Filomat 38:23 (2024), 8047–8052
https://doi.org/10.2298/FIL2423047K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Isometry property and inversion of the Radon transform over a family of paraboloids

Jeongmin Kim[a], Sunghwan Moon[b,*]

[a]*Department of Mathematical sciences, College of Natural Science, Seoul National University, Seoul 08826, Republic of Korea*
[b]*Department of Mathematics, College of Natural Science, Kyungpook National University, Daegu 41566, Republic of Korea*

Abstract. Integral geometry problems involve finding a desired function from its integrals on a surface. These problems are closely intertwined with the generalized Radon transform, and obtaining an inversion formula for it is pivotal in solving integral geometry problems. The applications of integral geometry span various fields, including tomography, radar, and radiology. Particularly noteworthy is the recovery of a function from integrals over a parabola, which holds significance in reflection seismology. In our study, we concentrate on the transform that maps a real-valued smooth function with compact support to integrals over the paraboloid. This transform, along with its dual, can be expressed as convolutions of kernels and given functions, and we have derived inversion formulas based on their isometric properties.

1. introduction

The integral geometry problem involves reconstructing a function from its integrals over a surface or manifold. The Radon transform serves as the mathematical foundation for solving these problems and is closely associated with the generalized Radon transform. Deriving the inversion formula for the Radon transform, whether general or specific to certain integration regions, is crucial for solving integral geometry problems, considering the geometric properties of the region.

For instance, variations of the Radon transform exist depending on the integration region, such as the spherical Radon transform [11, 15, 20] and the conical Radon transform [2, 9, 21]. Applications of solving integral geometry problems using the Radon transform extend to fields like tomography, radar, seismology, and radiology. Cormack first proposed applying the method of reconstructing a function from the regular Radon transform to radiology [3].

In reflection seismology, the Radon transform, also known as slant stack, is utilized to estimate geophysical properties from reflected seismic waves [12, 16]. Notably, the parabolic slant stack has been explored for its advantages in several studies [6–8]. For this reason, addressing the integral geometry problem described below is meaningful.

2020 *Mathematics Subject Classification*. Primary 44A12; Secondary 44A35.
Keywords. Inversion formula, Paraboloid, Radon transform, tomography
Received: 01 March 2024; Accepted: 29 April 2024
Communicated by Hari M. Srivastava
The work was supported by NRF-2022R1C1C1003464 and RS-2023-00217116.
* Corresponding author: Sunghwan Moon
Email addresses: yknowjm21@snu.ac.kr (Jeongmin Kim), sunghwan.moon@knu.ac.kr (Sunghwan Moon)

Problem. Let f be a smooth real valued function on $\mathbb{R}^{n+1}$ with compact support. Determine the function f from the integrals

$$\mathcal{R}f(\mathbf{u},v)=\int_{\mathcal{P}_{\mathbf{u},v}} f(\mathbf{x},y)\mathrm{d}\mathbf{x}=\int_{\mathbb{R}^n} f\left(\mathbf{x},v+p|\mathbf{x}-\mathbf{u}|^2\right)\mathrm{d}\mathbf{x}$$

for $\mathbf{u}\in\mathbb{R}^n, v\in\mathbb{R}$ and a fixed $p>0$, where

$$\mathcal{P}_{\mathbf{u},v}=\{(\mathbf{x},y)\in\mathbb{R}^n\times\mathbb{R}: y-v=p|\mathbf{x}-\mathbf{u}|^2\}.$$

For each vertex $(\mathbf{u},v)\in\mathbb{R}^n\times\mathbb{R}$, the transform $\mathcal{R}$ maps f to the integral of f on the paraboloid $y-v=\mathcal{P}_{\mathbf{u},v}|\mathbf{x}-\mathbf{u}|^2$ which has the fixed focal length p, with respect to the variable $\mathbf{x}$, not the surface element. This is often referred to as the seismic parabolic Radon transform [23]. Numerous studies have explored recovering a function from integrals on a parabola or paraboloid. For instance, Cormack investigated the method to recover a function from integrals on a curve with a central axis rotating around the origin [4, 5]. Denecker, Van Overloop, and Sommen proved the support theorem from integrals over the family of isofocal parabolas, deriving the inversion formula [7]. Jollivet, Nguyen, and Truong found the inversion formula from integrals on the parabola with a fixed symmetry axis [13]. There are studies on the method to reconstruct a function from the connection of seismic parabolic Radon transform and the regular Radon transform [18, 22]. Recent studies have focused on finding the inversion formula using the Fourier slice theorem and the stability with Sobolev norm, as well as deriving it using the first kind of Volterra integral equation [23].

Our study provides the method to find the inversion formula for $\mathcal{R}$ using the isometry property. Indeed, several studies have explored methods using the isometry property for the spherical Radon transform[1, 14, 19]. We also provide the inversion formula for the dual of $\mathcal{R}$ in the same manner. Subsequent subsections introduce relevant definitions, including those for the dual of $\mathcal{R}$, while Section 2 presents the isometry properties and inversion formulas for $\mathcal{R}$ and its dual.

1.1. Preliminaries

For $g\in C^\infty(\mathbb{R}^{n+1})$ with compact support, the dual $\mathcal{R}^\#$ of $\mathcal{R}$ is defined by

$$\mathcal{R}^\# g(\mathbf{x},y)=\int_{\mathbb{R}^n}\int_{\mathbb{R}} g(\mathbf{u},v)\delta\left(y-v-p|\mathbf{x}-\mathbf{u}|^2\right)\mathrm{d}v\mathrm{d}\mathbf{u}=\int_{\mathbb{R}^n} g\left(\mathbf{u},y-p|\mathbf{x}-\mathbf{u}|^2\right)\mathrm{d}\mathbf{u};$$

then we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}}\mathcal{R}f(\mathbf{u},v)g(\mathbf{u},v)\mathrm{d}\mathbf{u}\mathrm{d}v=\int_{\mathbb{R}^n}\int_{\mathbb{R}} f(\mathbf{x},y)\left[\int_{\mathbb{R}^n}\int_{\mathbb{R}} g(\mathbf{u},y-p|\mathbf{x}-\mathbf{u}|^2)\mathrm{d}v\mathrm{d}\mathbf{u}\right]\mathrm{d}y\mathrm{d}\mathbf{x}.$$

For $\mathbf{x}\in\mathbb{R}^n$ and $y\in\mathbb{R}$, if we set

$$\mathcal{P}^\#_{\mathbf{x},y}=\{(\mathbf{u},v)\in\mathbb{R}^n\times\mathbb{R}: v-y=-p|\mathbf{u}-\mathbf{x}|^2, \mathbf{u}\in\mathbb{R}^n\},$$

then one can find

$$\mathcal{R}^\# g(\mathbf{x},y)=\int_{\mathcal{P}^\#_{\mathbf{x},y}} g(\mathbf{u},v)\mathrm{d}\mathbf{u}.$$

The transform $\mathcal{R}$ maps f to the integral of f on the convex downward paraboloid, but its dual $\mathcal{R}^\#$ maps g to the integral of g on the convex upward paraboloid. The transform $\mathcal{R}$ is the exactly same with the operator $\mathcal{R}_1$ dealt in [23]. Also, the dual $\mathcal{R}^\#$ coincides with the operator $\mathcal{R}_2$ in [23], except the condition of a given function.

On the other hand, let us denote the (partial) Riesz potential of $f \in C^\infty(\mathbb{R}^{n+1})$ with compact support, acting on the last variable, by $I^\alpha f$, given by

$$\mathcal{F}(I^\alpha f)(\boldsymbol{\xi}, \zeta) = |\zeta|^{-\alpha}\mathcal{F} f(\boldsymbol{\xi}, \zeta), \quad \alpha < 1,$$

where the Fourier transform is defined by

$$\mathcal{F} f(\boldsymbol{\xi}, \zeta) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}} f(\mathbf{x}, y)e^{-\mathrm{i}y\zeta}e^{-\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}}\mathrm{d}y\mathrm{d}\mathbf{x}.$$

2. Main results

The following theorem gives that each of $\mathcal{R}f$ and $\mathcal{R}^\# f$ coincides exactly with the convolution of the given function and a kernel.

Theorem 2.1. *For $f \in C^\infty(\mathbb{R}^{n+1})$ with compact support, we obtain*

$$\mathcal{R}f = K * f \text{ and } \mathcal{R}^\# f = K^\# * f,$$

where $K(\mathbf{x}, y) = \delta(p|\mathbf{x}|^2 + y)$ and $K^\#(\mathbf{x}, y) = K(\mathbf{x}, -y)$.

From this with the symmetry of convolution, for $f \in C^\infty(\mathbb{R}^{n+1})$ with compact support, we also obtain $\mathcal{R}\mathcal{R}^\# f = \mathcal{R}^\#\mathcal{R}f$ when one of the two sides is well-defined.

Proof. For $\mathbf{u} \in \mathbb{R}^n$ and $v \in \mathbb{R}$, simple integral computation gives

$$\mathcal{R}f(\mathbf{u}, v) = \int\limits_{\mathbb{R}^n} f(\mathbf{x}, v + p|\mathbf{x} - \mathbf{u}|^2)\mathrm{d}\mathbf{x} = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}} f(\mathbf{x}, y)\delta(y - v - p|\mathbf{x} - \mathbf{u}|^2)\mathrm{d}y\mathrm{d}\mathbf{x} = K * f(\mathbf{u}, v).$$

Similarly, for $\mathbf{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$, we can easily get

$$\mathcal{R}^\# f(\mathbf{x}, y) = \int\limits_{\mathbb{R}^n} f(\mathbf{u}, y - p|\mathbf{x} - \mathbf{u}|^2)\mathrm{d}\mathbf{x} = K^\# * f(\mathbf{x}, y).$$

□

Theorem 1 provides the isometry properties of $\mathcal{R}$ and $\mathcal{R}^\# f$ with respect to the regular L^2 norm

$$\|f\|_{L^2} = \left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}} |f(\mathbf{x}, y)|^2\mathrm{d}y\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}.$$

Theorem 2.2. *For $f \in C^\infty(\mathbb{R}^{n+1})$ with compact support, we have*

$$\left\|f\right\|_{L^2} = \left(\frac{p}{\pi}\right)^{\frac{n}{2}} \left\|I^{-\frac{n}{2}}\left(\mathcal{R}f\right)\right\|_{L^2} = \left(\frac{p}{\pi}\right)^{\frac{n}{2}} \left\|I^{-\frac{n}{2}}\left(\mathcal{R}^\# f\right)\right\|_{L^2}.$$

Proof. We can compute

$$\mathcal{F}K(\boldsymbol{\xi}, \zeta) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}} \delta\left(p|\mathbf{x}|^2 + y\right)e^{-\mathrm{i}y\zeta}e^{-\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}}\mathrm{d}y\mathrm{d}\mathbf{x} = \int\limits_{\mathbb{R}^n} e^{\mathrm{i}p\zeta|\mathbf{x}|^2}e^{-\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}}\mathrm{d}\mathbf{x} = \left(\frac{\pi\mathrm{i}}{p\zeta}\right)^{\frac{n}{2}} e^{-\frac{\mathrm{i}|\boldsymbol{\xi}|^2}{4p\zeta}}$$

since analytic continuation provides (see [17, page 19])

$$\mathcal{F}[e^{-ax^2}](\xi) = \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}} \text{ for } \operatorname{Re}(a) \geq 0 \text{ and } \xi \in \mathbb{R}.$$

It follows

$$\mathcal{F}(\mathcal{R}f)(\boldsymbol{\xi}, \zeta) = \mathcal{F}(K * f)(\boldsymbol{\xi}, \zeta) = \mathcal{F}K(\boldsymbol{\xi}, \zeta)\mathcal{F}f(\boldsymbol{\xi}, \zeta) = \left(\frac{\pi \mathrm{i}}{p\zeta}\right)^{\frac{n}{2}} e^{-\frac{\mathrm{i}|\xi|^2}{4p\zeta}} \mathcal{F}f(\boldsymbol{\xi}, \zeta).$$

and similarly,

$$\mathcal{F}\left(\mathcal{R}^{\#}f\right)(\boldsymbol{\xi}, \zeta) = \mathcal{F}\left(K^{\#} * f\right)(\boldsymbol{\xi}, \zeta) = \left(\frac{\pi}{\mathrm{i}p\zeta}\right)^{\frac{n}{2}} e^{\frac{\mathrm{i}|\xi|^2}{4p\zeta}} \mathcal{F}f(\boldsymbol{\xi}, \zeta).$$

Using the Plancherel's theorem, we get

$$\begin{aligned}\|f\|_{L^2}^2 &= \frac{1}{(2\pi)^{n+1}} \|\mathcal{F}f\|_{L^2}^2 = \frac{1}{(2\pi)^{n+1}} \left(\frac{p}{\pi}\right)^n \left\||\zeta|^{\frac{n}{2}} \mathcal{F}(\mathcal{R}f)\right\|_{L^2}^2 = \frac{1}{(2\pi)^{n+1}} \left(\frac{p}{\pi}\right)^n \left\|\mathcal{F}\left[I^{-\frac{n}{2}}(\mathcal{R}f)\right]\right\|_{L^2}^2 \\ &= \left(\frac{p}{\pi}\right)^n \left\|I^{-\frac{n}{2}}(\mathcal{R}f)\right\|_{L^2}^2\end{aligned}$$

and

$$\|f\|_{L^2}^2 = \frac{1}{(2\pi)^{n+1}} \left(\frac{p}{\pi}\right)^n \left\||\zeta|^{\frac{n}{2}} \mathcal{F}\left(\mathcal{R}^{\#}f\right)\right\|_{L^2}^2 = \left(\frac{p}{\pi}\right)^n \left\|I^{-\frac{n}{2}}\left(\mathcal{R}^{\#}f\right)\right\|_{L^2}^2.$$

□

Before finding the inversion formula of $\mathcal{R}$, we observe the following fact. This helps find the inversion formula of $\mathcal{R}^{\#}$ from that of $\mathcal{R}$.

Proposition 2.3. *For $\alpha < 1$ and for $f \in C^\infty(\mathbb{R}^{n+1})$ with compact support, we have*

$$I^\alpha \mathcal{R} f = \mathcal{R} I^\alpha f \text{ and } I^\alpha \mathcal{R}^{\#} f = \mathcal{R}^{\#} I^\alpha f.$$

Proof. For any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}$, we obtain

$$\begin{aligned}\mathcal{F}(I^\alpha \mathcal{R}f)(\boldsymbol{\xi}, \zeta) &= |\zeta|^{-\alpha} \mathcal{F}(\mathcal{R}f)(\boldsymbol{\xi}, \zeta) = |\zeta|^{-\alpha} \mathcal{F}K(\boldsymbol{\xi}, \zeta)\mathcal{F}f(\boldsymbol{\xi}, \zeta) = \mathcal{F}K(\boldsymbol{\xi}, \zeta)\mathcal{F}(I^\alpha f)(\boldsymbol{\xi}, \zeta) \\ &= \mathcal{F}(\mathcal{R}I^\alpha f)(\boldsymbol{\xi}, \zeta).\end{aligned}$$

In the same way, for any $\boldsymbol{\xi} \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}$, we can get

$$\mathcal{F}\left(I^\alpha \mathcal{R}^{\#} f\right)(\boldsymbol{\xi}, \zeta) = |\zeta|^{-\alpha} \mathcal{F}K^{\#}(\boldsymbol{\xi}, \zeta)\mathcal{F}f(\boldsymbol{\xi}, \zeta) = \mathcal{F}\left(\mathcal{R}^{\#} I^\alpha f\right)(\boldsymbol{\xi}, \zeta).$$

□

Now we provide the inversion formulas for $\mathcal{R}$ and $\mathcal{R}^{\#}$:

Corollary 2.4. *For $f \in C^\infty(\mathbb{R}^{n+1})$ with compact support, we have*

$$f = \left(\frac{p}{\pi}\right)^n \mathcal{R}^{\#}\left[I^{-n}(\mathcal{R}f)\right] = \left(\frac{p}{\pi}\right)^n \mathcal{R}\left[I^{-n}\left(\mathcal{R}^{\#}f\right)\right].$$

Proof. it suffices to prove the first equality. In fact, if the first equality holds, then $\mathcal{R}^{\#}\mathcal{R}I^{-n}f$ is well-defined and the second equality holds with Proposition 3 and commutativity of $\mathcal{R}$ and $\mathcal{R}^{\#}$ mentioned below Theorem

1. From Theorem 2, we have for $g \in C^\infty\left(\mathbb{R}^{n+1}\right)$ with compact support

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}} f(\mathbf{x},y)g(\mathbf{x},y)\mathrm{d}y\mathrm{d}\mathbf{x} &= \left(\frac{p}{\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}} I^{-\frac{n}{2}}(\mathcal{R}f)(\mathbf{u},v) I^{-\frac{n}{2}}(\mathcal{R}g)(\mathbf{u},v)\mathrm{d}v\mathrm{d}\mathbf{u} \\
&= \frac{1}{(2\pi)^{n+1}}\left(\frac{p}{\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}} \mathcal{F}\left[I^{-n}(\mathcal{R}f)\right](\xi,\zeta)\mathcal{F}(K*g)(\xi,\zeta)\mathrm{d}\zeta\mathrm{d}\xi \\
&= \left(\frac{p}{\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}} I^{-n}(\mathcal{R}f)(\mathbf{u},r)(K*g)(\mathbf{u},v)\mathrm{d}v\mathrm{d}\mathbf{u} \\
&= \left(\frac{p}{\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}} I^{-n}(\mathcal{R}f)(\mathbf{u},v)K(\mathbf{u}-\mathbf{x},v-y)g(\mathbf{x},y)\mathrm{d}v\mathrm{d}\mathbf{u}\mathrm{d}y\mathrm{d}\mathbf{x}.
\end{aligned}$$

and hence

$$\begin{aligned}
f(\mathbf{x},y) &= \left(\frac{p}{\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}} I^{-n}(\mathcal{R}f)(\mathbf{u},v)K^{\#}(\mathbf{x}-\mathbf{u},y-v)\,\mathrm{d}v\mathrm{d}\mathbf{u} \\
&= \left(\frac{p}{\pi}\right)^n \mathcal{R}^{\#}\left[I^{-n}(\mathcal{R}f)\right](\mathbf{x},y).
\end{aligned}$$

□

Now we find the inversion formula when n is either odd or even. Since $I^{-n} = (-1)^{\frac{n}{2}}\partial_v^n$ for even n, we have

$$f = \left(\frac{p}{\pi}\right)^n (-1)^{\frac{n}{2}}\mathcal{R}^{\#}\left[\partial_v^n(\mathcal{R}f)\right] = \left(\frac{p}{\pi}\right)^n (-1)^{\frac{n}{2}}\mathcal{R}\left[\partial_v^n\left(\mathcal{R}^{\#}f\right)\right].$$

On the other hand, noting $I^{-n} = (-1)^{\frac{n-1}{2}}H\partial_v^n$ for odd n, where H is the Hilbert transform, and therefore

$$f = \left(\frac{p}{\pi}\right)^n (-1)^{\frac{n-1}{2}}\mathcal{R}^{\#}\left[H\partial_v^n(\mathcal{R}f)\right] = \left(\frac{p}{\pi}\right)^n (-1)^{\frac{n-1}{2}}\mathcal{R}\left[H\partial_v^n\left(\mathcal{R}^{\#}f\right)\right].$$

In particular, inversion formulas for $n = 1$ and $n = 2$ are, respectively,

$$\begin{aligned}
f(x,y) &= \frac{p}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\partial_v(\mathcal{R}f)(u,v)}{y-v-p(x-u)^2}\mathrm{d}v\mathrm{d}u \quad \text{and} \\
f(x_1,x_2,y) &= -\left(\frac{p}{\pi}\right)^2 \int_{\mathbb{R}}\int_{\mathbb{R}} \partial_v^2(\mathcal{R}f)\left(u_1,u_2,y-p\left[(x_1-u_1)^2+(x_2-u_2)^2\right]\right)\mathrm{d}u_1\mathrm{d}u_2.
\end{aligned}$$

3. Conclusion

Our results provide three facts: the first is that the transform $\mathcal{R}$ and its dual $\mathcal{R}^{\#}$ are convolutions of kernels and the given smooth functions on $\mathbb{R}^{n+1}$ with compact support in $\mathbb{R}^{n+1}$; the second is that there are isometry properties of $I^{-\frac{n}{2}}\mathcal{R}$ and $I^{-\frac{n}{2}}\mathcal{R}^{\#}$; the last is that we get the inversion formula of $\mathcal{R}$. By the commutativity of the Riezs potential, the transform $\mathcal{R}$, and its dual $\mathcal{R}^{\#}$, we also obtain the inversion formula of $\mathcal{R}^{\#}$.

References

[1] A. L. Bukhgeim, V. B. Kardakov, Solution of the inverse problem for the equation of elastic waves by the method of spherical means. *Sib. Math. J.*, 19(4):528-535, 1978.

[2] J. Cebeiro, M. Morvidone, M. K. Nguyen, Back-projection inversion of a conical Radon transform. *Inverse Probl. Sci. Eng.*, 24(2):328-352, 2016.

[3] A. M. Cormack, Representation of a function by its line integrals, with Some radiological applications. *J. Appl. Phys.*, 34(9):2722-2727, 1963.

[4] A. M. Cormack, The Radon transform on a family of curves in the plane. *Proc. Amer. Math. Soc.*, 83(2):325-330, 1981.

[5] A. M. Cormack, The Radon transform on a family of curves in the plane. II. *Proc. Amer. Math. Soc.*, 86(2):293-298, 1982.
[6] G. Darche, Spatial interpolation using a fast parabolic transform. *SEG Technical Program Expanded Abstracts*, 1647-1650, 1990.
[7] K. Denecker, J. V. Overloop, F. Sommen, The general quadratic Radon transform. *Inverse Problems*, 14(3):615, 1998.
[8] D. J. Foster, C. C. Mosher, Suppression of multiple reflections using the Radon transform. *Geophysics*, 57(3):386-395, 1992.
[9] R. Gouia-Zarrad and G. Ambartsoumian. Exact inversion of the conical Radon transform with a fixed opening angle. *Inverse Problems*, 30(4):045007, 2014.
[10] R. Gouia-Zarrad, S. Moon, Uniqueness and stability for a seismic-type generalized Radon transform. *Integral Transforms Spec. Funct.*, 1-13, 2024.
[11] M. Haltmeier and S. Moon. The spherical Radon transform with centers on cylindrical surfaces. *J. Math. Anal. Appl.*, 448(1):567 – 579, 2017.
[12] D. Hampson, Inverse velocity stacking for multiple elimination. *SEG Technical Program Expanded Abstracts*, 422-424, 1986.
[13] A. Jollivet, M. K. Nguyen, T. T. Truong, Properties and inversion of a new Radon transform on parabolas with fixed axis direction in $\mathbb{R}^2$. *preprint*, available at https://math.univ-lille.fr/, 2011.
[14] J. Kim, S. Moon, Isometry property and inversion of a spherical mean Radon transform with centers on a hyperplane. *AIP Advances*, 14(1):015032, 2024.
[15] L.A. Kunyansky. A series solution and a fast algorithm for the inversion of the spherical mean Radon transform. *Inverse Problems*, 23(6):S11, 2007.
[16] M.M. Nurul Kabir, D.J. Verschuur, Restoration of missing offsets by parabolic Radon transform. *Geophys. Prospecting*, 43(3):347-368, 1995.
[17] F. Linares, G. Ponce, *Introduction to nonlinear dispersive equations, 2nd edition.* Universitext, Springer, New York, 2015.
[18] S. Moon, Inversion of the seismic parabolic Radon transform and the seismic hyperbolic Radon transform. *Inverse Probl. Sci. Eng.*, 24(2):317-327, 2016.
[19] E. K. Narayanan, Rakesh, Spherical means with centers on a hyperplane in even dimensions. *Inverse Problems*, 26(3):035014, 2010.
[20] E. T. Quinto. Null spaces and ranges for the classical and spherical Radon transforms. *J. Math. Anal. Appl.*, 90(2):408 – 420, 1982.
[21] F. Terzioglu, P. Kuchment, and L. Kunyansky. Compton camera imaging and the cone transform: a brief overview. *Inverse Problems*, 34(5):054002, 2018.
[22] Z. Ustaoglu, On the inversion of a generalized Radon transform of seismic type. *J. Math. Anal. Appl.*, 453(1):287-303, 2017.
[23] Z. Ustaoglu, On the inversion of generalized Radon transforms over n-dimensional paraboloids. *J. Math. Anal. Appl.*, 531(1):127799, 2024.