with a numerical example and show the solutions on the graphics. At the end of the paper, conclusions are given.

1. Introduction

Fuzzy differential equations are very useful for solving differential equations arising in the fields of engineering, physical mathematics and mathematics. Therefore, the topic of fuzzy differential equations has been growing rapidly in recent years.

There are several approaches to study fuzzy differential equations [1, 9, 16, 19].The first approach was to use Hukuhara differentiability for fuzzy number-valued functions. But, this approach has a drawback: the solution becomes fuzzier as time goes by [4]. Therefore, the fuzzy solution behaves quite differently from the crisp solution.

Bede and Gal introduced the strongly generalized differentiability [5]. The strongly generalized differentiability was studied in [4, 6, 11, 12, 20]. This concept allows us to resolve the above-mentioned shortcoming. The strongly generalized differentiability is defined for a larger class of fuzzy valued function than the Hukuhara differentiability and fuzzy differential equations can have solutions which have a decreasing length of their support.

Fuzzy Laplace transform method to solve first order fuzzy differential equations was introduced by Allahviranloo and Ahmadi in 2010 [2]. Salahshour and Allahviranloo gave conditions for the existence of the fuzzy Laplace transform of fuzzy processes and results related to solving fuzzy initial value problems [22]. Many authors studied the method of fuzzy Laplace transform in their papers [7, 10, 13–15, 23, 24].

The aim of this article is to investigate the fuzzy problem for fuzzy differential equation with fuzzy coefficient using the method of fuzzy Laplace transform under the concept of generalized Hukuhara differentiability.

The rest of the article is organized as follows. In Sect. 2, we present fundamentals of fuzzy set theory and give some basic definitions and theorems. In Sect. 3, we research the problem for fuzzy differential

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equation with fuzzy coefficient by the fuzzy Laplace transform and give the results on solutions to a fuzzy problem. For illustration, we provide a numerical example in Sect. 4. In the last section, we present our conclusions.

2. Preliminaries and fundamentals

Definition 2.1 ([4]). *A fuzzy number is a mapping $\hat{u}:\mathbb{R}\rightarrow[0,1]$ satisfying the properties: $\hat{u}$ is upper semi-continuous on $\mathbb{R}$, normal and convex fuzzy set. Also, $\overline{\{x\in\mathbb{R}|\,\hat{u}(x)>0\}}$ is compact.*
$\mathbb{R}_F$ denote the set of all fuzzy numbers.

Definition 2.2 ([18]). *Let $\hat{u}\in\mathbb{R}_F$. The α-level set of $\hat{u}$ is $[\hat{u}]^\alpha=\left[\widehat{\underline{u}}_\alpha,\overline{\hat{u}}_\alpha\right]=\{x\in\mathbb{R}|\,\hat{u}(x)\geq\alpha\}\,,0<\alpha\leq1.$*

Definition 2.3 ([18]).

$$[\hat{u}]^\alpha=\left[\underline{u}+\left(\frac{\overline{u}-\underline{u}}{2}\right)\alpha,\overline{u}-\left(\frac{\overline{u}-\underline{u}}{2}\right)\alpha\right]$$

is the $\alpha-$level set of symmetric triangular fuzzy number $\hat{u}$, where $\left[\underline{u},\overline{u}\right]$ is the support of $\hat{u}$.

Definition 2.4 ([2]). *The α-level set $\left[\widehat{\underline{u}}_\alpha,\overline{\hat{u}}_\alpha\right]$ of $\hat{u}$ fuzzy number satisfy the following conditions:*
1. $\widehat{\underline{u}}_\alpha$ is right-continuous for $\alpha=0$ and bounded, non-decreasing and left-continuous on $(0,1]$,
2. $\overline{\hat{u}}_\alpha$ is right-continuous for $\alpha=0$ and bounded, non-increasing and left-continuous on $(0,1]$,
3. $\widehat{\underline{u}}_\alpha\leq\overline{\hat{u}}_\alpha,0\leq\alpha\leq1$.

Definition 2.5 ([3]). *Let $\hat{u},\ \hat{v}\in\mathbb{R}_F$. The generalized Hukuhara difference between $\hat{u}$ and $\hat{v}$ is the set $\hat{w}\in\mathbb{R}_F$ which $\hat{u}\ominus_g\hat{v}=\hat{w}$ if and only if $\hat{u}=\hat{v}+\hat{w}$ or $\hat{v}=\hat{u}+(-1)\,\hat{w}$.*

Definition 2.6 ([17]). *Let $\hat{g}:[a_1,a_2]\longrightarrow\mathbb{R}_F$ and $x_0\in[a_1,a_2]$.*

1) If there exists $\hat{g}'(x_0)\in\mathbb{R}_F$ such that for all $h>0$ sufficiently small,

$$\exists\hat{g}(x_0+h)\ominus\hat{g}(x_0)\,,\exists\hat{g}(x_0)\ominus\hat{g}(x_0-h)$$

and the limits

$$\lim_{h\longrightarrow0^+}\frac{\hat{g}(x_0+h)\ominus\hat{g}(x_0)}{h}=\lim_{h\longrightarrow0^+}\frac{\hat{g}(x_0)\ominus\hat{g}(x_0-h)}{h}=\hat{g}'(x_0)\,,$$

$\hat{g}$ is said to be (1)-differentiable at x_0.
2) If there exists $\hat{g}'(x_0)\in\mathbb{R}_F$ such that for all $h>0$ sufficiently small,

$$\exists\hat{g}(x_0)\ominus\hat{g}(x_0+h)\,,\exists\hat{g}(x_0-h)\ominus\hat{g}(x_0)$$

and the limits

$$\lim_{h\longrightarrow0^+}\frac{\hat{g}(x_0)\ominus\hat{g}(x_0+h)}{-h}=\lim_{h\longrightarrow0^+}\frac{\hat{g}(x_0-h)\ominus\hat{g}(x_0)}{-h}=\hat{g}'(x_0)\,,$$

$\hat{g}$ is said to be (2)-differentiable at x_0.

Theorem 2.7 ([8]). *Let $\hat{g}:[a_1,a_2]\longrightarrow\mathbb{R}_F$, where $[\hat{g}(x)]^\alpha=\left[\underline{\hat{g}}_\alpha(x)\,,\overline{\hat{g}}_\alpha(x)\right]$ for each $0\leq\alpha\leq1$.*

1) If $\hat{g}$ is (1) differentiable, $\underline{\hat{g}}_\alpha$, $\overline{\hat{g}}_\alpha$ are differentiable and $\left[\hat{g}'(x)\right]^\alpha=\left[\underline{\hat{g}}'_\alpha(x)\,,\overline{\hat{g}}'_\alpha(x)\right]$.

2) If $\hat{g}$ is (2) differentiable, $\underline{\hat{g}}_\alpha$, $\overline{\hat{g}}_\alpha$ are differentiable and $\left[\hat{g}'(x)\right]^\alpha=\left[\overline{\hat{g}}'_\alpha(x)\,,\underline{\hat{g}}'_\alpha(x)\right]$.

Definition 2.8 ([22]). *Let $\hat{g} : [a_1, a_2] \longrightarrow \mathbb{R}_F$. The fuzzy Laplace transform of $\hat{g}$ is*

$$\hat{G}(s) = \hat{L}(\hat{g}(x)) = \int_0^\infty e^{-sx}\hat{g}(x)\,dx = \left[\lim_{\rho\longrightarrow\infty}\int_0^\rho e^{-sx}\underline{\hat{g}}(x)\,dx\,,\lim_{\rho\longrightarrow\infty}\int_0^\rho e^{-sx}\overline{\hat{g}}(x)\,dx\right],$$

$$\hat{G}(s,\alpha) = \hat{L}\left([\hat{g}(x)]^\alpha\right) = \left[\hat{L}\left(\underline{\hat{g}}_\alpha(x)\right), L\left(\overline{\hat{g}}_\alpha(x)\right)\right],$$

$$\hat{L}\left(\underline{\hat{g}}_\alpha(x)\right) = \int_0^\infty e^{-sx}\underline{\hat{g}}_\alpha(x)\,dx = \lim_{\rho\longrightarrow\infty}\int_0^\rho e^{-sx}\underline{\widehat{g}}_\alpha(x)\,dx\,,$$

$$\hat{L}\left(\overline{\hat{g}}_\alpha(x)\right) = \int_0^\infty e^{-sx}\overline{\hat{g}}_\alpha(x)\,dx = \lim_{\rho\longrightarrow\infty}\int_0^\rho e^{-sx}\overline{\hat{g}}(x)\,dx.$$

Theorem 2.9 ([21]). *Let $\hat{g}''(x)$ be an integrable fuzzy function and $\hat{g}(x)$, $\hat{g}'(x)$ are primitive of $\hat{g}'(x)$, $\hat{g}''(x)$ on $[0,\infty)$.*
1. If the functions $\hat{g}$, $\hat{g}'$ are (1)-differentiable,

$$\hat{L}\left(\hat{g}''(x)\right) = s^2\hat{L}(\hat{g}(x)) \ominus s\hat{g}(0) \ominus \hat{g}'(0).$$

2. If the functions $\hat{g}$, $\hat{g}'$ are (2)-differentiable,

$$\hat{L}\left(\hat{g}''(x)\right) = s^2\hat{L}(g(x)) \ominus s\hat{g}(0) - \hat{g}'(0).$$

3. If $\hat{g}$ is (1)-differentiable, then $\hat{g}'$ is (2)-differentiable,

$$\hat{L}\left(\hat{g}''(x)\right) = \ominus\left(-s^2\right)\hat{L}(\hat{g}(x)) - s\hat{g}(0) - \hat{g}'(0).$$

4. If $\hat{g}$ is (2)-differentiable, then $\hat{g}'$ is (1)-differentiable,

$$\hat{L}\left(\hat{g}''(x)\right) = \ominus\left(-s^2\right)\hat{L}(\hat{g}(x)) - s\hat{g}(0) \ominus \hat{g}'(0).$$

3. Main results

We study the solutions of the fuzzy problem

$$\begin{cases} [\widehat{\chi}]^\alpha \hat{y}'' = \hat{y} \\ \hat{y}(0) = \left[\widehat{\xi}\right]^\alpha \\ \hat{y}'(0) = [\widehat{\kappa}]^\alpha \end{cases}, \tag{1}$$

$[\widehat{\chi}]^\alpha = \left[\underline{\widehat{\chi}}_\alpha, \overline{\widehat{\chi}}_\alpha\right], \left[\widehat{\xi}\right]^\alpha = \left[\underline{\widehat{\xi}}_\alpha, \overline{\widehat{\xi}}_\alpha\right]$, $[\widehat{\kappa}]^\alpha = \left[\underline{\widehat{\kappa}}_\alpha, \overline{\widehat{\kappa}}_\alpha\right]$ are positive symmetric triangular fuzzy numbers,$\hat{y}$ is positive fuzzy function.

In this paper, $\hat{L}(\hat{y}(x)) = \hat{Y}(s)$ is the fuzzy Laplace transform of $\hat{y}$ and (i,j)-solution means that $\hat{y}$ is (i)-differentiable and $\hat{y}'$ is (j)-differentiable, i,j=1,2.

3.1. (1,1)-solution

Since $\hat{y}'$ and $\hat{y}$ and are (1)-differentiable, from the equation

$$[\widehat{\chi}]^{\alpha}\left(s^2\hat{Y}(s)\ominus s\hat{y}(0)\ominus\hat{y}'(0)\right)=\hat{Y}(s),$$

the following equations are obtained.

$$\underline{\widehat{\chi}}_{\alpha}\left(s^2\underline{\hat{Y}}_{\alpha}(s)-s\underline{\hat{y}}_{\alpha}(0)-\underline{\hat{y}}'_{\alpha}(0)\right)=\underline{\hat{Y}}_{\alpha}(s), \tag{2}$$

$$\overline{\widehat{\chi}}_{\alpha}\left(s^2\overline{\hat{Y}}_{\alpha}(s)-s\overline{\hat{y}}_{\alpha}(0)-\overline{\hat{y}}'_{\alpha}(0)\right)=\overline{\hat{Y}}_{\alpha}(s). \tag{3}$$

The equations (2) and (3) yield

$$\underline{\hat{Y}}_{\alpha}(s)=\frac{s\underline{\widehat{\xi}}_{\alpha}}{s^2-\frac{1}{\underline{\widehat{\chi}}_{\alpha}}}+\frac{\underline{\widehat{\kappa}}_{\alpha}}{s^2-\frac{1}{\underline{\widehat{\chi}}_{\alpha}}},$$

$$\overline{\hat{Y}}_{\alpha}(s)=\frac{s\overline{\widehat{\xi}}_{\alpha}}{s^2-\frac{1}{\overline{\widehat{\chi}}_{\alpha}}}+\frac{\overline{\widehat{\kappa}}_{\alpha}}{s^2-\frac{1}{\overline{\widehat{\chi}}_{\alpha}}}.$$

From this, $\underline{\hat{y}}_{\alpha}(x)$ and $\overline{\hat{y}}_{\alpha}(x)$ are obtained as

$$\underline{\hat{y}}_{\alpha}(x)=\underline{\widehat{\xi}}_{\alpha}cosh\left(\frac{1}{\sqrt{\underline{\widehat{\chi}}_{\alpha}}}x\right)+\sqrt{\underline{\widehat{\chi}}_{\alpha}}\,\underline{\widehat{\kappa}}_{\alpha}sinh\left(\frac{1}{\sqrt{\underline{\widehat{\chi}}_{\alpha}}}x\right),$$

$$\overline{\hat{y}}_{\alpha}(x)=\overline{\widehat{\xi}}_{\alpha}cosh\left(\frac{1}{\sqrt{\overline{\widehat{\chi}}_{\alpha}}}x\right)+\sqrt{\overline{\widehat{\chi}}_{\alpha}}\,\overline{\widehat{\kappa}}_{\alpha}sinh\left(\frac{1}{\sqrt{\overline{\widehat{\chi}}_{\alpha}}}x\right).$$

3.2. (2,2)-solution

Similar to (1,1)-solution, (2,2)-solution is obtained as

$$\underline{\hat{y}}_{\alpha}(x)=\underline{\widehat{\xi}}_{\alpha}cosh\left(\frac{1}{\sqrt{\underline{\widehat{\chi}}_{\alpha}}}x\right)+\sqrt{\underline{\widehat{\chi}}_{\alpha}}\,\overline{\widehat{\kappa}}_{\alpha}sinh\left(\frac{1}{\sqrt{\underline{\widehat{\chi}}_{\alpha}}}x\right)$$

$$\overline{\hat{y}}_{\alpha}(x)=\overline{\widehat{\xi}}_{\alpha}cosh\left(\frac{1}{\sqrt{\overline{\widehat{\chi}}_{\alpha}}}x\right)+\sqrt{\overline{\widehat{\chi}}_{\alpha}}\,\underline{\widehat{\kappa}}_{\alpha}sinh\left(\frac{1}{\sqrt{\overline{\widehat{\chi}}_{\alpha}}}x\right)$$

$$[\hat{y}(x)]^{\alpha}=\left[\underline{\hat{y}}_{\alpha}(x),\overline{\hat{y}}_{\alpha}(x)\right].$$

3.3. ***(1,2)-solution***

Since $\hat{y}$ is (1)-differentiable and $\hat{y}'$ is (2)-differentiable, we have the equation

$$[\widehat{\chi}]\left(\ominus\left(-s^2\right)\hat{Y}(s) - s\hat{y}(0) - \hat{y}'(0)\right) = \hat{Y}(s),$$

that is, we have the equations

$$\underline{\widehat{\chi}}_\alpha\left(s^2\overline{\hat{Y}}_\alpha(s) - s\overline{\hat{y}}_\alpha(0) - \overline{\hat{y}}'_\alpha(0)\right) = \underline{\hat{Y}}_\alpha(s)$$
$$\overline{\widehat{\chi}}_\alpha\left(s^2\underline{\hat{Y}}_\alpha(s) - s\underline{\hat{y}}_\alpha(0) - \underline{\hat{y}}'_\alpha(0)\right) = \overline{\hat{Y}}_\alpha(s).$$

From this, using the initial conditions, the equations

$$s^2\underline{\widehat{\chi}}_\alpha\overline{\hat{Y}}_\alpha(s) - \underline{\hat{Y}}_\alpha(s) = s\underline{\widehat{\chi}}_\alpha\overline{\widehat{\xi}}_\alpha + \underline{\widehat{\chi}}_\alpha\overline{\widehat{\kappa}}_\alpha \tag{4}$$
$$s^2\overline{\widehat{\chi}}_\alpha\underline{\hat{Y}}_\alpha(s) - \overline{\hat{Y}}_\alpha(s) = s\overline{\widehat{\chi}}_\alpha\underline{\widehat{\xi}}_\alpha + \overline{\widehat{\chi}}_\alpha\underline{\widehat{\kappa}}_\alpha \tag{5}$$

are obtained. From the equations (4) and (5), we have

$$\underline{\hat{Y}}_\alpha(s) = \frac{1}{s^4 - \frac{1}{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}\left(s^3\underline{\widehat{\xi}}_\alpha + s^2\underline{\widehat{\kappa}}_\alpha + \frac{s\overline{\widehat{\xi}}_\alpha}{\overline{\widehat{\chi}}_\alpha} + \frac{\overline{\widehat{\kappa}}_\alpha}{\overline{\widehat{\chi}}_\alpha}\right),$$

$$\overline{\hat{Y}}_\alpha(s) = \frac{1}{s^4 - \frac{1}{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}\left(s^3\overline{\widehat{\xi}}_\alpha + s^2\overline{\widehat{\kappa}}_\alpha + \frac{s\underline{\widehat{\xi}}_\alpha}{\underline{\widehat{\chi}}_\alpha} + \frac{\underline{\widehat{\kappa}}_\alpha}{\underline{\widehat{\chi}}_\alpha}\right).$$

Then, (1,2)-solution is obtained as

$$\underline{\hat{y}}_\alpha(x) = \frac{\underline{\widehat{\xi}}_\alpha}{2}\left(cosh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right) + cos\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right)\right) + \frac{\underline{\widehat{\kappa}}_\alpha\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}{2}\left(sinh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right) + sin\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right)\right)$$
$$-\frac{\overline{\widehat{\xi}}_\alpha}{2}\sqrt{\frac{\underline{\widehat{\chi}}_\alpha}{\overline{\widehat{\chi}}_\alpha}}\left(cosh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right) - cos\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right)\right) - \frac{\overline{\widehat{\kappa}}_\alpha}{2}\frac{\sqrt[4]{\underline{\widehat{\chi}}_\alpha^{\,3}}}{\sqrt[4]{\overline{\widehat{\chi}}_\alpha}}\left(sinh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right) - sin\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right)\right),$$

$$\overline{\hat{y}}_\alpha(x) = \frac{\overline{\widehat{\xi}}_\alpha}{2}\left(cosh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right) + cos\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right)\right) + \frac{\overline{\widehat{\kappa}}_\alpha\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}{2}\left(sinh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right) + sin\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right)\right)$$
$$-\frac{\underline{\widehat{\xi}}_\alpha}{2}\sqrt{\frac{\overline{\widehat{\chi}}_\alpha}{\underline{\widehat{\chi}}_\alpha}}\left(cosh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right) - cos\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right)\right) - \frac{\underline{\widehat{\kappa}}_\alpha}{2}\frac{\sqrt[4]{\overline{\widehat{\chi}}_\alpha^{\,3}}}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha}}\left(sinh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right) - sin\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_\alpha\overline{\widehat{\chi}}_\alpha}}x\right)\right),$$

$$[\hat{y}(x)]^\alpha = \left[\underline{\hat{y}}_\alpha(x), \overline{\hat{y}}_\alpha(x)\right].$$

3.4. *(2,1)-solution*

Similar to (1,2)-solution, (2,1)-solution is obtained as

$$\underline{\hat{y}}_{\alpha}(x) = \frac{\underline{\widehat{\xi}}_{\alpha}}{2}\left(\cosh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right) + \cos\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right)\right) + \frac{\overline{\widehat{\kappa}}_{\alpha}\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}{2}\left(\sinh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right) + \sin\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right)\right)$$

$$-\frac{\overline{\widehat{\xi}}_{\alpha}}{2}\sqrt{\frac{\underline{\widehat{\chi}}_{\alpha}}{\overline{\widehat{\chi}}_{\alpha}}}\left(\cosh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right) - \cos\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right)\right) - \frac{\underline{\widehat{\kappa}}_{\alpha}}{2}\frac{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}^{3}}}{\sqrt[4]{\overline{\widehat{\chi}}_{\alpha}}}\left(\sinh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right) - \sin\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right)\right),$$

$$\overline{\hat{y}}_{\alpha}(x) = \frac{\overline{\widehat{\xi}}_{\alpha}}{2}\left(\cosh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right) + \cos\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right)\right) + \frac{\underline{\widehat{\kappa}}_{\alpha}\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}{2}\left(\sinh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right) + \sin\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right)\right)$$

$$-\frac{\underline{\widehat{\xi}}_{\alpha}}{2}\sqrt{\frac{\overline{\widehat{\chi}}_{\alpha}}{\underline{\widehat{\chi}}_{\alpha}}}\left(\cosh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right) - \cos\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right)\right) - \frac{\overline{\widehat{\kappa}}_{\alpha}}{2}\frac{\sqrt[4]{\overline{\widehat{\chi}}_{\alpha}^{3}}}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}}}\left(\sinh\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right) - \sin\left(\frac{1}{\sqrt[4]{\underline{\widehat{\chi}}_{\alpha}\overline{\widehat{\chi}}_{\alpha}}}x\right)\right),$$

$$[\hat{y}(x)]^{\alpha} = \left[\underline{\hat{y}}_{\alpha}(x), \overline{\hat{y}}_{\alpha}(x)\right].$$

4. Numerical example

Example 4.1. *Consider the fuzzy problem*

$$\begin{cases} \left[\hat{1}\right]^{\alpha}\hat{y}'' = \hat{y} \\ \hat{y}(0) = [1+\alpha, 3-\alpha] \\ \hat{y}'(0) = [2+\alpha, 4-\alpha] \end{cases},$$

where $\left[\hat{1}\right]^{\alpha} = [\alpha, 2-\alpha]$.

(1,1)-solution is

$$\underline{\hat{y}}_{\alpha}(x) = (1+\alpha)\cosh\left(\frac{1}{\sqrt{\alpha}}x\right) + \sqrt{\alpha}(2+\alpha)\sinh\left(\frac{1}{\sqrt{\alpha}}x\right),$$

$$\overline{\hat{y}}_{\alpha}(x) = (3-\alpha)\cosh\left(\frac{1}{\sqrt{2-\alpha}}x\right) + \sqrt{2-\alpha}(4-\alpha)\sinh\left(\frac{1}{\sqrt{2-\alpha}}x\right),$$

$$[\hat{y}(x)]^{\alpha} = \left[\underline{\hat{y}}_{\alpha}(x), \overline{\hat{y}}_{\alpha}(x)\right].$$

(2,2)-solution is

$$\underline{\hat{y}}_{\alpha}(x) = (1+\alpha)\cosh\left(\frac{1}{\sqrt{\alpha}}x\right) + \sqrt{\alpha}(4-\alpha)\sinh\left(\frac{1}{\sqrt{\alpha}}x\right),$$

$$\overline{\hat{y}}_{\alpha}(x) = (3-\alpha)\cosh\left(\frac{1}{\sqrt{2-\alpha}}x\right) + \sqrt{2-\alpha}(2+\alpha)\sinh\left(\frac{1}{\sqrt{2-\alpha}}x\right),$$

$$[\hat{y}(x)]^{\alpha} = \left[\underline{\hat{y}}_{\alpha}(x), \overline{\hat{y}}_{\alpha}(x)\right].$$

(1,2)-solution is

$$\underline{\hat{y}}_{\alpha}(x) = \frac{(1+\alpha)}{2}\left(cosh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) + cos\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$+\frac{(2+\alpha)\sqrt[4]{\alpha(2-\alpha)}}{2}\left(sinh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) + sin\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$-\frac{(3-\alpha)}{2}\sqrt{\frac{\alpha}{2-\alpha}}\left(cosh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) - cos\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$-\frac{(4-\alpha)}{2}\frac{\sqrt[4]{\alpha^3}}{\sqrt[4]{2-\alpha}}\left(sinh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) - sin\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right),$$

$$\overline{\hat{y}}_{\alpha}(x) = \frac{(3-\alpha)}{2}\left(cosh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) + cos\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$+\frac{(4-\alpha)\sqrt[4]{\alpha(2-\alpha)}}{2}\left(sinh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) + sin\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$-\frac{(1+\alpha)}{2}\sqrt{\frac{2-\alpha}{\alpha}}\left(cosh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) - cos\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$-\frac{(2+\alpha)}{2}\frac{\sqrt[4]{(2-\alpha)^3}}{\sqrt[4]{\alpha}}\left(sinh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) - sin\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right),$$

$$[\hat{y}(x)]^{\alpha} = \left[\underline{\hat{y}}_{\alpha}(x), \overline{\hat{y}}_{\alpha}(x)\right].$$

(2,1)-solution is

$$\underline{\hat{y}}_{\alpha}(x) = \frac{(1+\alpha)}{2}\left(cosh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) + cos\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$+\frac{(4-\alpha)\sqrt[4]{\alpha(2-\alpha)}}{2}\left(sinh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) + sin\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$-\frac{(3-\alpha)}{2}\sqrt{\frac{\alpha}{2-\alpha}}\left(cosh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) - cos\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$-\frac{(2+\alpha)}{2}\frac{\sqrt[4]{\alpha^3}}{\sqrt[4]{2-\alpha}}\left(sinh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) - sin\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right),$$

$$\overline{y}_{\alpha}(x) = \frac{(3-\alpha)}{2}\left(\cosh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) + \cos\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$+\frac{(2+\alpha)\sqrt[4]{\alpha(2-\alpha)}}{2}\left(\sinh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) + \sin\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$-\frac{(1+\alpha)}{2}\sqrt{\frac{2-\alpha}{\alpha}}\left(\cosh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) - \cos\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right)$$

$$-\frac{(4-\alpha)}{2}\frac{\sqrt[4]{(2-\alpha)^3}}{\sqrt[4]{\alpha}}\left(\sinh\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right) - \sin\left(\frac{1}{\sqrt[4]{\alpha(2-\alpha)}}x\right)\right),$$

$$[\hat{y}(x)]^{\alpha} = \left[\underline{\hat{y}}_{\alpha}(x), \overline{\hat{y}}_{\alpha}(x)\right].$$

From Definition 2.4 , (1,2)-solution is a valid alpha-level set for $x \in (0,\ 1.44275)$ *in figure 5 and (2,1)-solution is a valid alpha-level set for* $x \in (0,\ 0.590438)$ *in figure 7. Also, (1,2)-solution is a valid alpha-level set for* $x \in (0,\ 1.67157)$ *in figure 6 and (2,1)-solution is a valid alpha-level set for* $x \in (0,\ 0.628153)$ *in figure 8. But, according to figure 1-4, (1,1) and (2,2)-solutions are not valid alpha-level sets.*

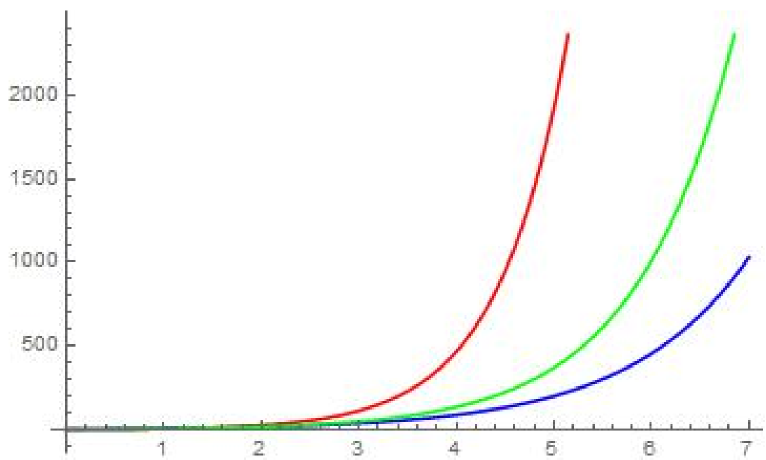

Figure 1. Graphic of (1,1)-solution for $\alpha = 0.5$

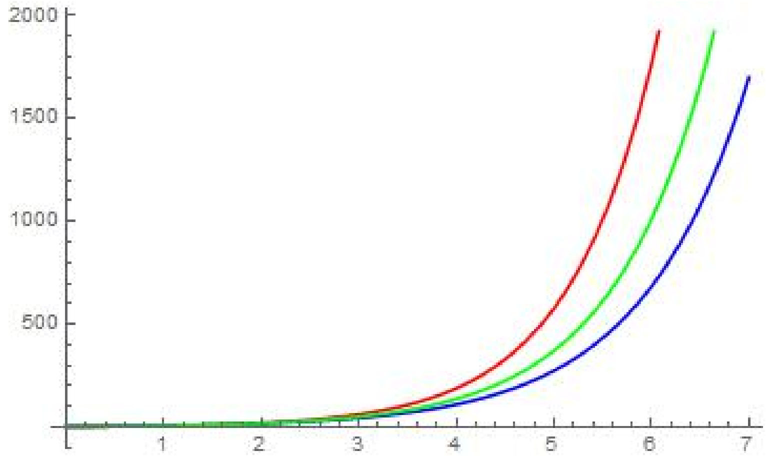

Figure 2. Graphic of (1,1)-solution for $\alpha = 0.8$

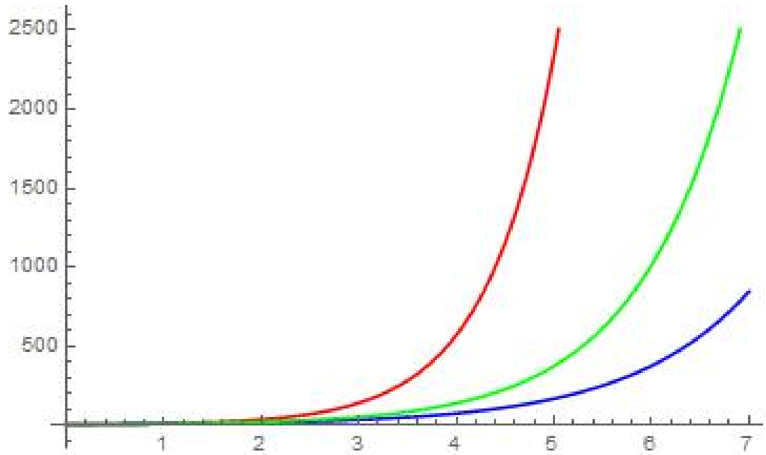

Figure 3. Graphic of (2,2)-solution for $\alpha = 0.5$

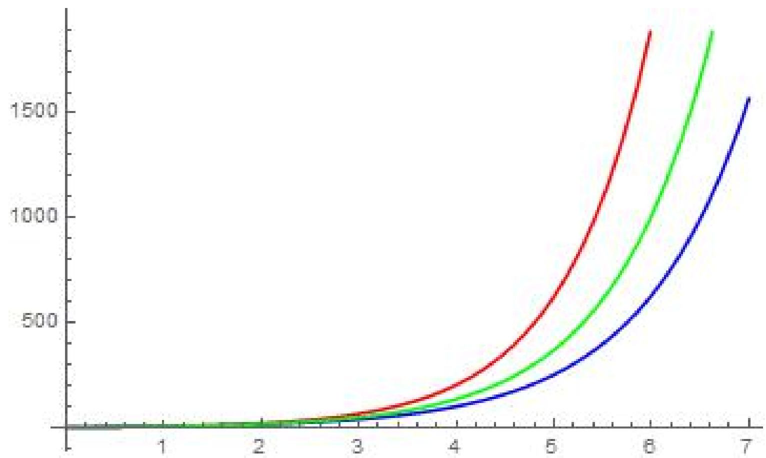

Figure 4. Graphic of (2,2)-solution for $\alpha = 0.8$

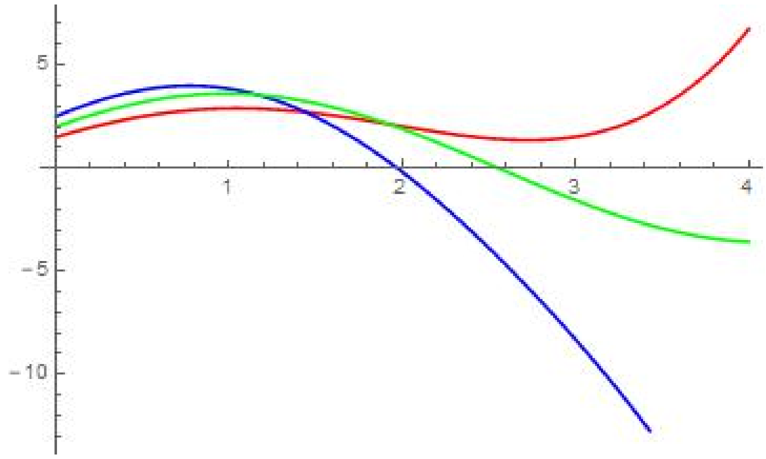

Figure 5. Graphic of (1,2)-solution for $\alpha = 0.5$

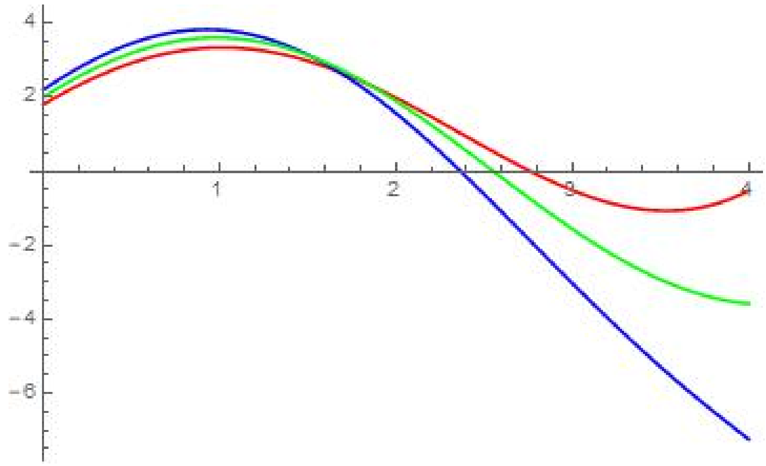

Figure 6. Graphic of (1,2)-solution for $\alpha = 0.8$

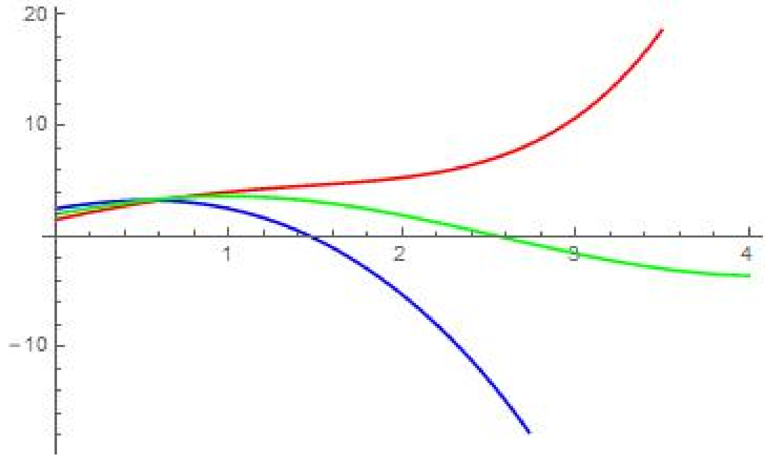

Figure 7. Graphic of (2,1)-solution for $\alpha = 0.5$

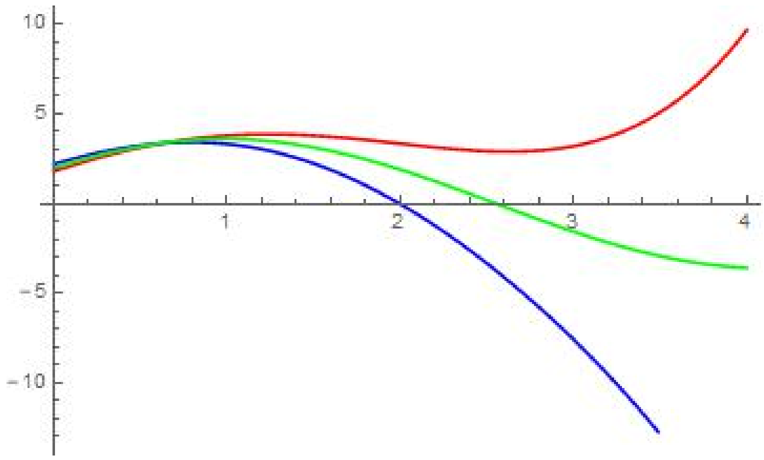

Figure 8. Graphic of (2,1)-solution for $\alpha = 0.8$

$$red \rightarrow \underline{\hat{y}}_{\alpha}(x), blue \rightarrow \overline{\hat{y}}_{\alpha}(x), green \rightarrow \underline{\hat{y}}_{1}(x) = \overline{\hat{y}}_{1}(x)$$

5. Conclusions

In this work, we study the fuzzy problem for fuzzy differential equation with fuzzy coefficient. Four different solutions of the problem are found. Fuzzy Laplace transform method is used in this paper. The results of the problem are given. Also, to illustrate the problem, a numerical example is solved. Then, the graphics of the solutions are drawn. In this paper, it is seen that (1,2) and (2,1)-solutions are valid alpha-level sets in different intervals for different alpha-level sets. In addition, as the alpha-level set increases, the range in which the problem is a valid fuzzy function expands. But, (1,1) and (2,2)-solutions are not valid alpha-level sets.

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