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The area of Hügelschäffer curves via Taylor series

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Abstract. In this paper, we give new Taylor approximative formulae for the area of the egg-shaped parts of Hügelschäffer curves. Based on a parametrization of the Hügelschäffer curve, a formula for the area of the egg-shaped part of such a curve is derived via elliptic integrals of the first and second kind. Furthermore, new approximative formulae for calculating this area derived from standard and double Taylor approximations are given. A representation of the value $\frac{1}{\pi}$ was also obtained using an appropriate series.

1. Hügelschäffer curve $\mathcal{F}_{q,\,egg}$

The Hügelschäffer curve [11] is an algebraic cubic curve given by the following equation

$$\mathcal{F}: \quad 2wxy^2 + b^2x^2 + (a^2 + w^2)y^2 - a^2b^2 = 0, \tag{1}$$

where $a, b, w > 0$. In the papers [30], [31] and [33], a decomposition of this cubic curve is described:

$$\mathcal{F} = \mathcal{F}_{egg} \cup \mathcal{F}_{hyp}. \tag{2}$$

The egg-shaped part $\mathcal{F}_{egg}$ of the curve is defined over $[-a, a]$, and the hyperbolic part $\mathcal{F}_{hyp}$ of the curve (which consists of two branches) is defined over $(-\infty, \gamma)$, where $\gamma = -\frac{a^2+w^2}{2w}$, see Fig. 1. It is easy to check that $\gamma < -a \Leftrightarrow (a - w)^2 > 0$.

Let us consider just the non-degenerative cases of the Hügelschäffer cubic curve (1) as in [30] and [33]. Let w be the distance between the two circle centers when constructing the curve ($w = |O_1O_2|$, see Fig. 2). Then, we consider the two cases (I) $w < a$ and (II) $w > a$. The abscissa $\boldsymbol{u}$ at which the Hügelschäffer curve reaches its extremes over the segment $[-a, a]$ is defined in [33] with the following formula:

$$\boldsymbol{u} = \begin{cases} -w & : \quad w < a, \\ -a^2/w & : \quad w > a. \end{cases} \tag{3}$$

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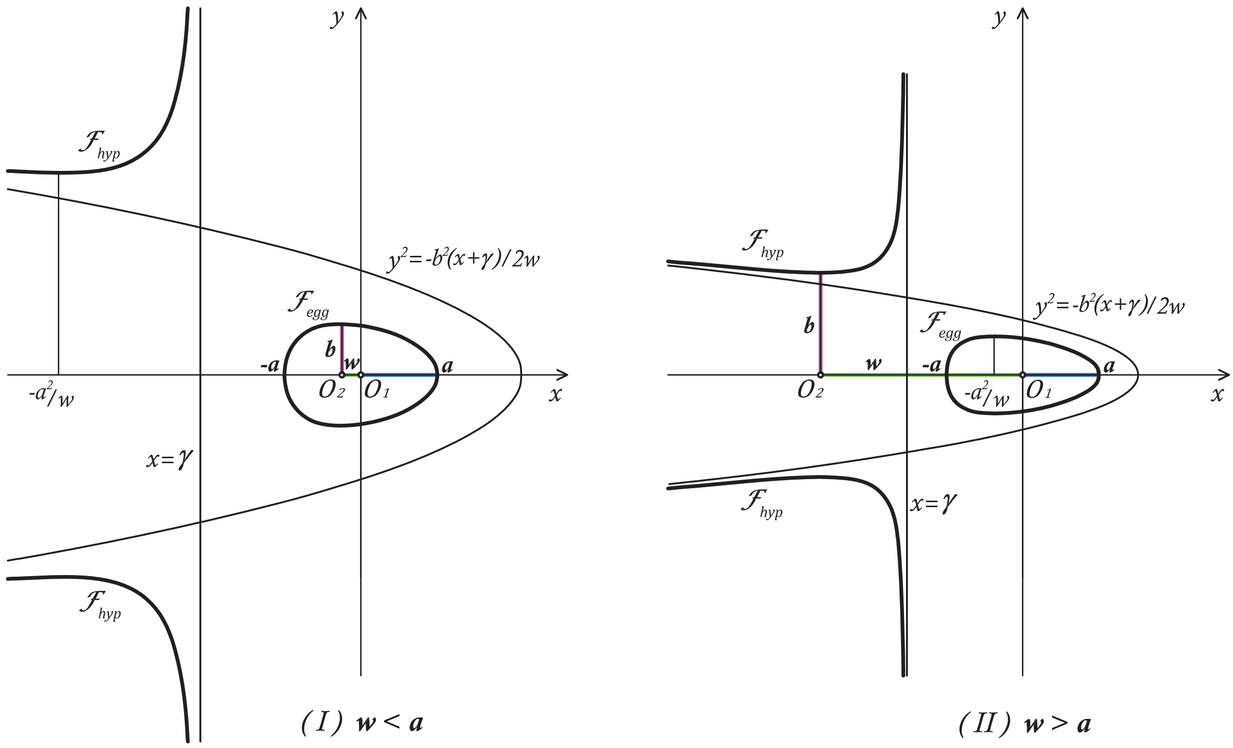

Figure 1: A Hügelschäffer curve $\mathcal{F} = \mathcal{F}_{egg} \cup \mathcal{F}_{hyp}$; Source: © First Author

Introducing the parameter $\boldsymbol{q}$ as:

$$\boldsymbol{q} = \left\{ \begin{array}{ccc} 1 & : & w < a, \\ a/w & : & w > a \end{array} \right\} = \frac{\sqrt{a}}{\sqrt{w}} \, \frac{\min\{\sqrt{a},\ \sqrt{w}\}}{\max\{\sqrt{a},\ \sqrt{w}\}}. \tag{4}$$

Then, it is true that $\boldsymbol{q} \in (0, 1]$.
The relationship between $\boldsymbol{u}$ and $\boldsymbol{q}$ (i.e. (3) and (4)) is given as

$$\boldsymbol{u} = -\boldsymbol{q}^2 w. \tag{5}$$

Only case (I) of the cubic curve $\mathcal{F}_{egg}$ was considered by F. Hügelschäffer in the work [11], while case (II) was introduced by M. Petrović in the thesis [30]. In this section we show that it is possible to unify these two cases:

$$\mathcal{F}_{\boldsymbol{q}}: \quad 2\boldsymbol{q}^2 wxy^2 + \boldsymbol{q}^2 b^2 x^2 + (a^2 + \boldsymbol{q}^4 w^2)y^2 - a^2 b^2 \boldsymbol{q}^2 = 0, \tag{6}$$

using the parameter $\boldsymbol{q}$ given by the formula (4). For the curve (6), the following:

$$\mathcal{F}_{\boldsymbol{q}} \equiv \mathcal{F} \tag{7}$$

holds if and only if

$$\frac{2\boldsymbol{q}^2 w}{2w} = \frac{\boldsymbol{q}^2 b^2}{b^2} = \frac{a^2 + \boldsymbol{q}^4 w^2}{a^2 + w^2} = \frac{a^2 b^2 \boldsymbol{q}^2}{a^2 b^2} \quad \Longleftrightarrow \quad \left(\boldsymbol{q}^2 - 1\right)\left(\boldsymbol{q}^2 w^2 - a^2\right) = 0, \tag{8}$$

i.e. the parameter $\boldsymbol{q}$ holds as defined in (4). Furthermore,

$$\mathcal{F}_{\boldsymbol{q}} = \mathcal{F}_{\boldsymbol{q},egg} \cup \mathcal{F}_{\boldsymbol{q},hyp}, \tag{9}$$

where $\mathcal{F}_{\boldsymbol{q},egg}$ is the egg-shaped part of $\mathcal{F}_{\boldsymbol{q}}$ over $[-a, a]$ and $\mathcal{F}_{\boldsymbol{q},hyp}$ is the hyperbolic-shaped part of $\mathcal{F}_{\boldsymbol{q}}$ over $(-\infty, \gamma_{\boldsymbol{q}})$, for $\gamma_{\boldsymbol{q}} = \gamma$.

From the geometric point of view, Hügelschäffer's construction of the egg-shaped part of the curve is defined using two non-concentric circles as considered in [11], [12], [30], [31], [37] and [38]. An analogous construction for $\mathcal{F}_{\boldsymbol{q},egg}$ is defined using the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ (see Fig. 2) given by the parametric equations

$$\mathcal{K}_1 : \quad \begin{cases} x_1(t) = a\cos t, \\ y_1(t) = a\sin t \end{cases} \qquad \text{and} \qquad \mathcal{K}_2 : \quad \begin{cases} x_2(t) = -\boldsymbol{q}^2 w + \boldsymbol{q}\, b\cos t, \\ y_2(t) = \boldsymbol{q}\, b\sin t \end{cases} \tag{10}$$

for $t \in [0, 2\pi]$, such that the points $P_t = \big(x(t), y(t)\big) \in \mathcal{F}_{\boldsymbol{q},egg}$ have the following coordinates

$$\mathcal{F}_{\boldsymbol{q},egg} : \quad \begin{cases} x(t) = -\boldsymbol{q}^2 w\sin^2 t + \cos t\sqrt{a^2 - \boldsymbol{q}^4 w^2 \sin^2 t}, \\ y(t) = \boldsymbol{q}\, b\sin t, \end{cases} \tag{11}$$

for $t \in [0, 2\pi]$.

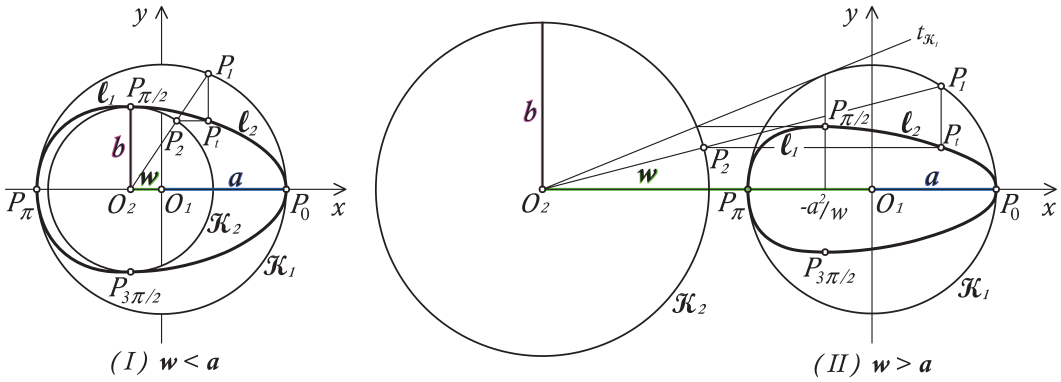

Figure 2: Hügelschäffer's construction of an egg curve $\mathcal{F}_{\boldsymbol{q},egg}$; Source: © First Author

The parametrization (11), for $0 \le t \le \pi$, determines the upper part $\mathcal{F}^+_{\boldsymbol{q},egg}$ of curve $\mathcal{F}_{\boldsymbol{q},egg}$ from point $P_0 = (a, 0)$ to point $P_\pi = (-a, 0)$; while, for $\pi \le t \le 2\pi$, the lower part $\mathcal{F}^-_{\boldsymbol{q},egg}$ of curve $\mathcal{F}_{\boldsymbol{q},egg}$ is determined from point $P_\pi = (-a, 0)$ to point $P_{2\pi} = P_0 = (a, 0)$. Let us note that (5) holds for the abscissa of point $P_{\pi/2} = (-\boldsymbol{q}^2 w, \boldsymbol{q}\, b)$.

The upper part $\mathcal{F}^+_{\boldsymbol{q},egg}$ can be further decomposed into an union of two disjunct portions

$$\mathcal{F}^+_{\boldsymbol{q},egg} = \ell_1 \cup \ell_2, \tag{12}$$

such that for $0 \le t < \pi/2$ we get ℓ_2 and for $\pi/2 \le t \le \pi$ we get ℓ_1 (see Fig. 2).

Let us note that in the paper [33], surfaces $\mathcal{A}_1$ and $\mathcal{A}_2$ were considered for the egg-shaped part $\mathcal{F}_{egg}$ of the Hügelschäffer curve from (2) (i.e. $\mathcal{F}_{\boldsymbol{q},egg}$ from (9) because (7) holds when $\boldsymbol{q} = 1$) wherein the arc ℓ_1 is part of the boundary of surface $\mathcal{A}_1$ and arc ℓ_2 is part of the boundary of surface $\mathcal{A}_2$.

2. The area of curve $\mathcal{F}_{\boldsymbol{q},egg}$

In this paper, we give a new formula for calculating the area of the surface bound by the curve $\mathcal{F}_{\boldsymbol{q},egg}$ using the elliptic integral of the first kind, [13]:

$$\boldsymbol{K}(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}, \quad 0 \le k^2 < 1,$$

and the elliptic integral of the second kind, [13]:

$$\boldsymbol{E}(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta, \quad 0 \le k^2 \le 1.$$

Let it be noted that for $\boldsymbol{K}$ and $\boldsymbol{E}$, it holds that

$$\boldsymbol{K}(k) = F\left(\frac{\pi}{2}, k\right) \qquad \text{and} \qquad \boldsymbol{E}(k) = E\left(\frac{\pi}{2}, k\right)$$

where $F(\theta, k)$ and $E(\theta, k)$ are elliptic integrals defined in [13] with the formulae 8.112/1 and 8.112/2. Let us also note that inequalities for these elliptic integrals were given recently in [16].

If the values of the parameter t from π to 0 are considered, then the points $(x(t), y(t))$ are located on $\mathcal{F}^{+}_{\boldsymbol{q},egg}$, from $P_\pi = (-a, 0)$ to $P_0 = (a, 0)$. In view of this, let $\mathcal{A}_{\boldsymbol{q},egg}$ denote the area of the surface bounded by the curve $\mathcal{F}_{\boldsymbol{q},egg}$ (see (11)), as given by the integral

$$\mathcal{A}_{\boldsymbol{q},egg} = 2\int_\pi^0 y(t)\, x'(t)\, dt = -2\int_0^\pi y(t)\, x'(t)\, dt\,. \tag{13}$$

Let

$$\mathcal{A}_{\boldsymbol{q},egg} = \mathcal{A}_{\boldsymbol{q},2} + \mathcal{A}_{\boldsymbol{q},1}, \tag{14}$$

where

$$\mathcal{A}_{\boldsymbol{q},2} = -2\int_0^{\pi/2} y(t)\, x'(t)\, dt \tag{15}$$

and

$$\mathcal{A}_{\boldsymbol{q},1} = -2\int_{\pi/2}^{\pi} y(t)\, x'(t)\, dt\,. \tag{16}$$

1. Firstly, $\mathcal{A}_{\boldsymbol{q},2}$ is calculated. Substituting $x(t)$ and $y(t)$ from (11) in (15), we get

$$\mathcal{A}_{\boldsymbol{q},2} = 2\int_0^{\pi/2} \boldsymbol{q}\, b \sin t \left(2\, \boldsymbol{q}^2 w \sin t \cos t + \sin t \sqrt{a^2 - \boldsymbol{q}^4 w^2 \sin^2 t} + \frac{\boldsymbol{q}^4 w^2 \sin t \cos^2 t}{\sqrt{a^2 - \boldsymbol{q}^4 w^2 \sin^2 t}} \right) dt\,.$$

Let

$$I_1 = \int_0^{\pi/2} \sin^2 t \cos t\, dt, \tag{17}$$

$$I_2 = \int_0^{\pi/2} \sin^2 t \sqrt{1 - \frac{\boldsymbol{q}^4 w^2}{a^2} \sin^2 t}\, dt, \tag{18}$$

$$I_3 = \int_0^{\pi/2} \frac{\sin^2 t \cos^2 t}{\sqrt{1 - \frac{\boldsymbol{q}^4 w^2}{a^2} \sin^2 t}}\, dt. \tag{19}$$

Then

$$\mathcal{A}_{q,2} = 4\,\boldsymbol{q}^3 w\, b\, I_1 + 2\,\boldsymbol{q}\, a\, b\, I_2 + 2\,\frac{\boldsymbol{q}^5 b\, w^2}{a}\, I_3\,. \tag{20}$$

It holds that

$$I_1 = \frac{1}{3} \tag{21}$$

For I_2, formula 2.583/4 from [13] is used and it follows that

$$\begin{aligned} I_2 &= \left(-\frac{\sin t \cos t}{3}\sqrt{1 - k^2 \sin^2 t} + \frac{1-k^2}{3k^2} F(t,k) + \frac{2k^2-1}{3k^2} E(t,k)\right)\Bigg|_0^{\pi/2} \\ &= \frac{1-k^2}{3k^2} F(\tfrac{\pi}{2},k) + \frac{2k^2-1}{3k^2} E(\tfrac{\pi}{2},k) - \frac{1-k^2}{3k^2} F(0,k) - \frac{2k^2-1}{3k^2} E(0,k) \\ &= \frac{1-k^2}{3k^2} \boldsymbol{K}(k) + \frac{2k^2-1}{3k^2} \boldsymbol{E}(k), \end{aligned} \tag{22}$$

where $k^2 = \dfrac{\boldsymbol{q}^4 w^2}{a^2}$ and it is true that $0 < k^2 < 1$.

For I_3 formula 2.584/13 from [13] is used and it follows that

$$\begin{aligned} I_3 &= \left(-\frac{\sin t \cos t}{3k^2}\sqrt{1 - k^2 \sin^2 t} + \frac{2k^2-2}{3k^4} F(t,k) + \frac{2-k^2}{3k^4} E(t,k)\right)\Bigg|_0^{\pi/2} \\ &= \frac{2k^2-2}{3k^4} F(\tfrac{\pi}{2},k) + \frac{2-k^2}{3k^4} E(\tfrac{\pi}{2},k) - \frac{2k^2-2}{3k^4} F(0,k) - \frac{2-k^2}{3k^4} E(0,k) \\ &= \frac{2k^2-2}{3k^4} \boldsymbol{K}(k) + \frac{2-k^2}{3k^4} \boldsymbol{E}(k), \end{aligned} \tag{23}$$

where $k^2 = \dfrac{\boldsymbol{q}^4 w^2}{a^2}$ and it is true that $0 < k^2 < 1$. Based on the previous, it holds that

$$\begin{aligned} \mathcal{A}_{q,2} &= 4\,\boldsymbol{q}^3 w\, b\, I_1 + 2\,\boldsymbol{q}\, a\, b\, I_2 + 2\,\frac{\boldsymbol{q}^5 b\, w^2}{a}\, I_3 \\ &= \frac{4}{3}\,\boldsymbol{q}^3 w\, b + 2\,\boldsymbol{q}\, a\, b \left(\frac{1-k^2}{3k^2}\boldsymbol{K}(k) + \frac{2k^2-1}{3k^2}\boldsymbol{E}(k)\right) + \\ &\quad + 2\,\frac{\boldsymbol{q}^5 b\, w^2}{a}\left(\frac{2k^2-2}{3k^4}\boldsymbol{K}(k) + \frac{2-k^2}{3k^4}\boldsymbol{E}(k)\right) \\ &= \frac{2}{3}\, a\, b\, \boldsymbol{q}\left(\left(1 - \frac{1}{k^2}\right)\boldsymbol{K}(k) + \left(1 + \frac{1}{k^2}\right)\boldsymbol{E}(k) + 2k\right). \end{aligned} \tag{24}$$

2. To calculate $\mathcal{A}_{q,1}$, substituting $x(t)$ and $y(t)$ from (11) in (16), we get

$$\mathcal{A}_{q,1} = 2\int_{\pi/2}^{\pi} \boldsymbol{q}\, b \sin t \left(2\,\boldsymbol{q}^2 w \sin t \cos t + \sin t\sqrt{a^2 - \boldsymbol{q}^4 w^2 \sin^2 t} + \frac{\boldsymbol{q}^4 w^2 \sin t \cos^2 t}{\sqrt{a^2 - \boldsymbol{q}^4 w^2 \sin^2 t}}\right) dt\,.$$

Let

$$J_1 = \int_{\pi/2}^{\pi} \sin^2 t \cos t \, dt, \tag{25}$$

$$J_2 = \int_{\pi/2}^{\pi} \sin^2 t \sqrt{1 - \frac{\boldsymbol{q}^4 w^2}{a^2} \sin^2 t} \, dt, \tag{26}$$

$$J_3 = \int_{\pi/2}^{\pi} \frac{\sin^2 t \cos^2 t}{\sqrt{1 - \frac{\boldsymbol{q}^4 w^2}{a^2} \sin^2 t}} \, dt. \tag{27}$$

Then

$$\mathcal{A}_{\boldsymbol{q},1} = 4\,\boldsymbol{q}^3 w\, b\, J_1 + 2\,\boldsymbol{q}\, a\, b\, J_2 + 2\,\frac{\boldsymbol{q}^5 b\, w^2}{a}\, J_3\,. \tag{28}$$

It holds that

$$J_1 = -\frac{1}{3}, \quad J_2 = \int_0^{\pi/2} \cos^2 t \sqrt{1 - \frac{\boldsymbol{q}^4 w^2}{a^2} \cos^2 t} \, dt, \quad J_3 = \int_0^{\pi/2} \frac{\cos^2 t \sin^2 t}{\sqrt{1 - \frac{\boldsymbol{q}^4 w^2}{a^2} \cos^2 t}} \, dt. \tag{29}$$

Analogously to the previous

$$J_2 = \frac{1-k^2}{3k^2}\boldsymbol{K}(k) + \frac{2k^2-1}{3k^2}\boldsymbol{E}(k) = I_2, \quad J_3 = \frac{2k^2-2}{3k^4}\boldsymbol{K}(k) + \frac{2-k^2}{3k^4}\boldsymbol{E}(k) = I_3, \tag{30}$$

where $k^2 = \dfrac{\boldsymbol{q}^4 w^2}{a^2}$ and it is true that $0 < k^2 < 1$. Thus, we obtain that

$$\begin{aligned} \mathcal{A}_{\boldsymbol{q},1} &= 4\,\boldsymbol{q}^3 w\, b\, J_1 + 2\,\boldsymbol{q}\, a\, b\, J_2 + 2\,\frac{q^5 b\, w^2}{a}\, J_3 \\ &= \frac{2}{3}\, a\, b\, \boldsymbol{q} \left(\left(1 - \frac{1}{k^2}\right)\boldsymbol{K}(k) + \left(1 + \frac{1}{k^2}\right)\boldsymbol{E}(k) - 2k \right). \end{aligned} \tag{31}$$

In all, the following theorem has been proven.

Theorem 2.1. *For the area $\mathcal{A}_{\boldsymbol{q},egg}$ of curve $\mathcal{F}_{\boldsymbol{q},egg}$ it holds that*:

$$\mathcal{A}_{\boldsymbol{q},egg} = \mathcal{A}_{\boldsymbol{q},2} + \mathcal{A}_{\boldsymbol{q},1} = \frac{4}{3}\, a\, b\, \boldsymbol{q} \left(\left(1 - \frac{1}{k^2}\right)\boldsymbol{K}(k) + \left(1 + \frac{1}{k^2}\right)\boldsymbol{E}(k) \right) \tag{32}$$

and

$$\mathcal{A}_{\boldsymbol{q},2} - \mathcal{A}_{\boldsymbol{q},1} = \frac{8}{3}\, a\, b\, \boldsymbol{q}\, k = \frac{8}{3}\, w\, b\, \boldsymbol{q}^3 = \begin{cases} 8\,w\,b/3 & : \quad w < a, \\ \dfrac{8}{3\,w^2}\, b\, a^3 & : \quad w > a; \end{cases} \tag{33}$$

where a, b, w are Hügelschäffer curve parameters and $k = \dfrac{\boldsymbol{q}^2 w}{a}$.

Remark 2.2. *Let us note that, for cases (I) and (II), the equality (33) is derived in [33] (p. 185).*

Alongside the initial applications of Hügelschäffer curves in aero-engineering (see [7], [11]), recently, there has been research on the applications of these curves in: architecture and civil engineering (see [30], [32]); poultry industry, ornithology and bioengineering (see [15], [23], [24], [25], [26], [27], [28]); traffic engineering (see [34]) and hydro-engineering (see [14], [33], [44]). To aid in the application of Hügelschäffer curves and the practical usage of the area formulae for these curves, we have developed the applet [21].

3. Taylor approximations of elliptic integrals

Let $f:(\alpha,\beta)\longrightarrow\mathbb{R}$ be a real function. We state some definitions and characteristics according to [19].

Definition 3.1. *Let $T_n^{f,\alpha_+}(x)$ be a Taylor polynomial for function $f(x)$, of degree $n\in\mathbb{N}_0$, in the right neighborhood of point α. If, for the real function $f:(\alpha,\beta)\longrightarrow\mathbb{R}$, there exist finite limits $f^{(i)}(\alpha_+)=\lim_{x\to\alpha_+}f^{(i)}(x)$, for $i\in\{0,1,\dots,n\}$ then*

$$T_n^{f,\alpha_+}(x)=\sum_{i=0}^{n}\frac{f^{(i)}(\alpha_+)}{i!}(x-\alpha)^i \tag{34}$$

is the first Taylor approximation of function f in the right neighborhood of point α, for $n\in\mathbb{N}_0$, where

$$R_n^{f,\alpha_+}(x)=f(x)-T_{n-1}^{f,\alpha_+}(x) \tag{35}$$

is the remainder of the first Taylor approximation in the right neighborhood of point α.

Definition 3.2. *Let $T_n^{f,\beta_-}(x)$ be a Taylor polynomial for function $f(x)$, of degree $n\in\mathbb{N}_0$, in the left neighborhood of point β. If, for the real function $f:(\alpha,\beta)\longrightarrow\mathbb{R}$, there exist finite limits $f^{(i)}(\beta_-)=\lim_{x\to\beta_-}f^{(i)}(x)$, for $i\in\{0,1,\dots,n\}$ then*

$$T_n^{f,\beta_-}(x)=\sum_{i=0}^{n}\frac{f^{(i)}(\beta_-)}{i!}(x-\beta)^i \tag{36}$$

is the first Taylor approximation of function f in the left neighborhood of point β, for $n\in\mathbb{N}_0$, where

$$R_n^{f,\beta_-}(x)=f(x)-T_{n-1}^{f,\beta_-}(x) \tag{37}$$

is the remainder of the first Taylor approximation in the left neighborhood of point β.

Definition 3.3. *For the polynomial of the form*

$$\mathbb{T}_n^{f;\alpha_+,\beta_-}(x)=\begin{cases}T_{n-1}^{f,\alpha_+}(x)+\dfrac{(x-\alpha)^n}{(\beta-\alpha)^n}R_n^{f,\alpha_+}(\beta_-) & : \quad n\geq 1\\ f(\beta_-) & : \quad n=0\end{cases} \tag{38}$$

we say that it is the second Taylor approximation of function f in the right neighborhood of point α, for $n\in\mathbb{N}_0$, while the polynomial

$$\mathbb{T}_n^{f;\beta_-,\alpha_+}(x)=\begin{cases}T_{n-1}^{f,\beta_-}(x)+\dfrac{(x-\beta)^n}{(\alpha-\beta)^n}R_n^{f,\beta_-}(\alpha_+) & : \quad n\geq 1\\ f(\alpha_+) & : \quad n=0\end{cases} \tag{39}$$

is the second Taylor approximation of function f in the left neighborhood of point β, for $n\in\mathbb{N}_0$.

Theorem 3.4 (Theorem WD). *Suppose that function f is real over (α,β), i.e. $f:(\alpha,\beta)\longrightarrow\mathbb{R}$ and let n be a whole natural number such that $f^{(i)}(\alpha_+)$ and $f^{(i)}(\beta_-)$ exist, for $i\in\{0,1,\dots,n\}$.*

Also suppose that $(-1)^{(n)}f^{(n)}(x)$ is increasing over (α,β). Then, for each $x\in(\alpha,\beta)$ the following inequality holds:

$$\mathbb{T}_n^{f;\beta_-,\alpha_+}(x)<f(x)<T_n^{f;\beta_-}(x) \tag{40}$$

and supposing that $f^{(n)}(x)$ is increasing over (α,β), then for each $x\in(\alpha,\beta)$ it holds that:

$$\mathbb{T}_n^{f;\alpha_+,\beta_-}(x)>f(x)>T_n^{f;\alpha+}(x) \tag{41}$$

When the function $(-1)^{(n)}f^{(n)}(x)$ is decreasing over (α,β), or when $f^{(n)}(x)$ is decreasing over (α,β) then for each $x\in(\alpha,\beta)$ the reverse inequalities hold.

The preceding theorem was proven by S. Wu and L. Debnath u [43]. In [19] and [20] some applications of this statement were considered within the Theory of analytical inequalities, see also papers [4] and [9].

3.1 Taylor approximations of *K* and *E* elliptic integrals

Let us apply the previous consideration to elliptic integrals $\boldsymbol{K}(x)$ over $(-1,1)$ and $\boldsymbol{E}(x)$ over $[-1,1]$. We will use the well-known series expansions:

$$\boldsymbol{K}(x) = \frac{\pi}{2}\sum_{i=0}^{\infty}\left(\frac{(2i-1)!!}{(2i)!!}\right)^2 x^{2i}, \tag{42}$$

for $x\in(-1,1)$, see formula 8.113/1 in [13] and

$$\boldsymbol{E}(x) = \frac{\pi}{2} - \frac{\pi}{2}\sum_{i=1}^{\infty}\left(\frac{(2i-1)!!}{(2i)!!}\right)^2 \frac{1}{2i-1}\, x^{2i}, \tag{43}$$

for $x\in[-1,1]$, see formula 8.114/1 in [13]. Let us note that the series expansion (43) has a radius of 1 and that $x = 1$ can be included in the convergence domain, and that the series expansion (42) has a radius of 1 and that $x = 1$ can not be included in the convergence domain.

For the elliptic integral $\boldsymbol{K}(x)$, based on the series expansion (42), the first and second Taylor approximations are obtained. The first Taylor approximation is:

$$T_n^{\boldsymbol{K},0}(x) = \frac{\pi}{2}\sum_{i=0}^{[n/2]}\left(\frac{(2i-1)!!}{(2i)!!}\right)^2 x^{2i}, \tag{44}$$

where $x\in(-1,1)$. For a fixed $\beta\in(0,1)$, the second Taylor approximation in the right neighborhood of point $\alpha = 0$ is:

$$\mathbb{T}_n^{\boldsymbol{K};0,\beta}(x) = \begin{cases} T_{n-1}^{\boldsymbol{K},0}(x) + \dfrac{x^n}{\beta^n}\left(\boldsymbol{K}(\beta) - T_{n-1}^{\boldsymbol{K},0}(\beta)\right) & : \quad n \geq 1 \\ \boldsymbol{K}(\beta) & : \quad n = 0 \end{cases}, \tag{45}$$

where $x\in[-\beta,\beta]$.

For the elliptic integral $\boldsymbol{E}(x)$, based on the series expansion (43), the first and second Taylor approximations are obtained. The first Taylor approximation is:

$$T_n^{\boldsymbol{E},0}(x) = \frac{\pi}{2} - \frac{\pi}{2}\sum_{i=1}^{[n/2]}\left(\frac{(2i-1)!!}{(2i)!!}\right)^2 \frac{1}{2i-1}\, x^{2i}, \tag{46}$$

where $x\in[-1,1]$. For a fixed $\beta\in(0,1]$, the second Taylor approximation in the right neighborhood of point $\alpha = 0$ is:

$$\mathbb{T}_n^{\boldsymbol{E};0,\beta}(x) = \begin{cases} T_{n-1}^{\boldsymbol{E},0}(x) + \dfrac{x^n}{\beta^n}\left(\boldsymbol{E}(\beta) - T_{n-1}^{\boldsymbol{E},0}(\beta)\right) & : \quad n \geq 1 \\ \boldsymbol{E}(\beta) & : \quad n = 0 \end{cases}, \tag{47}$$

where $x\in[-\beta,\beta]$.

According to [19], the following inequalities hold for the elliptic integral $\boldsymbol{K}(x)$:

$$\begin{gathered} \frac{\pi}{2} = T_0^{\boldsymbol{K},0}(x) = T_1^{\boldsymbol{K},0}(x) \leq T_2^{\boldsymbol{K},0}(x) = T_3^{\boldsymbol{K},0}(x) \leq \ldots \leq T_{2i}^{\boldsymbol{K},0}(x) = T_{2i+1}^{\boldsymbol{K},0}(x) \leq \ldots \\ \leq \boldsymbol{K}(x) \leq \\ \ldots \leq \mathbb{T}_j^{\boldsymbol{K};0,\beta}(x) \leq \ldots \leq \mathbb{T}_3^{\boldsymbol{K};0,\beta}(x) \leq \mathbb{T}_2^{\boldsymbol{K};0,\beta}(x) \leq \mathbb{T}_1^{\boldsymbol{K};0,\beta}(x) \leq \mathbb{T}_0^{\boldsymbol{K};0,\beta}(x) = \boldsymbol{K}(\beta) \end{gathered} \tag{48}$$

for a fixed $\beta \in (0,1)$ and an arbitrary $x \in [-\beta,\beta]$. Furthermore, according to [19] the following inequalities hold for the elliptic integral $\boldsymbol{E}(x)$:

$$\begin{aligned} \frac{\pi}{2} = T_0^{\boldsymbol{E},0}(x) = T_1^{\boldsymbol{E},0}(x) \geq T_2^{\boldsymbol{E},0}(x) = T_3^{\boldsymbol{E},0}(x) \geq \ldots \geq T_{2i}^{\boldsymbol{E},0}(x) = T_{2i+1}^{\boldsymbol{E},0}(x) \geq \ldots \\ \geq \boldsymbol{E}(x) \geq \\ \ldots \geq \mathbb{T}_j^{\boldsymbol{E};0,\beta}(x) \geq \ldots \geq \mathbb{T}_3^{\boldsymbol{E};0,\beta}(x) \geq \mathbb{T}_2^{\boldsymbol{E};0,\beta}(x) \geq \mathbb{T}_1^{\boldsymbol{E};0,\beta}(x) \geq \mathbb{T}_0^{\boldsymbol{E};0,\beta}(x) = \boldsymbol{E}(\beta) \end{aligned} \tag{49}$$

for a fixed $\beta \in (0,1]$ and an arbitrary $x \in [-\beta,\beta]$.

In table 1, we list the explicit forms of the polynomials $T_j^{\boldsymbol{K},0}(x)$ and $\mathbb{T}_j^{\boldsymbol{K};0,\beta}(x)$, for $\beta \in (0,1)$ and $j = 0,1,\ldots,10$; while, in table 2, we list the explicit forms of the polynomials $T_j^{\boldsymbol{E},0}(x)$ and $\mathbb{T}_j^{\boldsymbol{E};0,1}(x)$, for $j = 0,1,\ldots,10$.

j	$T_j^{\boldsymbol{K},0}(x)$	$\mathbb{T}_j^{\boldsymbol{K};0,\beta}(x)$ and $C_K = \boldsymbol{K}(\beta)$ for $\beta \in (0,1)$
0	$\frac{\pi}{2}$	C_K
1	$\frac{\pi}{2}$	$T_0^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta} - \frac{\pi}{2\beta}\right)x$
2	$\frac{\pi}{2} + \frac{\pi}{8}x^2$	$T_1^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta^2} - \frac{\pi}{2\beta^2}\right)x^2$
3	$\frac{\pi}{2} + \frac{\pi}{8}x^2$	$T_2^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta^3} - \frac{\pi}{2\beta^3} - \frac{\pi}{8\beta}\right)x^3$
4	$\frac{\pi}{2} + \frac{\pi}{8}x^2 + \frac{9\pi}{128}x^4$	$T_3^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta^4} - \frac{\pi}{2\beta^4} - \frac{\pi}{8\beta^2}\right)x^4$
5	$\frac{\pi}{2} + \frac{\pi}{8}x^2 + \frac{9\pi}{128}x^4$	$T_4^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta^5} - \frac{\pi}{2\beta^5} - \frac{\pi}{8\beta^3} - \frac{9\pi}{128\beta}\right)x^5$
6	$\frac{\pi}{2} + \frac{\pi}{8}x^2 + \frac{9\pi}{128}x^4 + \frac{25\pi}{512}x^6$	$T_5^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta^6} - \frac{\pi}{2\beta^6} - \frac{\pi}{8\beta^4} - \frac{9\pi}{128\beta^2}\right)x^6$
7	$\frac{\pi}{2} + \frac{\pi}{8}x^2 + \frac{9\pi}{128}x^4 + \frac{25\pi}{512}x^6$	$T_6^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta^7} - \frac{\pi}{2\beta^7} - \frac{\pi}{8\beta^5} - \frac{9\pi}{128\beta^3} - \frac{25\pi}{512\beta}\right)x^7$
8	$\frac{\pi}{2} + \frac{\pi}{8}x^2 + \frac{9\pi}{128}x^4 + \frac{25\pi}{512}x^6 + \frac{1225\pi}{32768}x^8$	$T_7^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta^8} - \frac{\pi}{2\beta^8} - \frac{\pi}{8\beta^6} - \frac{9\pi}{128\beta^4} - \frac{25\pi}{512\beta^2}\right)x^8$
9	$\frac{\pi}{2} + \frac{\pi}{8}x^2 + \frac{9\pi}{128}x^4 + \frac{25\pi}{512}x^6 + \frac{1225\pi}{32768}x^8$	$T_8^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta^9} - \frac{\pi}{2\beta^9} - \frac{\pi}{8\beta^7} - \frac{9\pi}{128\beta^5} - \frac{25\pi}{512\beta^3} - \frac{1225\pi}{32768\beta}\right)x^9$
10	$\frac{\pi}{2} + \frac{\pi}{8}x^2 + \frac{9\pi}{128}x^4 + \frac{25\pi}{512}x^6 + \frac{1225\pi}{32768}x^8 + \frac{3969\pi}{131072}x^{10}$	$T_9^{\boldsymbol{K};0}(x) + \left(\frac{C_K}{\beta^{10}} - \frac{\pi}{2\beta^{10}} - \frac{\pi}{8\beta^8} - \frac{9\pi}{128\beta^6} - \frac{25\pi}{512\beta^4} - \frac{1225\pi}{32768\beta^2}\right)x^{10}$

Table 1.

j	$T_j^{\boldsymbol{E},0}(x)$	$\mathbb{T}_j^{\boldsymbol{E};0,1}(x)$ and $\beta=1$
0	$\frac{\pi}{2}$	1
1	$\frac{\pi}{2}$	$T_0^{\boldsymbol{E},0}(x)+\left(1-\frac{\pi}{2}\right)x$
2	$\frac{\pi}{2}-\frac{\pi}{8}x^2$	$T_1^{\boldsymbol{E},0}(x)+\left(1-\frac{\pi}{2}\right)x^2$
3	$\frac{\pi}{2}-\frac{\pi}{8}x^2$	$T_2^{\boldsymbol{E},0}(x)+\left(1-\frac{3\pi}{8}\right)x^3$
4	$\frac{\pi}{2}-\frac{\pi}{8}x^2-\frac{3\pi}{128}x^4$	$T_3^{\boldsymbol{E},0}(x)+\left(1-\frac{3\pi}{8}\right)x^4$
5	$\frac{\pi}{2}-\frac{\pi}{8}x^2-\frac{3\pi}{128}x^4$	$T_4^{\boldsymbol{E},0}(x)+\left(1-\frac{45\pi}{128}\right)x^5$
6	$\frac{\pi}{2}-\frac{\pi}{8}x^2-\frac{3\pi}{128}x^4-\frac{5\pi}{512}x^6$	$T_5^{\boldsymbol{E},0}(x)+\left(1-\frac{45\pi}{128}\right)x^6$
7	$\frac{\pi}{2}-\frac{\pi}{8}x^2-\frac{3\pi}{128}x^4-\frac{5\pi}{512}x^6$	$T_6^{\boldsymbol{E},0}(x)+\left(1-\frac{175\pi}{512}\right)x^7$
8	$\frac{\pi}{2}-\frac{\pi}{8}x^2-\frac{3\pi}{128}x^4-\frac{5\pi}{512}x^6-\frac{175\pi}{32768}x^8$	$T_7^{\boldsymbol{E},0}(x)+\left(1-\frac{175\pi}{512}\right)x^8$
9	$\frac{\pi}{2}-\frac{\pi}{8}x^2-\frac{3\pi}{128}x^4-\frac{5\pi}{512}x^6-\frac{175\pi}{32768}x^8$	$T_8^{\boldsymbol{E},0}(x)+\left(1-\frac{11025\pi}{32768}\right)x^9$
10	$\frac{\pi}{2}-\frac{\pi}{8}x^2-\frac{3\pi}{128}x^4-\frac{5\pi}{512}x^6-\frac{175\pi}{32768}x^8-\frac{441\pi}{131072}x^{10}$	$T_9^{\boldsymbol{E},0}(x)+\left(1-\frac{11025\pi}{32768}\right)x^{10}$

Table 2.

3.2 Taylor approximations of *D*-elliptic integrals

For the process of determining a formula for the area $\mathcal{A}_{q,egg}$ of curve $\mathcal{F}_{q,egg}$, Taylor approximations of D-elliptic integrals are of special interest. D-elliptic integrals are determined as follows

$$\boldsymbol{D}(x)=\begin{cases}\dfrac{\boldsymbol{K}(x)-\boldsymbol{E}(x)}{x^2} & : \quad x\in(-1,1)\backslash\{0\}\\ 0 & : \quad x=0\end{cases}, \tag{50}$$

see formula 8.112/5 in [13]. For D-elliptic integrals, the series expansion

$$\boldsymbol{D}(x)=\pi\sum_{i=0}^{\infty}\frac{i+1}{2i+1}\left(\frac{(2i+1)!!}{(2i+2)!!}\right)^2x^{2i}, \tag{51}$$

holds for $x\in(-1,1)$, see formula 8.115 in [13]. Let us note that the series expansion (51) has a radius of 1 and that $x=1$ can not be included in the convergence domain.

For the elliptic integral $\boldsymbol{D}(x)$, based on the series expansion (51), the first and second Taylor approximations are obtained. The first Taylor approximation is:

$$T_n^{\boldsymbol{D},0}(x)=\pi\sum_{i=0}^{[n/2]}\frac{i+1}{2i+1}\left(\frac{(2i+1)!!}{(2i+2)!!}\right)^2x^{2i}, \tag{52}$$

where $x \in (-1,1)$. For a fixed $\beta \in (0,1)$, the second Taylor expansion in the right neighborhood of point $\alpha = 0$ is:

$$\mathbb{T}_n^{\boldsymbol{D};0,\beta}(x) = \begin{cases} T_{n-1}^{\boldsymbol{D},0}(x) + \dfrac{x^n}{\beta^n}\left(\boldsymbol{D}(\beta) - T_{n-1}^{\boldsymbol{D},0}(\beta)\right) & : \quad n \geq 1 \\ \boldsymbol{D}(\beta) & : \quad n = 0 \end{cases}, \tag{53}$$

where $x \in [-\beta, \beta]$.

According to [19], the following inequalities hold for the elliptic integral $\boldsymbol{D}(x)$:

$$\begin{array}{c} \dfrac{\pi}{4} = T_0^{\boldsymbol{D},0}(x) = T_1^{\boldsymbol{D},0}(x) \leq T_2^{\boldsymbol{D},0}(x) = T_3^{\boldsymbol{D},0}(x) \leq \ldots \leq T_{2i}^{\boldsymbol{D},0}(x) = T_{2i+1}^{\boldsymbol{D},0}(x) \leq \ldots \\ \leq \boldsymbol{D}(x) \leq \\ \ldots \leq \mathbb{T}_j^{\boldsymbol{D};0,\beta}(x) \leq \ldots \leq \mathbb{T}_3^{\boldsymbol{D};0,\beta}(x) \leq \mathbb{T}_2^{\boldsymbol{D};0,\beta}(x) \leq \mathbb{T}_1^{\boldsymbol{D};0,\beta}(x) \leq \mathbb{T}_0^{\boldsymbol{D};0,\beta}(x) = \boldsymbol{D}(\beta) \end{array} \tag{54}$$

for a fixed $\beta \in (0,1)$ and an arbitrary $x \in [-\beta, \beta]$. In the following table, we list the explicit forms of the polynomials $T_j^{\boldsymbol{D},0}(x)$ and $\mathbb{T}_j^{\boldsymbol{D};0,\beta}(x)$, for $\beta \in (0,1)$ and $j = 0, 1, \ldots, 10$:

j	$T_j^{\boldsymbol{D},0}(x)$	$\mathbb{T}_j^{\boldsymbol{D};0,\beta}(x)$ and $C_D = \boldsymbol{D}(\beta)$ for $\beta \in (0,1)$
0	$\frac{\pi}{4}$	C_D
1	$\frac{\pi}{4}$	$T_0^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta}-\frac{\pi}{4\beta}\right)x$
2	$\frac{\pi}{4}+\frac{3\pi}{32}x^2$	$T_1^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta^2}-\frac{\pi}{4\beta^2}\right)x^2$
3	$\frac{\pi}{4}+\frac{3\pi}{32}x^2$	$T_2^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta^3}-\frac{\pi}{4\beta^3}-\frac{3\pi}{32\beta}\right)x^3$
4	$\frac{\pi}{4}+\frac{3\pi}{32}x^2+\frac{15\pi}{256}x^4$	$T_3^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta^4}-\frac{\pi}{4\beta^4}-\frac{3\pi}{32\beta^2}\right)x^4$
5	$\frac{\pi}{4}+\frac{3\pi}{32}x^2+\frac{15\pi}{256}x^4$	$T_4^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta^5}-\frac{\pi}{4\beta^5}-\frac{3\pi}{32\beta^3}-\frac{15\pi}{256\beta}\right)x^5$
6	$\frac{\pi}{4}+\frac{3\pi}{32}x^2+\frac{15\pi}{256}x^4+\frac{175\pi}{4096}x^6$	$T_5^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta^6}-\frac{\pi}{4\beta^6}-\frac{3\pi}{32\beta^4}-\frac{15\pi}{256\beta^2}\right)x^6$
7	$\frac{\pi}{4}+\frac{3\pi}{32}x^2+\frac{15\pi}{256}x^4+\frac{175\pi}{4096}x^6$	$T_6^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta^7}-\frac{\pi}{4\beta^7}-\frac{3\pi}{32\beta^5}-\frac{15\pi}{256\beta^3}-\frac{175\pi}{4096\beta}\right)x^7$
8	$\frac{\pi}{4}+\frac{3\pi}{32}x^2+\frac{15\pi}{256}x^4+\frac{175\pi}{4096}x^6+\frac{2205\pi}{65536}x^8$	$T_7^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta^8}-\frac{\pi}{4\beta^8}-\frac{3\pi}{32\beta^6}-\frac{15\pi}{256\beta^4}-\frac{175\pi}{4096\beta^2}\right)x^8$
9	$\frac{\pi}{4}+\frac{3\pi}{32}x^2+\frac{15\pi}{256}x^4+\frac{175\pi}{4096}x^6+\frac{2205\pi}{65536}x^8$	$T_8^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta^9}-\frac{\pi}{4\beta^9}-\frac{3\pi}{32\beta^7}-\frac{15\pi}{256\beta^5}-\frac{175\pi}{4096\beta^3}-\frac{2205\pi}{65536\beta}\right)x^9$
10	$\frac{\pi}{4}+\frac{3\pi}{32}x^2+\frac{15\pi}{256}x^4+\frac{175\pi}{4096}x^6+\frac{2205\pi}{65536}x^8+\frac{14553\pi}{524288}x^{10}$	$T_9^{\boldsymbol{D};0}(x)+\left(\frac{C_D}{\beta^{10}}-\frac{\pi}{4\beta^{10}}-\frac{3\pi}{32\beta^8}-\frac{15\pi}{256\beta^6}-\frac{175\pi}{4096\beta^4}-\frac{2205\pi}{65536\beta^2}\right)x^{10}$

Table 3.

Let us note that there is recent research into elliptic integrals of the first and second kind (and their convexity, monotonicity, approximations, inequalities, applications, ...), see the papers: [1], [2], [5], [10], [16], [18], [22], [29], [35], [40], [41], [45]-[50]. An approach to the approximate computation of complete elliptic integrals for practical use in water engineering is given in the paper [39].

4. Taylor approximations for the area of Hügelschäffer curve $\mathcal{F}_{q,egg}$

Let $\mathcal{A}(k)$ be the size of the area $\mathcal{A}_{q,egg}$ of curve $\mathcal{F}_{q,egg}$ as a function of $k \in (0,1)$. For $k = 0$, a degenerative case is obtained wherein the egg-shaped part of the curve is reduced to an ellipse. Then, we define that $\mathcal{A}(0) = ab\boldsymbol{q}\pi$. Additionally, for $k = 1$, another degenerative case is obtained wherein the egg-shaped part of the curve is bounded by a part of a parabola and a line. Then, we define that $\mathcal{A}(1) = \frac{8}{3}ab\boldsymbol{q}$. Based on the degenerative cases and the formula for the non-degenerative case (32), with the use of the $\boldsymbol{D}$-elliptic integral, it follows that

$$\mathcal{A}(k) = \begin{cases} ab\boldsymbol{q}\pi & : \quad k = 0, \\ \dfrac{4}{3}\, a\, b\, \boldsymbol{q}\Big(\boldsymbol{K}(k) + \boldsymbol{E}(k) - \boldsymbol{D}\,(k)\Big) & : \quad k \in (0,1), \\ \dfrac{8}{3}ab\boldsymbol{q} & : \quad k = 1. \end{cases} \tag{55}$$

For $k \in (0,1)$, using the formulae (43), (42) i (51) we obtain a series expansion:

$$\mathcal{A}(k) = a\,b\,\boldsymbol{q}\,\pi - a\,b\,\boldsymbol{q}\,\pi \sum_{i=1}^{\infty} \frac{1}{(2i-1)(i+1)} \left(\frac{(2i-1)!!}{(2i)!!}\right)^2 k^{2i}. \tag{56}$$

Let us note that the series expansion (56) has a radius of 1 and that the value $k = 1$ can be included in the convergence domain in accordance with Raabe's test. Furthermore, for $k = 0$ and for $k = 1$ the formulae (55) and (56) give the same results.

Remark 4.1. *Based on the two previous expressions, (55) and (56), for $k = 1$:*

$$\mathcal{A}(1) = \frac{8}{3}ab\boldsymbol{q} \tag{57}$$

and

$$\mathcal{A}(1) = a\,b\,\boldsymbol{q}\,\pi - a\,b\,\boldsymbol{q}\,\pi \sum_{i=1}^{\infty} \frac{1}{(2i-1)(i+1)} \left(\frac{(2i-1)!!}{(2i)!!}\right)^2 \tag{58}$$

the following representation holds:

$$\frac{1}{\pi} = \frac{3}{8}\left(1 - \sum_{i=1}^{\infty} \frac{1}{(2i-1)(i+1)} \left(\frac{(2i-1)!!}{(2i)!!}\right)^2\right). \tag{59}$$

The approximation of the numbers π and $\frac{1}{\pi}$ was the topic of research in the papers [3], [6], [8], [17], [36] and [42].

For $\mathcal{A}(k)$, based on the series expansion (56), the first and second Taylor approximations are obtained, which we denote with the formulae (60) and (61) in the remainder of this paper.

The first Taylor approximation is:

$$T_n^{\mathcal{A},0}(k) = a\,b\,\boldsymbol{q}\,\pi - a\,b\,\boldsymbol{q}\,\pi \sum_{i=1}^{[n/2]} \frac{1}{(2i-1)(i+1)} \left(\frac{(2i-1)!!}{(2i)!!}\right)^2 k^{2i}, \quad k \in (0,1). \tag{60}$$

For a fixed $\beta \in (0,1]$, the second Taylor approximation in the right neighborhood of point $\alpha = 0$ is:

$$\mathbb{T}_n^{\mathcal{A};0,\beta}(k) = \begin{cases} T_{n-1}^{\mathcal{A},0}(k) + \dfrac{x^n}{\beta^n}\left(\mathcal{A}(\beta) - T_{n-1}^{\mathcal{A},0}(\beta)\right) & : \quad n \geq 1 \\ \mathcal{A}(\beta) & : \quad n = 0 \end{cases}, \quad k \in (0,\beta]. \tag{61}$$

According to [19], the following inequalities hold:

$$
\begin{aligned}
a\,b\,\boldsymbol{q}\,\pi = T_0^{\mathcal{A},0}(k) = T_1^{\mathcal{A},0}(k) \geq T_2^{\mathcal{A},0}(k) = T_3^{\mathcal{A},0}(k) \geq \ldots \geq T_{2i}^{\mathcal{A},0}(k) = T_{2i+1}^{\mathcal{A},0}(k) \geq \ldots \\
\geq \mathcal{A}(k) \geq \qquad\qquad\qquad\qquad\qquad (62)\\
\ldots \geq \mathbb{T}_j^{\mathcal{A};0,\beta}(k) \geq \ldots \geq \mathbb{T}_3^{\mathcal{A};0,\beta}(k) \geq \mathbb{T}_2^{\mathcal{A};0,\beta}(k) \geq \mathbb{T}_1^{\mathcal{A};0,\beta}(k) \geq \mathbb{T}_0^{\mathcal{A};0,\beta}(k) = \mathcal{A}(\beta)
\end{aligned}
$$

for a fixed $\beta \in (0,1)$ and an arbitrary $k \in (0,\beta]$.

We specify a list of Taylor approximations for $\mathcal{A}(k)$ where $k = \dfrac{\boldsymbol{q}^2 w}{a}$, i.e. $k \in (0,1]$, in table 4:

j	$T_j^{\mathcal{A},0}(k)$	$\mathbb{T}_j^{\mathcal{A};0,\beta}(k)$ and $\beta = 1$
0	$a\,b\,\boldsymbol{q}\,\pi$	$\frac{8}{3}a\,b\,\boldsymbol{q}$
1	$a\,b\,\boldsymbol{q}\,\pi$	$T_0^{\mathcal{A},0}(k) + a\,b\,\boldsymbol{q}\left(\frac{8}{3} - \pi\right)k$
2	$a\,b\,\boldsymbol{q}\,\pi\left(1 - \frac{1}{8}k^2\right)$	$T_1^{\mathcal{A},0}(k) + a\,b\,\boldsymbol{q}\left(\frac{8}{3} - \pi\right)k^2$
3	$a\,b\,\boldsymbol{q}\,\pi\left(1 - \frac{1}{8}k^2\right)$	$T_2^{\mathcal{A},0}(k) + a\,b\,\boldsymbol{q}\left(\frac{8}{3} - \frac{7}{8}\pi\right)k^3$
4	$a\,b\,\boldsymbol{q}\,\pi\left(1 - \frac{1}{8}k^2 - \frac{1}{64}k^4\right)$	$T_3^{\mathcal{A},0}(k) + a\,b\,\boldsymbol{q}\left(\frac{8}{3} - \frac{7}{8}\pi\right)k^4$
5	$a\,b\,\boldsymbol{q}\,\pi\left(1 - \frac{1}{8}k^2 - \frac{1}{64}k^4\right)$	$T_4^{\mathcal{A},0}(k) + a\,b\,\boldsymbol{q}\left(\frac{8}{3} - \frac{55}{64}\pi\right)k^5$
6	$a\,b\,\boldsymbol{q}\,\pi\left(1 - \frac{1}{8}k^2 - \frac{1}{64}k^4 - \frac{5}{1024}k^6\right)$	$T_5^{\mathcal{A},0}(k) + a\,b\,\boldsymbol{q}\left(\frac{8}{3} - \frac{55}{64}\pi\right)k^6$
7	$a\,b\,\boldsymbol{q}\,\pi\left(1 - \frac{1}{8}k^2 - \frac{1}{64}k^4 - \frac{5}{1024}k^6\right)$	$T_6^{\mathcal{A},0}(k) + a\,b\,\boldsymbol{q}\left(\frac{8}{3} - \frac{875}{1024}\pi\right)k^7$
8	$a\,b\,\boldsymbol{q}\,\pi\left(1 - \frac{1}{8}k^2 - \frac{1}{64}k^4 - \frac{5}{1024}k^6 - \frac{35}{16384}k^8\right)$	$T_7^{\mathcal{A},0}(k) + a\,b\,\boldsymbol{q}\left(\frac{8}{3} - \frac{875}{1024}\pi\right)k^8$
9	$a\,b\,q\,\pi\left(1 - \frac{1}{8}k^2 - \frac{1}{64}k^4 - \frac{5}{1024}k^6 - \frac{35}{16384}k^8\right)$	$T_8^{\mathcal{A},0}(k) + a\,b\,q\left(\frac{8}{3} - \frac{13965}{16384}\pi\right)k^9$
10	$a\,b\,q\,\pi\left(1 - \frac{1}{8}k^2 - \frac{1}{64}k^4 - \frac{5}{1024}k^6 - \frac{35}{16384}k^8 - \frac{147}{131072}k^{10}\right)$	$T_9^{\mathcal{A},0}(k) + a\,b\,q\left(\frac{8}{3} - \frac{13965}{16384}\pi\right)k^{10}$

Table 4.

5. Approximative formulae for the area of curve $\mathcal{F}_{q,egg}$

Theorem 5.1. *(Theorem for the area of Hügelschäffer egg curves). The following two estimations hold:*

$$\frac{8}{3}a\,b\,\boldsymbol{q} \leq \mathcal{A}(k) \leq \pi\,a\,b\,\boldsymbol{q} \tag{63}$$

and

$$\frac{8}{3}a\,b\,\boldsymbol{q} + \Delta_q \leq \mathcal{A}(k) \leq \pi\,a\,b\,\boldsymbol{q} - \nabla_q \tag{64}$$

where

$$\Delta_q = a\,b\,\boldsymbol{q}\,\pi\,(1-k) = \begin{cases} b\,\pi\,(a-w) & : \ w < a \\ \dfrac{a^2}{w^2}\,b\,\pi\,(w-a) & : \ w > a \end{cases}, \qquad \nabla_q = \frac{\pi}{8}\,a\,b\,\boldsymbol{q}\,k^2 = \begin{cases} \dfrac{\pi}{8\,a}\,b\,w^2 & : \ w < a \\ \dfrac{\pi}{8\,w^3}\,a^4\,b & : \ w > a \end{cases} \tag{65}$$

for $k \in [0,1]$.

Proof. The series expansion $\mathcal{A}(k)$ given by (56) is a monotonously decreasing function when $k \in [0,1]$ (based on the term by term differentiation of the series). Thus, the estimate (63) is obtained based on (55) and (56).

Based on (62) and table 4, the following is a proof of the estimate (64):

$$\frac{8}{3}a\,b\,\boldsymbol{q} \le a\,b\,\boldsymbol{q}\,\pi + a\,b\,\boldsymbol{q}\left(\frac{8}{3}-\pi\right)k \le \mathcal{A}(k) \le \pi\,a\,b\,\boldsymbol{q}\left(1-\frac{1}{8}k^2\right) \le \pi\,a\,b\,\boldsymbol{q} \tag{66}$$

i.e.

$$\frac{8}{3}a\,b\,\boldsymbol{q} \le \frac{8}{3}a\,b\,\boldsymbol{q} + a\,b\,\boldsymbol{q}\,\pi\,(1-k) \le \mathcal{A}(k) \le \pi\,a\,b\,\boldsymbol{q} - \frac{\pi}{8}\,a\,b\,\boldsymbol{q}\,k^2 \le \pi\,a\,b\,\boldsymbol{q}. \tag{67}$$

The estimate (63) is graphically illustrated in Fig. 3.

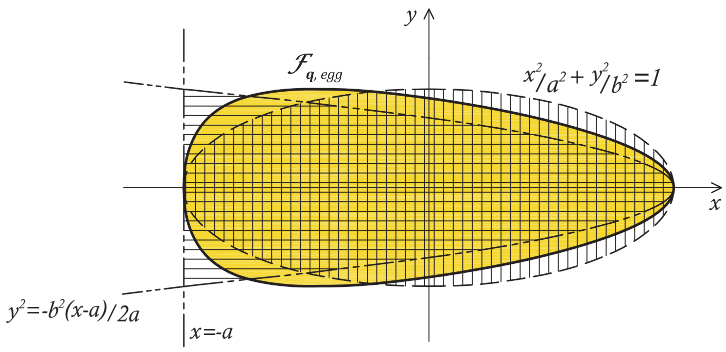

Figure 3: A comparison of the areas of a parabola, Hügelschäffer egg curve $\mathcal{F}_{\boldsymbol{q},egg}$, and an ellipse; Source: © First Author

Remark 5.2. *Based on the series of inequalities (62), it is possible to obtain even better estimates for $\mathcal{A}(k)$ with the appropriate polynomials given in table 4.*

6. Conclusion

Applying Taylor series and double Taylor series (Section 3), we have obtained novel approximations for $\boldsymbol{K}$, $\boldsymbol{E}$ and $\boldsymbol{D}$ elliptic integrals (see table 1 – 3), as well as approximative formulae for calculating the area $\mathcal{A}(k)$ of the egg-shaped part of Hügelschäffer curves (see table 4 in Section 4). Furthermore, based on the expression (57) which represents the area of a part of a parabola (see Fig. 3), and the expression (58) which represents the series expansion $\mathcal{A}(k)$ when $k=1$, a new representation (59) of the number $1/\pi$ has arisen.

With the development of new software tools and applets (see [21]), the use of the newly-introduced formulae (32) or (55) for the calculation of the area of the egg-shaped part of Hügelschäffer curves $\mathcal{A}_{\boldsymbol{q},egg}$ or $\mathcal{A}(k)$ could be significant in various areas of engineering, poultry industry and ornithology.

Conflict of interest. The authors declare that there are no conflicts of interest for this research.

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