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Some properties of pointwise k-slant submanifolds of Kähler manifolds

Adara M. Blaga[a,*], **Dan Radu Laţcu**[b]

[a] *West University of Timişoara, Timişoara, Romania*
[b] *Central University Library of Timişoara, Timişoara, Romania*

Abstract. We study some properties of pointwise k-slant submanifolds of almost Hermitian manifolds with a special view towards Kähler manifolds. In particular, we characterize the integrability of the component distributions, treating also the totally geodesic case.

1. Introduction and Preliminaries

Slant and *pointwise slant submanifolds*, introduced by Chen [3] and Etayo [5] (see also [4]), respectively, have been intensively investigated in different geometries. Recently, Laţcu [7] defined the more general notions of *k-slant* and *pointwise k-slant submanifold* of an almost product Riemannian, an almost Hermitian, and an almost contact or paracontact metric manifold, involving the decomposition of the tangent bundle of a submanifold into a sum of orthogonal slant or pointwise slant distributions. It's to be mentioned that Ronsse [8] and Chen [2, 3] considered, in the almost Hermitian case, the orthogonal decomposition of the tangent space in a point of a submanifold into the direct sum of the eigenspaces corresponding to the square of the tangential component of the structural tensor field. Accordingly, the submanifold was called [8] a *generic submanifold*, or, under some restrictions, a *skew CR submanifold*.

In this paper, we focus on some properties of pointwise k-slant submanifolds of almost Hermitian manifolds, with a special view toward the Kähler case. More precisely, we characterize the integrability of the component distributions, and we obtain some properties of such submanifolds with parallel tensor fields, discussing also the totally geodesic case.

Let $(\bar{M}, g)$ be a Riemannian manifold, and let φ be a $(1, 1)$-tensor field on $\bar{M}$. We recall that $(\bar{M}, \varphi, g)$ is said to be an *almost Hermitian manifold* if

$$\varphi^2 = -I \quad \text{and} \quad g(\varphi\cdot, \varphi\cdot) = g,$$

which further gives

$$g(\varphi\cdot, \cdot) = -g(\cdot, \varphi\cdot).$$

If the structural endomorphism φ satisfies $\bar{\nabla}\varphi = 0$, where $\bar{\nabla}$ is the Levi-Civita connection of g, then $\bar{M}$ is called a *Kähler manifold*.

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* Corresponding author: Adara M. Blaga
Email addresses: adarablaga@yahoo.com (Adara M. Blaga), latcu07@yahoo.com (Dan Radu Laţcu)

For a submanifold M of an almost Hermitian manifold $(\bar{M}, \varphi, g)$ defined by an injective immersion, we will denote the induced metric on M also with g and by ∇ the Levi-Civita connection on M. The Gauss and Weingarten equations are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \text{ and } \bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A is the shape operator, related by $g(h(X, Y), V) = g(A_V X, Y)$.

We have the orthogonal decomposition

$$T\bar{M} = TM \oplus T^{\perp}M,$$

and, for all $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we will write:

$$\varphi X = TX + NX \text{ and } \varphi V = tV + nV,$$

where TX, NX and tV, nV stand for the tangent and the normal component of φX and φV, respectively.

2. Pointwise k-slant submanifolds of almost Hermitian manifolds

We recall that a distribution $\mathcal{D} \subseteq TM$ is called a *pointwise slant distribution* if, at each point $p \in M$, the angle $\theta(p)$ between φX_p and $\mathcal{D}_p$ is nonzero and independent of the choice of the tangent vector $X_p \in \mathcal{D}_p \backslash \{0\}$ (but it depends on $p \in M$). In this case, the function θ is called the *slant function*.

Definition 2.1. [7] *A submanifold M, defined by an injective immersion, of an almost Hermitian manifold $(\bar{M}, \varphi, g)$ is said to be a pointwise k-slant submanifold of $\bar{M}$ ($k \in \mathbb{N}^*$) if there exist some orthogonal smooth regular distributions, $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_k$, satisfying:*

(i) $TM = \mathcal{D}_0 \oplus \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_k$;

(ii) $T(D_i) \subseteq \mathcal{D}_i$ for any $i \in \{1, \ldots, k\}$;

(iii) $\mathcal{D}_0$ is invariant (or even trivial) and $\mathcal{D}_i$, $i \in \{1, \ldots, k\}$, are nontrivial, pointwise slant distributions with their slant functions θ_i, $\theta_i(p) \in (0, \frac{\pi}{2}]$ for $p \in M$ and $i \in \{1, \ldots, k\}$, which are pointwise distinct (i.e., $\theta_i(p) \neq \theta_j(p)$ for all $p \in M$ and $i \neq j$).

Conventionally, we will denote by θ_0 the null angle, i.e., the "slant" angle of the invariant distribution $\mathcal{D}_0$ (if $\mathcal{D}_0$ is not trivial).

We notice [7] that the condition (ii) from the Definition 2.1 is equivalent to: $\varphi(\mathcal{D}_i) \perp \mathcal{D}_j$ for all $i \neq j$, $i, j \in \{1, \ldots, k\}$.

The slant functions θ_i are continuous (even smooth, under a certain assumption) [6], and, for all $X \in \Gamma(\mathcal{D}_i) \setminus \{0\}$ and $p \in M$, the angle $\theta_i(p)$ between φX_p and T_pM coincides with the angle between φX_p and $(\mathcal{D}_i)_p$, and it satisfies

$$\cos \theta_i(p) \cdot \|\varphi X_p\| = \|TX_p\|.$$

If θ_i is constant for all $i \in \{1, \ldots, k\}$, then the submanifold M is called a *k-slant submanifold* [7], so all the results for pointwise k-slant submanifolds are also valid for k-slant submanifolds.

Now, we will construct an example of a pointwise k-slant submanifold and one of a k-slant submanifold of a Kähler manifold.

Example 2.2. *Let us consider the Kähler manifold $\left(\mathbb{R}^{6k}, \varphi, \langle \cdot, \cdot \rangle\right)$, $k \geq 2$, with the standard Euclidean metric $\langle \cdot, \cdot \rangle$ and φ given by*

$$\varphi\left(\frac{\partial}{\partial u_i}\right) = -\frac{\partial}{\partial v_i}, \quad \varphi\left(\frac{\partial}{\partial v_i}\right) = \frac{\partial}{\partial u_i},$$

where $(u_1, v_1, \ldots, u_{3k}, v_{3k})$ are the canonical coordinates in $\mathbb{R}^{6k}$. We consider the submanifold M of $\mathbb{R}^{6k}$ defined by the immersion

$$f : \{z = (x_1, x_2, y_1, \ldots, y_{2k-1}) \in \mathbb{R}^{2k+1} : \|z\| < 1, x_1 > 0, x_2 > 0\} \to \mathbb{R}^{6k},$$

$$f(x_1,x_2,y_1,\dots,y_{2k-1}) := \Big(x_1\cos y_1, x_2\cos y_1, x_1\sin y_1, x_2\sin y_1, x_1, x_2, y_2, \frac{1}{2}y_2^2, y_2+y_3, y_2-y_3, y_3, \frac{1}{2}y_3^2, \dots,$$
$$(k-1)y_{2k-2}, \frac{1}{2}y_{2k-2}^2, y_{2k-2}+y_{2k-1}, y_{2k-2}-y_{2k-1}, (k-1)y_{2k-1}, \frac{1}{2}y_{2k-1}^2\Big).$$

Then, TM is spanned by

$$\begin{aligned}
X_1 &= \cos y_1\frac{\partial}{\partial u_1} + \sin y_1\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_3},\\
X_2 &= \cos y_1\frac{\partial}{\partial v_1} + \sin y_1\frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_3},\\
X_3 &= -x_1\sin y_1\frac{\partial}{\partial u_1} - x_2\sin y_1\frac{\partial}{\partial v_1} + x_1\cos y_1\frac{\partial}{\partial u_2} + x_2\cos y_1\frac{\partial}{\partial v_2},\\
X_{2i} &= (i-1)\frac{\partial}{\partial u_{3i-2}} + y_{2i-2}\frac{\partial}{\partial v_{3i-2}} + \frac{\partial}{\partial u_{3i-1}} + \frac{\partial}{\partial v_{3i-1}},\\
X_{2i+1} &= \frac{\partial}{\partial u_{3i-1}} - \frac{\partial}{\partial v_{3i-1}} + (i-1)\frac{\partial}{\partial u_{3i}} + y_{2i-1}\frac{\partial}{\partial v_{3i}}
\end{aligned}$$

for $i \in \{2,3,\dots,k\}$. *We notice that* $X_1, X_2, \dots, X_{2k+1}$ *are mutually orthogonal.*

Applying φ *to the base vector fields of TM, we get*

$$\begin{aligned}
\varphi X_1 &= -\cos y_1\frac{\partial}{\partial v_1} - \sin y_1\frac{\partial}{\partial v_2} - \frac{\partial}{\partial v_3},\\
\varphi X_2 &= \cos y_1\frac{\partial}{\partial u_1} + \sin y_1\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_3},\\
\varphi X_3 &= -x_2\sin y_1\frac{\partial}{\partial u_1} + x_1\sin y_1\frac{\partial}{\partial v_1} + x_2\cos y_1\frac{\partial}{\partial u_2} - x_1\cos y_1\frac{\partial}{\partial v_2},\\
\varphi X_{2i} &= y_{2i-2}\frac{\partial}{\partial u_{3i-2}} - (i-1)\frac{\partial}{\partial v_{3i-2}} + \frac{\partial}{\partial u_{3i-1}} - \frac{\partial}{\partial v_{3i-1}},\\
\varphi X_{2i+1} &= -\frac{\partial}{\partial u_{3i-1}} - \frac{\partial}{\partial v_{3i-1}} + y_{2i-1}\frac{\partial}{\partial u_{3i}} - (i-1)\frac{\partial}{\partial v_{3i}}
\end{aligned}$$

for $i \in \{2,3,\dots,k\}$. *We immediately obtain* $\varphi X_1 = -X_2$ *and* $\varphi X_2 = X_1$; *therefore, the distribution* $\mathcal{D}_0 = \mathrm{Span}\{X_1, X_2\}$ *is an invariant distribution. Also, we have*

$$\frac{\left|\langle \varphi X_{2i}, X_{2i+1}\rangle\right|}{\left\|\varphi X_{2i}\right\|\cdot\|X_{2i+1}\|} = \frac{\left|\langle \varphi X_{2i+1}, X_{2i}\rangle\right|}{\left\|\varphi X_{2i+1}\right\|\cdot\|X_{2i}\|} = \frac{2}{\sqrt{\left(2+(i-1)^2+y_{2i-2}^2\right)\left(2+(i-1)^2+y_{2i-1}^2\right)}};$$

hence, since X_{2i} *and* X_{2i+1} *are orthogonal, the distribution* $\mathcal{D}_i = \mathrm{Span}\{X_{2i}, X_{2i+1}\}$, $i \in \{2,3,\dots,k\}$, *is a pointwise slant distribution with the slant function defined by*

$$\theta_i(f(x_1,x_2,y_1,\dots,y_{2k-1})) = \arccos\frac{2}{\sqrt{\left(2+(i-1)^2+y_{2i-2}^2\right)\left(2+(i-1)^2+y_{2i-1}^2\right)}}.$$

By a direct computation, we find

$$\langle \varphi X_3, X_1\rangle = \langle \varphi X_3, X_2\rangle = \langle \varphi X_3, X_3\rangle = \langle \varphi X_3, X_{2i}\rangle = \langle \varphi X_3, X_{2i+1}\rangle = 0$$

for any $i \in \{2,3,\dots,k\}$, *and so the distribution* $\mathcal{D}_1 = \mathrm{Span}\{X_3\}$ *is an anti-invariant distribution.*

Therefore, we can conclude that M is a pointwise k-slant submanifold of the Kähler manifold $\left(\mathbb{R}^{6k}, \varphi, \langle\cdot,\cdot\rangle\right)$.

Example 2.3. *The submanifold obtained by replacing $\frac{1}{2}y_j^2$ with y_j, $j \in \{2,3,\dots,2k-1\}$, in the expression of the immersion from the previous example is a k-slant submanifold of the standard Kähler manifold $\left(\mathbb{R}^{6k}, \varphi, \langle\cdot,\cdot\rangle\right)$, with the slant angles*

$$\theta_i = \arccos \frac{2}{3+(i-1)^2}$$

for the slant distributions $\mathcal{D}_i$, $i \in \{2,3,\dots,k\}$, respectively, and $\theta_1 = \frac{\pi}{2}$ for $\mathcal{D}_1$.

If M is a pointwise k-slant submanifold of an almost Hermitian manifold $(\bar{M}, \varphi, g)$, then we have the following decompositions [7] of the tangent and normal bundles of M:

$$TM = \oplus_{i=0}^{k} \mathcal{D}_i, \;\; T^{\perp}M = \oplus_{i=1}^{k} N(\mathcal{D}_i) \oplus H,$$

where $\varphi(H) = H$. Let P_i be the projection from TM onto $\mathcal{D}_i$, $i \in \{0,\dots,k\}$, Q_i be the projection from $T^{\perp}M$ onto $N(\mathcal{D}_i)$, $i \in \{1,\dots,k\}$, and Q_0 be the projection from $T^{\perp}M$ onto H. Then, any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ can be written as:

$$X = \sum_{i=0}^{k} P_i X, \;\; V = \sum_{i=0}^{k} Q_i V.$$

By a direct computation, we immediately obtain (see also [7]) the following

Lemma 2.4. *If M is a pointwise k-slant submanifold of an almost Hermitian manifold $(\bar{M}, \varphi, g)$, then:*

$$(i)\; g(TX,Y) = -g(X,TY), \;\; g(NX,V) = -g(X,tV), \;\; g(nV,W) = -g(V,nW)$$

for all $X, Y \in \Gamma(TM)$ and $V, W \in \Gamma(T^{\perp}M)$;

$$(ii)\; T^2 = -\sum_{i=0}^{k} \cos^2\theta_i \cdot P_i, \;\; n^2 = -\sum_{i=0}^{k} \cos^2\theta_i \cdot Q_i;$$

$$(iii)\; g(TX,TY) = \sum_{i=0}^{k} \cos^2\theta_i \cdot g(P_iX, P_iY), \;\; g(NX,NY) = \sum_{i=1}^{k} \sin^2\theta_i \cdot g(P_iX, P_iY)$$

for all $X, Y \in \Gamma(TM)$.

3. On the integrability of the component distributions

For a pointwise k-slant submanifold M of an almost Hermitian manifold $(\bar{M}, \varphi, g)$, we will denote $TM = \oplus_{i=0}^{k} \mathcal{D}_i$. Similarly to the almost contact metric case [1], the integrability of the component distributions $\mathcal{D}_i$, $i \in \{0,\dots,k\}$, can be characterized in the Kähler case as follows.

We recall that a distribution $\mathcal{D}$ is called *integrable* if $[X,Y] \in \Gamma(\mathcal{D})$ for all $X, Y \in \Gamma(\mathcal{D})$, and *completely integrable* if $\nabla_X Y \in \Gamma(\mathcal{D})$ for all $X, Y \in \Gamma(\mathcal{D})$.

Theorem 3.1. *If M is a pointwise k-slant submanifold of a Kähler manifold $(\bar{M}, \varphi, g)$, then:*
(i) for $i \in \{0,\dots,k\}$, $\mathcal{D}_i$ is an integrable distribution if and only if

$$g(X, \nabla_Y Z) = g(Y, \nabla_X Z)$$

for all $X, Y \in \Gamma(\mathcal{D}_i)$ and $Z \in \Gamma(\mathcal{D}_j)$, $j \in \{0,\dots,k\}$ with $j \neq i$;

(ii) $\mathcal{D}_0$ is an integrable distribution if and only if

$$h(X,TY) = h(TX,Y)$$

for all $X,Y \in \Gamma(\mathcal{D}_0)$;

(iii) for $i \in \{0,\dots,k\}$ with $\theta_j(p) \neq \frac{\pi}{2}$ for all $p \in M$ and $j \neq i$, $j \in \{0,\dots,k\}$, $\mathcal{D}_i$ is an integrable distribution if and only if

$$\nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X \in \Gamma(\mathcal{D}_i)$$

for all $X,Y \in \Gamma(\mathcal{D}_i)$.

Proof. We have

$$\bar{\nabla}_X \varphi Y = \varphi(\bar{\nabla}_X Y)$$

for all X, $Y \in \Gamma(TM)$, and, using Gauss and Weingarten equations, we get

$$\begin{aligned}\nabla_X TY + h(X,TY) &= T(\nabla_X Y) + N(\nabla_X Y) + A_{NY}X - \nabla^{\perp}_X NY + th(X,Y) + nh(X,Y)\\ &= \sum_{i=0}^{k} TP_i(\nabla_X Y) + \sum_{i=1}^{k} NP_i(\nabla_X Y) + A_{NY}X - \nabla^{\perp}_X NY + th(X,Y) + nh(X,Y).\end{aligned}$$

Identifying the tangent and the normal components in the previous relation, we obtain:

$$\begin{aligned}\nabla_X TY &= \sum_{i=0}^{k} TP_i(\nabla_X Y) + A_{NY}X + th(X,Y),\\ h(X,TY) &= \sum_{i=1}^{k} NP_i(\nabla_X Y) - \nabla^{\perp}_X NY + nh(X,Y)\end{aligned}$$

for all X, $Y \in \Gamma(TM)$.

The distribution $\mathcal{D}_i$ is integrable if and only if $g([X,Y],Z) = 0$ for all $X,Y \in \Gamma(\mathcal{D}_i)$ and $Z \in \Gamma(\mathcal{D}_j)$, $j \in \{0,\dots,k\}$ with $j \neq i$. Since $g([X,Y],Z) = g(X,\nabla_Y Z) - g(Y,\nabla_X Z)$, we get (i).

Let $X,Y \in \Gamma(\mathcal{D}_0)$. Then, $NX = NY = 0$, and we obtain

$$h(X,TY) - h(TX,Y) = \sum_{i=1}^{k} NP_i[X,Y].$$

If the distribution $\mathcal{D}_0$ is integrable, then $[X,Y] \in \Gamma(\mathcal{D}_0)$; therefore, $P_i[X,Y] = 0$ for all $i \in \{1,\dots,k\}$, hence the conclusion. Conversely, if $h(X,TY) = h(TX,Y)$ for all $X,Y \in \Gamma(\mathcal{D}_0)$, then $\sum_{i=1}^{k} NP_i[X,Y] = 0$, hence $P_i[X,Y] = 0$ for all $i \in \{1,\dots,k\}$, and we get (ii).

Let $X,Y \in \Gamma(\mathcal{D}_i)$. Then, we obtain

$$T[X,Y] = T(\nabla_X Y) - T(\nabla_Y X) = \nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X.$$

If the distribution $\mathcal{D}_i$ is integrable, then $[X,Y] \in \Gamma(\mathcal{D}_i)$; therefore, $T[X,Y] \in \Gamma(\mathcal{D}_i)$, hence the conclusion.

Conversely, if $\nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X \in \Gamma(\mathcal{D}_i)$, then $T[X,Y] \in \Gamma(\mathcal{D}_i)$, and, since $\theta_j(p) \neq \frac{\pi}{2}$ for all $p \in M$ and $j \neq i$, we get (iii). □

In particular, for a totally geodesic submanifold (i.e., for $h = 0$), we deduce

Corollary 3.2. *If M is a totally geodesic pointwise k-slant submanifold of a Kähler manifold $(\bar{M},\varphi,g)$, then:*

(i) $\mathcal{D}_0$ is an integrable distribution;

(ii) for $i \in \{0,\dots,k\}$ with $\theta_j(p) \neq \frac{\pi}{2}$ for all $p \in M$ and $j \neq i$, $j \in \{0,\dots,k\}$, $\mathcal{D}_i$ is an integrable distribution if and only if

$$\nabla_X TY - \nabla_Y TX \in \Gamma(\mathcal{D}_i)$$

for all $X,Y \in \Gamma(\mathcal{D}_i)$.

For a submanifold M of an almost Hermitian manifold $(\bar{M}, \varphi, g)$ defined by an injective immersion, by using the Gauss and Weingarten equations, for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we obtain:

$$(\bar{\nabla}_X \varphi)Y = \nabla_X TY - T(\nabla_X Y) - A_{NY}X - th(X,Y) + \nabla^{\perp}_X NY - N(\nabla_X Y) + h(X,TY) - nh(X,Y),$$
$$(\bar{\nabla}_X \varphi)V = \nabla_X tV - t(\nabla^{\perp}_X V) - A_{nV}X + T(A_V X) + \nabla^{\perp}_X nV - n(\nabla^{\perp}_X V) + h(X,tV) + N(A_V X).$$

Denoting:

$$(\nabla_X T)Y := \nabla_X TY - T(\nabla_X Y), \qquad (\nabla_X N)Y := \nabla^{\perp}_X NY - N(\nabla_X Y),$$
$$(\nabla_X t)V := \nabla_X tV - t(\nabla^{\perp}_X V), \qquad (\nabla_X n)V := \nabla^{\perp}_X nV - n(\nabla^{\perp}_X V)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, by identifying the tangent and the normal components in the Kähler case, we get the following

Lemma 3.3. *If $(\bar{M}, \varphi, g)$ is a Kähler manifold, then, for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have:*
(i) $(\nabla_X T)Y = A_{NY}X + th(X,Y)$;
(ii) $(\nabla_X N)Y = -h(X,TY) + nh(X,Y)$;
(iii) $(\nabla_X t)V = A_{nV}X - T(A_V X)$;
(iv) $(\nabla_X n)V = -h(X,tV) - N(A_V X)$.
Moreover, if M is totally geodesic, then we get: $\nabla T = 0$, $\nabla N = 0$, $\nabla t = 0$, and $\nabla n = 0$.

We recall that a $(1,1)$-tensor field J on M is called *parallel* if $(\nabla_X J)Y = 0$ for all $X, Y \in \Gamma(TM)$. We will characterize the property of T and N to be parallel tensor fields as follows.

Proposition 3.4. *If $(\bar{M}, \varphi, g)$ is a Kähler manifold, then:*
(i) $\nabla T = 0$ is equivalent to: $A_{NY}X = A_{NX}Y$ for all $X, Y \in \Gamma(TM)$;
(ii) $\nabla N = 0$ is equivalent to any of the following assertions:
(1) $h(TX, Y) = h(X, TY)$ for all $X, Y \in \Gamma(TM)$;
(2) $T(A_V X) = -A_V(TX)$ for all $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$;
(3) $A_{nV}X = -A_V(TX)$ for all $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. Since T is skew-symmetric, the condition $\nabla T = 0$ is equivalent to: $(\nabla_X T)Y = (\nabla_Y T)X$ for all $X, Y \in \Gamma(TM)$, which, by means of Lemma 3.3 (i), is equivalent to: $A_{NY}X = A_{NX}Y$ for all $X, Y \in \Gamma(TM)$, and we get (i).

We shall prove now:

$$\nabla N = 0 \Longrightarrow (1) \Longrightarrow (2) \Longrightarrow \nabla N = 0 \text{ and } \nabla N = 0 \Longrightarrow (3) \Longrightarrow (1).$$

If $\nabla N = 0$, we immediately get (1) from Lemma 3.3 (ii). So,

$$g(A_V(TX), Y) = g(h(TX, Y), V) = g(h(X, TY), V) = g(A_V X, TY) = -g(T(A_V X), Y)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, and we obtain (2). Further,

$$g(h(X, TY), V) = g(A_V TY, X) = -g(T(A_V Y), X) = g(A_V Y, TX) = g(h(TX, Y), V)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$; hence, from Lemma 3.3 (ii), we obtain $(\nabla_X N)Y = (\nabla_Y N)X$ for all $X, Y \in \Gamma(TM)$, which, since N is skew-symmetric, is equivalent to: $\nabla N = 0$.

Also, $\nabla N = 0$ is equivalent to $h(X, TY) = nh(X, Y)$ for all $X, Y \in \Gamma(TM)$; hence,

$$g(A_V(TY), X) = g(h(X, TY), V) = g(nh(X, Y), V) = -g(h(X, Y), nV) = -g(A_{nV}Y, X)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, and we deduce (3). Further,

$$g(h(TX, Y), V) = g(A_V(TX), Y) = -g(A_{nV}X, Y) = -g(h(X, Y), nV) = -g(A_{nV}Y, X) = g(A_V(TY), X) = g(h(X, TY), V)$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, hence (1). □

Theorem 3.5. *Let M be a pointwise k-slant submanifold of a Kähler manifold $(\bar{M}, \varphi, g)$. If $\nabla T = 0$, then:*
(i) $\mathcal{D}_0$ and $\oplus_{i=1}^k \mathcal{D}_i$ are completely integrable distributions;
(ii) for $i \in \{0, \ldots, k\}$ with $\theta_j(p) \neq \frac{\pi}{2}$ for all $p \in M$ and $j \neq i$, $j \in \{0, \ldots, k\}$, $\mathcal{D}_i$ is an integrable distribution if and only if, for all $X, Y \in \Gamma(\mathcal{D}_i)$,

$$\nabla_X TY - \nabla_Y TX \in \Gamma(\mathcal{D}_i);$$

(iii) either M is a $(\mathcal{D}_0, \mathcal{D}_0)$-totally geodesic submanifold of $\bar{M}$ (i.e., $h(X, Y) = 0$ for all $X, Y \in \Gamma(\mathcal{D}_0)$), or (-1) is an eigenvalue of n^2, and, for $X, Y \in \Gamma(\mathcal{D}_0)$, any nonzero $h(X, Y)$ is an eigenvector for it.

Proof. For all $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathcal{D}_0)$, we have

$$0 = (\bar{\nabla}_X \varphi)Y = -N(\nabla_X Y) + h(X, TY) - \varphi h(X, Y)$$

from Gauss and Weingarten equations, which, for all $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathcal{D}_0)$, implies

$$\sum_{i=1}^k NP_i(\nabla_X Y) = h(X, TY) - \varphi h(X, Y).$$

Since $\nabla T = 0$, we have $th(X, Y) = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathcal{D}_0)$, so

$$g(\varphi h(X, Y), Z) = -g(h(X, Y), \varphi Z) = -g(h(X, Y), NZ) = g(th(X, Y), Z) = 0$$

for all $X, Z \in \Gamma(TM)$ and $Y \in \Gamma(\mathcal{D}_0)$. On the other hand, for $Y \in \Gamma(\mathcal{D}_0)$, we have $TY \in \Gamma(\mathcal{D}_0)$, which gives $g(h(X, TY), NZ) = 0$, and we obtain

$$\Big\|N\Big(\sum_{i=1}^k P_i(\nabla_X Y)\Big)\Big\|^2 = g\Big(h(X, TY), N\Big(\sum_{i=1}^k P_i(\nabla_X Y)\Big)\Big) - g\Big(\varphi h(X, Y), \sum_{i=1}^k NP_i(\nabla_X Y)\Big) = 0$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathcal{D}_0)$. Since

$$g(NX, NX) = \sum_{i=1}^k \sin^2 \theta_i \cdot g(P_i X, P_i X)$$

for all $X \in \Gamma(TM)$, we have

$$\Big\|N\Big(\sum_{i=1}^k P_i(\nabla_X Y)\Big)\Big\|^2 = \sum_{i=1}^k \sin^2 \theta_i \cdot \Big\|P_i(\nabla_X Y)\Big\|^2,$$

and, from the fact that θ_i is nowhere zero, we obtain $P_i(\nabla_X Y) = 0$ for any $i \in \{1, \ldots, k\}$, which implies $\nabla_X Y \in \Gamma(\mathcal{D}_0)$ for all $X \in \Gamma(TM)$, $Y \in \Gamma(\mathcal{D}_0)$; hence, the distribution $\mathcal{D}_0$ is completely integrable.

Also, for any $X \in \Gamma(TM)$ and $Y \in \Gamma(\oplus_{i=1}^k \mathcal{D}_i)$, we have $\nabla_X Y \in \Gamma(\oplus_{i=1}^k \mathcal{D}_i)$ since

$$g(\nabla_X Y, Z) = -g(Y, \nabla_X Z) = 0$$

for all $Z \in \Gamma(\mathcal{D}_0)$. Therefore, we obtain (i).

If the distribution $\mathcal{D}_i$ is integrable, then, for all $X, Y \in \Gamma(\mathcal{D}_i)$, we have $T[X, Y] \in \Gamma(\mathcal{D}_i)$, which implies

$$\nabla_X TY - \nabla_Y TX = T(\nabla_X Y - \nabla_Y X) \in \Gamma(\mathcal{D}_i).$$

Conversely, if $\nabla_X TY - \nabla_Y TX \in \Gamma(\mathcal{D}_i)$ for all $X, Y \in \Gamma(\mathcal{D}_i)$, then

$$T[X, Y] = \nabla_X TY - \nabla_Y TX \in \Gamma(\mathcal{D}_i).$$

Applying T, we get $\sum_{j=0}^k \cos^2 \theta_j \cdot P_j[X, Y] \in \Gamma(\mathcal{D}_i)$ from Lemma 2.4 (ii), and, taking into account the orthogonality of the distributions and the fact that $\theta_j(p) \neq \frac{\pi}{2}$ for all $p \in M$ and $j \in \{0, \ldots, k\}$ with $j \neq i$, we obtain (ii).

Now, since $\mathcal{D}_0$ is completely integrable, from the Kähler condition, we deduce

$$nh(X, Y) - h(X, TY) = -N(\nabla_X Y) = 0$$

for all $X, Y \in \Gamma(\mathcal{D}_0)$. Writing this relation for TY instead of Y, we get

$$n^2 h(X, Y) = -h(X, Y)$$

for all $X, Y \in \Gamma(\mathcal{D}_0)$, and we obtain (iii). □

Theorem 3.6. *Let M be a pointwise k-slant submanifold of a Kähler manifold $(\bar{M}, \varphi, g)$. If $\nabla N = 0$, then:*
(i) M is a $(\mathcal{D}_0, \mathcal{D}_i)$-mixed totally geodesic submanifold of $\bar{M}$ (i.e., $h(X, Y) = 0$ for all $X \in \Gamma(\mathcal{D}_0)$ and $Y \in \Gamma(\mathcal{D}_i)$) for $i \in \{1, \dots, k\}$;
(ii) for $i \in \{0, \dots, k\}$, either M is a $(\mathcal{D}_i, \mathcal{D}_i)$-totally geodesic submanifold of $\bar{M}$, or $(-\cos^2\theta_i)$ is an eigenvalue function of n^2, and, for $X, Y \in \Gamma(\mathcal{D}_i)$, any nonzero $h(X, Y)$ is an eigenvector for it;
(iii) for $i \in \{0, \dots, k\}$, either $h(X, Y) = 0$ for any $X \in \Gamma(\mathcal{D}_i)$ and $Y \in \Gamma(TM)$, or $(-\cos^2\theta_i)$ is an eigenvalue function of T^2, and, for $X \in \Gamma(\mathcal{D}_i)$ and $V \in \Gamma(T^\perp M)$, any nonzero $A_V X$ is an eigenvector for it.

Proof. Since $\nabla N = 0$, we have $h(X, TY) = nh(X, Y)$ for all $X, Y \in \Gamma(TM)$, which implies

$$n^2 h(X, Y) = -\cos^2\theta_i \cdot h(X, Y)$$

for all $Y \in \Gamma(\mathcal{D}_i)$. For $X, Y \in \Gamma(\mathcal{D}_i)$, we deduce (ii). For $X \in \Gamma(\mathcal{D}_0)$ and $Y \in \Gamma(\mathcal{D}_i)$, we obtain

$$n^2 h(X, Y) = n^2 h(Y, X) = nh(Y, TX) = h(Y, T^2 X) = -h(Y, X) = -h(X, Y),$$

and we get

$$\sin^2\theta_i \cdot h(X, Y) = 0,$$

hence (i).

Applying T to the relation from Proposition 3.4 (ii)(2), we infer

$$T^2(A_V X) = A_V(T^2 X)$$

for all $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$; therefore, for all $X \in \Gamma(\mathcal{D}_i)$, we have

$$T^2(A_V X) = -\cos^2\theta_i \cdot A_V X,$$

and we get (iii). □

Theorem 3.7. *Let M be a connected pointwise k-slant submanifold of an almost Hermitian manifold $(\bar{M}, \varphi, g)$. Then:*
(i) $\nabla T^2 = 0$ if and only if M is a k-slant submanifold, and ∇ restricts to $\mathcal{D}_i$ (i.e., $\nabla_X Y \in \Gamma(\mathcal{D}_i)$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathcal{D}_i)$) for any $i \in \{0, \dots, k\}$;
(ii) for $i \in \{0, \dots, k\}$, $(\nabla_X T^2)Y = (\nabla_Y T^2)X$ for all $X, Y \in \Gamma(\mathcal{D}_i)$ if and only if $\mathcal{D}_i$ is a slant, integrable distribution. Furthermore, if T^2 is a Codazzi tensor field on $\mathcal{D}_i$ for all $i \in \{0, \dots, k\}$, then M is a k-slant submanifold.

Proof. (i) Following the same steps as in [1], we obtain

$$(\nabla_X T^2)Y = -\sum_{i=0}^{k} X(\cos^2\theta_i)P_i Y + \sum_{0 \le i,j \le k} (\cos^2\theta_i - \cos^2\theta_j)P_i(\nabla_X P_j Y)$$

for all $X, Y \in \Gamma(TM)$. Taking into account the orthogonality of the distributions, the condition $\nabla T^2 = 0$ is equivalent to:

$$\sum_{j=0}^{k} (\cos^2\theta_i - \cos^2\theta_j)P_i(\nabla_X P_j Y) - X(\cos^2\theta_i)P_i Y = 0$$

for all $X, Y \in \Gamma(TM)$ and any $i \in \{0, \ldots, k\}$. We get

$$X(\cos^2 \theta_i)Y = 0$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(\mathcal{D}_i)$, $i \in \{0, \ldots, k\}$, hence θ_i is constant for all $i \in \{1, \ldots, k\}$ (so M is a k-slant submanifold). Also, $P_i(\nabla_X Y) = 0$ for all $Y \in \Gamma(\mathcal{D}_j)$, $i \neq j$; hence, ∇ restricts to $\mathcal{D}_j$ for any $j \in \{0, \ldots, k\}$. The converse implication follows immediately since $\nabla_X P_j Y \in \Gamma(\mathcal{D}_j)$ for all $X, Y \in \Gamma(TM)$ and any $j \in \{0, \ldots, k\}$.

(ii) On the other hand, for all $X, Y \in \Gamma(TM)$, we get

$$(\nabla_X T^2)Y - (\nabla_Y T^2)X = \sum_{j=0}^{k} P_j\Big(- X(\cos^2 \theta_j)Y + Y(\cos^2 \theta_j)X\Big) - \sum_{j=0}^{k} P_j\Big(\sum_{l=0}^{k} \cos^2 \theta_l (\nabla_X P_l Y - \nabla_Y P_l X)\Big) + \sum_{j=0}^{k} P_j\Big(\cos^2 \theta_j (\nabla_X Y - \nabla_Y X)\Big).$$

In particular, for $X, Y \in \Gamma(\mathcal{D}_i)$, we obtain

$$(\nabla_X T^2)Y - (\nabla_Y T^2)X = \Big(- X(\cos^2 \theta_i)Y + Y(\cos^2 \theta_i)X\Big) - \sum_{j=0}^{k} P_j\Big(\cos^2 \theta_i (\nabla_X Y - \nabla_Y X)\Big) + \sum_{j=0}^{k} P_j\Big(\cos^2 \theta_j (\nabla_X Y - \nabla_Y X)\Big),$$

and we deduce that $(\nabla_X T^2)Y = (\nabla_Y T^2)X$ for all $X, Y \in \Gamma(\mathcal{D}_i)$ if and only if

$$\begin{cases} X(\cos^2 \theta_i)Y - Y(\cos^2 \theta_i)X = 0 \\ (\cos^2 \theta_j - \cos^2 \theta_i)P_j[X, Y] = 0 \end{cases}$$

for all $X, Y \in \Gamma(\mathcal{D}_i)$ and $j \neq i$. The first assertion is equivalent to the fact that θ_i is a constant, and the second one, since θ_i and θ_j are pointwise distinct for $i \neq j$, is equivalent to the integrability of $\mathcal{D}_i$. □

Hence, we recovered in Theorem 3.7 (i) a known result proved by Chen[2].

In particular, from the proof of Theorem 3.7, we deduce

Corollary 3.8. *Let M be a k-slant submanifold of an almost Hermitian manifold $(\bar{M}, \varphi, g)$. Then:*
(i) for $i \in \{0, \ldots, k\}$, $\mathcal{D}_i$ is completely integrable if and only if $(\nabla_X T^2)Y = 0$ for all $X, Y \in \Gamma(\mathcal{D}_i)$;
(ii) for $i \in \{0, \ldots, k\}$, $\mathcal{D}_i$ is integrable if and only if $(\nabla_X T^2)Y = (\nabla_Y T^2)X$ for all $X, Y \in \Gamma(\mathcal{D}_i)$.

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