Filomat 38:24 (2024), 8349–8365
https://doi.org/10.2298/FIL2424349J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterization of symmetrical H_q-Laguerre-Hahn orthogonal polynomials of class zero

Sobhi Jbeli[a], Lotfi Khériji[b]

[a]*Université de Tunis El Manar, Faculté des Sciences de Tunis, Tunisia. LR13ES06*
[b]*Université de Tunis El Manar, Institut Préparatoire aux Etudes d'Ingénieur El Manar, Tunis, Tunisia. LR13ES06*

Abstract. We study the H_q-Laguerre-Hahn forms u, that is to say those satisfying a q-quadratic q-difference equation with polynomial coefficients (Φ, Ψ, B): $H_q(\Phi(x)u) + \Psi(x)u + B(x)\left(x^{-1}u(h_q u)\right) = 0$, where H_q be the q-derivative operator. We give the definition of the class s of such form and the characterization of its corresponding orthogonal polynomials sequence $\{P_n\}_{n\geq 0}$ by the structure relation. As a consequence, we establish the system fulfilled by the coefficients of the structure relation, those of the polynomials Φ, Ψ, B and the recurrence coefficients β_n, γ_{n+1}, $n \geq 0$ of $\{P_n\}_{n\geq 0}$ for the class zero. In addition, we carry out the complete description of the symmetrical H_q-Laguerre-Hahn forms of class $s = 0$. The limiting cases are also recovered.

1. Introduction and preliminaries

The concept of the usual Laguerre-Hahn orthogonal polynomials that is to say the D-Laguerre-Hahn orthogonal polynomials, where D be the derivative operator, were extremely studied by many authors [1, 4, 7-9, 23, 26]. The D-Laguerre-Hahn set is is invariant under many types of perturbations such that association, co-dilation, co-recursion ... [2, 5, 12, 13, 27, 29]. In particular, D-semiclassical orthogonal polynomials are D-Laguerre-Hahn [2, 6, 24]. Moreover in [8, 9], the D-Laguerre-Hahn orthogonal polynomials of class zero were exhaustively described.

In [17], instead of the D operator, the authors used the q-derivative one denoted H_q and they established the basic theory of H_q-Laguerre-Hahn orthogonal polynomials. In addition, a few generic examples related to some standard transformation and perturbation of H_q-classical [20, 22] or more generally H_q-semiclassical polynomials [10, 16, 18, 21, 28] were studied in [17]. Recently in [19], the Christoffel transformation and the Geronimus one of a H_q-Laguerre-Hahn form (linear functional) were studied into detail. For other relevant works in the domain of q-Laguerre-Hahn orthogonal polynomials see [3, 14, 15].

The goal of this contribution is to respond to the following classification problem:" find all H_q-Laguerre-Hahn orthogonal polynomials $\{P_n\}_{n\geq 0}$ of class zero," that is to say that the corresponding form u satisfies the q-quadratic q-difference equation $H_q(\Phi(x)u) + \Psi(x)u + B(x)\left(x^{-1}u(h_q u)\right) = 0$ with Φ (monic), $\deg \Phi \leq 2$, $\deg \Psi \leq 1$ and $\deg B \leq 2$. Through the so-called structure relation of a H_q-Laguerre-Hahn orthogonal polynomials, we managed to get the system fulfilled by the coefficients of the structure relation, those of the

2020 *Mathematics Subject Classification*. Primary 33C45; Secondary 42C05.
Keywords. Orthogonal polynomials, q-derivative operator, q-Laguerre-Hahn character, structure relation.
Received: 22 December 2022; Revised: 25 December 2023; Accepted: 04 May 2024
Communicated by Dragan S. Djordjević
Email addresses: jbelisobhi@gmail.com (Sobhi Jbeli), kheriji@yahoo.fr (Lotfi Khériji)

polynomials Φ, Ψ, B and the recurrence coefficients β_n, γ_{n+1}, $n \geq 0$ of $\{P_n\}_{n\geq 0}$ for the class zero (see section 2). The system obtained is difficult to solve in general. In section 3, we have tried to solve it in the symmetrical case ($\beta_n = 0$, $n \geq 0$, $\Phi(x) = 1$, $\Phi(x) = x^2$, $\Phi(x) = x^2 + c_0$, $c_0 \neq 0$) and we provided a complete description of this symmetrical class as perturbations of symmetrical H_q-classical orthogonal polynomials [20, 22] and the appearance of some new situations. Also, we were able to rediscover the limiting cases D-Laguerre-Hahn of class zero ($q \to 1$) (which indicates the limiting case i.e q tends towards 1) [8, 9].

We denote by $\mathcal{P}$ the vector space of the polynomials with coefficients in $\mathbb{C}$ and by $\mathcal{P}'$ its dual space. The action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. For instance, for any form u, any polynomial g and any $(A, c) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$, we let $H_q u$, gu, $h_A u$, Du, $(x - c)^{-1}u$ and δ_c, be the forms defined as usually [24] and [20] for the results related to the operator H_q

$$\langle H_q u, f\rangle := -\langle u, H_q f\rangle \ , \ \langle gu, f\rangle := \langle u, gf\rangle \ , \ \langle h_A u, f\rangle := \langle u, h_A f\rangle \ ,$$

$$\langle Du, f\rangle := -\langle u, f'\rangle \quad , \quad \langle (x - c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle \quad , \quad \langle \delta_c, f\rangle := f(c) \quad ,$$

where for all $f \in \mathcal{P}$ and $q \in \widetilde{\mathbb{C}} := \left\{z \in \mathbb{C},\ z \neq 0, z^n \neq 1, n \geq 1\right\}$ [20]

$$(H_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x},\ x \neq 0,\ (H_q f)(0) = f'(0),\ (h_A f)(x) = f(Ax),\ (\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}.$$

Let us define

$$[n]_q := \frac{q^n - 1}{q - 1},\ n \geq 0 \quad ; \quad [-n]_q := -q^{-n}[n]_q,\ n \geq 0.$$

For $0 < q < 1$ or $q > 1$, we may extend the above definition for a complex number z by

$$[z]_q := \frac{q^z - 1}{q - 1}.$$

The well known formula holds [20]

$$H_q(fg)(x) = (h_q f)(x)(H_q g)(x) + g(x)(H_q f)(x),\ f, g \in \mathcal{P}. \tag{1}$$

It is obvious that when $q \to 1$, we meet again the derivative D.
For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial [24]

$$(uf)(x) := \langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta}\rangle.$$

This allows us to define the Cauchy's product of two forms:

$$\langle uv, f\rangle := \langle u, vf\rangle \ ,\ f \in \mathcal{P}.$$

A form u is said to be regular whenever there is a sequence of monic polynomials $\{P_n\}_{n\geq 0}$, $\deg P_n = n$, $n \geq 0$ MPS such that $\langle u, P_n P_m\rangle = 0$, $n, m \geq 0$, $n \neq m$ and $\langle u, P_n^2\rangle \neq 0$, $n \geq 0$. In this case, $\{P_n\}_{n\geq 0}$ is called a monic orthogonal polynomials sequence MOPS and it is characterized by the following three-term recurrence relation (Favard's theorem) (TTRR in short) [11, 24]

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ &P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0, \end{aligned} \tag{2}$$

where $\beta_n = \dfrac{\langle u, xP_n^2\rangle}{\langle u, P_n^2\rangle} \in \mathbb{C}$, $\gamma_{n+1} = \dfrac{\langle u, P_{n+1}^2\rangle}{\langle u, P_n^2\rangle} \in \mathbb{C} \setminus \{0\}$, $n \geq 0$.

The shifted MOPS $\{\widehat{P}_n := A^{-n}\big(h_A P_n\big)\}_{n\geq 0}$ is then orthogonal with respect to $\widehat{u} = h_{A^{-1}}u$ and satisfies (2) with [24]

$$\widehat{\beta}_n = \frac{\beta_n}{A} \quad , \quad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{A^2} \quad , \quad n \geq 0. \tag{3}$$

Moreover, the form u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any regular form will be normalized. When u is regular, $\{P_n\}_{n\geq 0}$ is a symmetrical MOPS if and only if $\beta_n = 0$, $n \geq 0$ or equivalently $(u)_{2n+1} = 0$, $n \geq 0$ [11, 24].

Given a regular form u and the corresponding MOPS $\{P_n\}_{n\geq 0}$, we define the associated sequence of the first kind $\{P_n^{(1)}\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ by [11, 24]

$$P_n{}^{(1)}(x) = \Big\langle u, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x-\xi} \Big\rangle = (u\theta_0 P_{n+1})(x),\ n \geq 0.$$

We will give now some future about the H_q-Laguerre-Hahn character.

Definition 1.1. [17] *A form u is called H_q-Laguerre-Hahn when it is regular and satisfies the q-quadratic q-difference equation*

$$H_q(\Phi(x)u) + \Psi(x)u + B(x)\left(x^{-1}u(h_q u)\right) = 0, \tag{4}$$

where Φ, Ψ, B are polynomials, with Φ monic. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called a H_q-Laguerre-Hahn MOPS.

Remark 1.2.
1. *When $B = 0$ and the form u is regular then u is H_q-semiclassical [21].*
2. *When u satisfies* (4)*, then $\widehat{u} = h_{A^{-1}}u$ fulfills the q-quadratic q-difference equation [17]*

$$H_q(A^{-\deg\Phi}\Phi(Ax)\widehat{u}) + A^{1-\deg\Phi}\Psi(Ax)\widehat{u} + A^{-\deg\Phi}B(Ax)\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big) = 0. \tag{5}$$

3. *Put $t = \deg\Phi$, $p = \deg\Psi$, $r = \deg B$ and $d = \max(t, r)$, we define the class of u the nonnegative integer s [17]*

$$s = \min\max(p-1, d-2),$$

where the minimum is taken over all triplets (Φ, Ψ, B) satisfying (4). Moreover, the regular form u H_q-Laguerre-Hahn satisfying (4) *is of class $s = \max(p-1, d-2)$ if and only if,*

$$\prod_{c\in\mathcal{Z}_\Phi}\Big\{\Big|q(h_q\Psi)(c) + (H_q\Phi)(c)\Big| + \Big|q(h_qB)(c)\Big| + \Big|\Big\langle u, q(\theta_{cq}\Psi) + (\theta_{cq}\circ\theta_c\Phi) + q\Big(h_q u(\theta_0\circ\theta_{cq}B)\Big)\Big\rangle\Big|\Big\} > 0, \tag{6}$$

where $\mathcal{Z}_\Phi$ is the set of roots of Φ [17].

Proposition 1.3. [17] *Let u be a regular form and $\{P_n\}_{n\geq 0}$ be its MOPS. The following statements are equivalent :*

(i) u is a H_q-Laguerre-Hahn form satisfying (4).

(ii) There exist an integer $s \geq 0$, two polynomials Φ (monic), B with $t = \deg\Phi \leq s+2$, $r = \deg B \leq s+2$ and a sequence of complex numbers $\{\lambda_{n,\nu}\}_{n,\nu\geq 0}$ such that (the structure relation)

$$\Phi(x)(H_qP_{n+1})(x) - h_q(BP_n^{(1)})(x) = \sum_{\nu=n-s}^{n+d}\lambda_{n,\nu}P_\nu(x), \quad n > s, \quad \lambda_{n,n-s} \neq 0. \tag{7}$$

Proposition 1.4. [17] *Let u be a symmetric H_q-Laguerre-Hahn form of class s satisfying (4) . The following statements hold*

(i) If s is odd, then the polynomials Φ and B are odd and Ψ is even.

(ii) If s is even, then the polynomials Φ *and* B *are even and* Ψ *is odd.*

Lastly, the following results will be needed in the sequel.

Lemma 1.5. [1] *Let* $\{P_n\}_{n\geq 0}$ *be a MOPS and* $M(x,n)$, $N(x,n)$ *two polynomials such that*

$$M(x,n)P_{n+1}(x) = N(x,n)P_n(x),\ n \geq 0.$$

Then, for any index n for which $\deg N(x,n) \leq n$, *we have*

$$N(x,n) = 0 \ \text{ and } \ M(x,n) = 0.$$

Lemma 1.6. [25] *Let* $(b_n)_{n\geq 0}$ *with* $b_n \neq 0,\ n \geq 0$, $(c_n)_{n\geq 0}$ *two sequences of complex numbers and* $(x_n)_{n\geq 0}$ *the sequence satisfying the recurrence relation:*

$$x_{n+1} = b_n x_n + c_n,\ n \geq 0,\ x_0 = a \in \mathbb{C} - \{0\}.$$

We have

$$x_{n+1} = \left(\prod_{k=0}^{n} b_k\right)\left\{a + \sum_{k=0}^{n}\left(\prod_{l=0}^{k} b_l\right)^{-1} c_k\right\}, \quad n \geq 0.$$

2. The system fulfilled by a H_q-Laguerre-Hahn MOPS of class zero

Let $\{P_n\}_{n\geq 0}$ be a MOPS H_q-Laguerre-Hahn of class $s = 0$ such that its corresponding regular form satisfies (4). We have for the polynomials Φ, B and Ψ

$$\Phi(x) = c_2x^2 + c_1x + c_0 \text{ (monic)},\ B(x) = b_2x^2 + b_1x + b_0,\ \Psi(x) = a_1x + a_0, \tag{8}$$

with

$$|a_1| + |b_2| + |c_2| \neq 0. \tag{9}$$

Furthermore, the MOPS $\{P_n\}_{n\geq 0}$ fulfills the TTRR (2). Thus, from definition, the MOPS $\{P_n^{(1)}\}_{n\geq 0}$ fulfills the TTRR

$$\begin{aligned} &P_0^{(1)}(x) = 1, \quad P_1^{(1)}(x) = x - \beta_1, \\ &P_{n+2}^{(1)}(x) = (x - \beta_{n+2})P_{n+1}^{(1)}(x) - \gamma_{n+2}P_n^{(1)}(x), \quad n \geq 0, \end{aligned} \tag{10}$$

By virtue of (2), we get for the structure relation (7)

$$\Phi(x)(H_qP_{n+1})(x) - B(qx)P_n^{(1)}(qx) = \Big(G_nx + E_n\Big)P_{n+1}(x) + F_nP_n(x),\ n \geq 0. \tag{11}$$

Indeed, (11) is valid for $n = 0$ since $\deg(\Phi - h_qB) \leq 2$ and then we may write $\Phi(x) - B(qx) = \Big(G_0x + E_0\Big)P_1(x) + F_0$. Applying the operator H_q to (2), on account of (1) and next multiplying by $\Phi(x)$ we get

$$\Phi(x)(H_qP_{n+2})(x) = (qx - \beta_{n+1})\Phi(x)(H_qP_{n+1})(x) - \gamma_{n+1}\Phi(x)(H_qP_n)(x) + \Phi(x)P_{n+1}(x). \tag{12}$$

Therefore,
on the one hand ($P_{-1}^{(1)} := 0$),

$$\begin{aligned} \Phi(x)(H_qP_{n+2})(x) - B(qx)P_{n+1}^{(1)}(qx) &\underset{(12),(10)}{=} (qx - \beta_{n+1})\Big\{\Phi(x)(H_qP_{n+1})(x) - B(qx)P_n^{(1)}(qx)\Big\} \\ &\quad -\gamma_{n+1}\Big\{\Phi(x)(H_qP_n)(x) - B(qx)P_{n-1}^{(1)}(qx)\Big\} + \Phi(x)P_{n+1}(x) \\ &\underset{(11),(2)}{=} (qx - \beta_{n+1})\Big\{\Big(G_nx + E_n\Big)P_{n+1}(x) + F_nP_n(x)\Big\} \\ &\quad -\gamma_{n+1}\Big\{\Big(G_{n-1}x + E_{n-1}\Big)P_n(x) + \tfrac{1}{\gamma_n}F_{n-1}\Big((x - \beta_n)P_n(x) - P_{n+1}(x)\Big)\Big\} + \Phi(x)P_{n+1}(x), \end{aligned}$$

and on the other hand,

$$\Phi(x)(H_qP_{n+2})(x) - B(qx)P^{(1)}_{n+1}(qx) \underset{(11)}{=} \Big(G_{n+1}x + E_{n+1}\Big)P_{n+2}(x) + F_{n+1}P_{n+1}(x)$$
$$\underset{(2)}{=} \Big((x-\beta_{n+1})(G_{n+1}x + E_{n+1}) + F_{n+1}\Big)P_{n+1}(x) - \gamma_{n+1}\Big(G_{n+1}x + E_{n+1}\Big)P_n(x).$$

Consequently,

$$M(x,n)P_{n+1}(x) = N(x,n)P_n(x), \quad n \geq 0, \tag{13}$$

where for all $n \geq 0$,

$$M(x,n) = \Big(c_2 + qG_n - G_{n+1}\Big)x^2 + \Big\{c_1 + \beta_{n+1}(G_{n+1} - G_n) + qE_n - E_{n+1}\Big\}x + c_0 + \beta_{n+1}(E_{n+1} - E_n) + \frac{\gamma_{n+1}}{\gamma_n}F_{n-1} - F_{n+1},$$

$$N(x,n) = \Big\{\gamma_{n+1}(G_{n-1} - G_{n+1}) + \frac{\gamma_{n+1}}{\gamma_n}F_{n-1} - qF_n\Big\}x + \gamma_{n+1}(E_{n-1} - E_{n+1}) + \beta_{n+1}F_n - \beta_n\frac{\gamma_{n+1}}{\gamma_n}F_{n-1},$$

with constraints $\gamma_0 := 1$ and $G_{-1} = E_{-1} = F_{-1} := 0$. Obviously,

$$G_{n+1} - qG_n - c_2 = 0, \quad n \geq 0. \tag{14}$$

By virtue of Lemma 1.5 we have

$$M(x,n) = 0,\ N(x,n) = 0, \quad n \geq 1,$$

which leads to

$$\beta_{n+1}(G_{n+1} - G_n) + qE_n - E_{n+1} + c_1 = 0, \quad n \geq 1, \tag{15}$$

$$\beta_{n+1}(E_{n+1} - E_n) + \frac{\gamma_{n+1}}{\gamma_n}F_{n-1} - F_{n+1} + c_0 = 0, \quad n \geq 1, \tag{16}$$

$$\gamma_{n+1}(G_{n+1} - G_{n-1}) - \frac{\gamma_{n+1}}{\gamma_n}F_{n-1} + qF_n = 0, \quad n \geq 1, \tag{17}$$

$$\gamma_{n+1}(E_{n+1} - E_{n-1}) + \beta_n\frac{\gamma_{n+1}}{\gamma_n}F_{n-1} - \beta_{n+1}F_n = 0, \quad n \geq 1. \tag{18}$$

For $n = 0$, equality (13) yields $M(x,0)(x - \beta_0) = N(x,0)$. Namely,

$$G_0 = c_2 - b_2q^2, \tag{19}$$

$$E_0 = \beta_0(c_2 - b_2q^2) + c_1 - b_1q, \tag{20}$$

$$F_0 = \Phi(\beta_0) - B(q\beta_0). \tag{21}$$

The structure relation (11) for $n = 1$ gives

$$(q+1)c_1 - q^2b_1 - E_1 + q\beta_0(c_2 - b_2q^2) + q\beta_1(c_2 - q(q-1)b_2) = 0, \tag{22}$$

$$\beta_0\Big\{c_1 - E_1 + \beta_1((q+1)c_2 - q^3b_2)\Big\} - (q+1)c_0 - \beta_1E_1 + F_1 = \gamma_1((q+1)c_2 - q^3b_2) - \beta_1(c_1 - qb_1) - qb_0, \tag{23}$$

$$\beta_0(\beta_1E_1 - F_1 + c_0) = \beta_1(b_0 - c_0) + \gamma_1E_1. \tag{24}$$

Lastly, the condition $\langle H_q(\Phi u) + \Psi u + B\Big(x^{-1}u(h_qu)\Big), x^n\rangle = 0$, $n \geq 0$ gives for $n = 0, 1$

$$\Psi(\beta_0) = -\Big((1+q)b_2\beta_0 + b_1\Big), \tag{25}$$

$$\Phi(\beta_0) - B(q\beta_0) = \Big(a_1 - c_2 + (1+q^2)b_2\Big)\gamma_1 \tag{26}$$

since $(u)_0 = 1$, $(u)_1 = \beta_0$, $(u)_2 = \beta_0^2 + \gamma_1$ and (8).

Proposition 2.1. *Denoting*

$$\xi_q := c_2 + b_2 q(1-q), \tag{27}$$

$$r_n := \frac{F_n}{\gamma_{n+1}}, \quad n \geq 0, \tag{28}$$

and

$$S_n = \sum_{k=0}^{n} \beta_k, \quad n \geq 0. \tag{29}$$

The system (14)-(26) become

$$G_n = c_2[n+1]_q - b_2 q^{n+2}, \quad n \geq 0, \tag{30}$$

$$r_n = q^{-n}\Big\{r_0 - q\xi_q[2n]_q\Big\}, \quad n \geq 0, \tag{31}$$

$$\beta_{n+1}\{q\xi_q[2n+1]_q - r_0\} - q^2\beta_n\{q\xi_q[2n-3]_q - r_0\} + q^n(q+1)c_1 = q^{n+1}(q-1)\{\beta_0(\xi_q + r_0) + b_1\}, \quad n \geq 1, \tag{32}$$

$$\beta_1(q\xi_q - r_0) - \beta_0(-(2q+1)c_2 + 2b_2q^3 - qr_0) + (q+1)c_1 - b_1q^2 = 0, \tag{33}$$

$$\begin{aligned}\gamma_{n+2}\{q\xi_q[2n+2]_q - r_0\} - q^2\gamma_{n+1}\{q\xi_q[2n-2]_q - r_0\} = -q^{n+1}\Phi(\beta_{n+1}) + \\ \frac{1-q}{1+q}a_0 q^{n+1}\beta_{n+1} + \Big\{q^{n+1}c_2 - \frac{q}{1+q}\Big((q-1)r_0 + q\xi_q(1+q^{2n})\Big)\Big\}\beta_{n+1}^2, \quad n \geq 1,\end{aligned} \tag{34}$$

$$\gamma_2\{(q+1)\xi_q - q^{-1}r_0\} - \gamma_1\{-(q+1)c_2 + b_2q^3 - r_0\} = -q\Phi(\beta_1) + (q-1)\{c_0 + \beta_1[q(b_1 + b_2q\beta_1) - \beta_0(c_2 - b_2q^2)]\}, \tag{35}$$

$$E_n = q^n\{\xi_q S_n + c_1 q^{-n}[n+1]_q - q(b_1 + b_2\beta_0)\}, \quad n \geq 0, \tag{36}$$

$$F_n = q^{-n}\Big(r_0 - q\xi_q[2n]_q\Big)\gamma_{n+1}, \quad n \geq 0, \tag{37}$$

$$r_0 = \frac{\Phi(\beta_0) - B(q\beta_0)}{\gamma_1}, \tag{38}$$

$$a_0 = -\beta_0(r_0 + \xi_q) - b_1, \tag{39}$$

$$a_1 = r_0 + \xi_q - (1+q)b_2. \tag{40}$$

Proof. The equalities in (30) and (31) are consequence from (14), (17) and (19) by applying Lemma 1.6. Next, we obtain (37)-(38) from (21), (28) and (31).
From (22) we have,

$$E_1 - qc_1 + b_1q^2 - q\beta_0(c_2 - b_2q^2) = c_1 + q\beta_1(c_2 - q(q-1)b_2).$$

Therefore, by (20) the last formula becomes

$$E_1 - qE_0 = c_1 + q\beta_1(c_2 - q(q-1)b_2). \tag{41}$$

Then, by eliminating E_1 in (23), and using (22), we get

$$\beta_1E_1 - F_1 + (q+1)c_0 = \beta_1(c_1 - b_1q) + b_0q - \gamma_1((q+1)c_2 - b_2q^3) + \beta_0\{q(b_1q - c_1) + (\beta_1 - q\beta_0)(c_2 - b_2q^2)\} \tag{42}$$

Consequently, by injecting (42) into (24), and replacing F_0, E_0 and E_1 by their expressions from (20)-(21) and (41), we get,

$$\gamma_1\beta_0((2q+1)c_2 - 2b_2q^3) + (q\beta_0 - \beta_1)F_0 + \gamma_1((q+1)c_1 - b_1q^2) + q\gamma_1\beta_1c_2 - \gamma_1\beta_1q^2(q-1)b_2 = 0. \tag{43}$$

On the other hand, by eliminating E_1 in (42) and by (22) we obtain

$$q\phi(\beta_1) - F_1 + \gamma_1((q+1)c_2 - b_2q^3) + F_0 = (q-1)\{B(q\beta_1) - \beta_0[c_1 - b_1q + (c_2 - b_2q^2)(\beta_0 + \beta_1)]\}. \tag{44}$$

On account of (27) and (28), the formulas (42)-(44) become (33) and (35).
Now, by virtue of (31), the equations (15)-(18) becomes successively

$$c_1 + q^{n+1}\beta_{n+1}\xi_q + qE_n - E_{n+1} = 0,\ n \geq 0, \tag{45}$$

$$\beta_{n+1}r_n - \beta_n r_{n-1} + E_{n-1} - E_{n+1} = 0,\ n \geq 1. \tag{46}$$

Therefore, subtracting (46) by (45) gives

$$\beta_{n+1}(r_n - q^{n+1}\xi_q) - \beta_n r_{n-1} - c_1 + E_{n-1} - qE_n = 0,\ n \geq 1. \tag{47}$$

Also, (45) for the order n yields

$$c_1 + q^n\beta_n\xi_q + qE_{n-1} - E_n = 0,\ n \geq 1. \tag{48}$$

So, (48) + (47) gives

$$\beta_{n+1}(r_n - q^{n+1}\xi_q) - \beta_n(r_{n-1} - q^n\xi_q) + (q+1)(E_{n-1} - E_n) = 0,\ n \geq 1. \tag{49}$$

By going to the sum on (49), we obtain

$$(1+q)E_n = (1+q)E_0 + \beta_{n+1}(r_n - q^{n+1}\xi_q) - \beta_1(r_0 - q\xi_q),\ n \geq 1. \tag{50}$$

The operation $(1+q)\times(48)$ and on account of (33) and (50) leads to

$$\beta_{n+1}\{q^{n+1}\xi_q - r_n\} + q\beta_n\{q^n\xi_q + r_{n-1}\} + (q+1)c_1 = q(q-1)\{\beta_0(\xi_q + r_0) + b_1\},\ n \geq 1.$$

Then, (31) leads to (32).
(36) is a consequence from Lemma 1.6, (29) and (31).
The operation $(48) - q\times(43)$ gives,

$$c_1 + \beta_n(qr_{n-1} + q^n\xi_q) - q\beta_{n+1}r_n + qE_{n+1} - E_n = 0,\ n \geq 1. \tag{51}$$

Subtracting (51) by (45) yields

$$(q+1)(E_{n+1} - E_n) = \beta_{n+1}(q^{n+1}\xi_q + qr_n) - \beta_n(q^n\xi_q + qr_{n-1}),\ n \geq 1. \tag{52}$$

On account of (32), (52) becomes

$$(q+1)(E_{n+1} - E_n) = \beta_{n+1}\{2q^{n+1}\xi_q + (q-1)r_n\} + (1+q)c_1 + q(1-q)\{\beta_0(\xi_q + r_0) + b_1\}. \tag{53}$$

Thanks to (53), the equation $(1+q)\times(16)$ gives (34).
Lastly, (39)-(40) are consequence from (25)-(26) and (38). □

Remark 2.2. *1. When $q \to 1$ in (30)-(40), we recover again the system of D-Laguerre-Hahn MOPSs of class zero [8, 9].*

2. There are three possibilities for the polynomial Φ

$$\Phi(x) = 1, \quad \Phi(x) = x - c,\ c \in \mathbb{C}, \quad \Phi(x) = (x-c)(x-d),\ c, d \in \mathbb{C},$$

but it is a very difficult problem to describe exhaustively the situtations in any case on account of the expressions in (33)-(34). However, in the next section, we are going to describe the symmetrical case that is to say $\beta_n = 0,\ n \geq 0$.

3. Description of the symmetrical H_q-Laguerre-Hahn MOPSs of class zero

First of all, to describe the symmetrical H_q-Laguerre-Hahn forms of class zero we may write for (8)

$$\Phi(x) = c_2x^2 + c_0 \text{ (monic)}, \; B(x) = b_2x^2 + b_0, \; \Psi(x) = a_1x, \quad |a_1| + |b_2| + |c_2| \neq 0, \tag{54}$$

since Proposition 1.4. and (9). Consequently, three possibilities for the polynomial Φ occurred

$$\Phi(x) = 1, \quad \Phi(x) = x^2, \quad \Phi(x) = x^2 + c_0, \quad c_0 \neq 0,$$

for the symmetrical case. Now, taking for all $n \geq 0$, $\beta_n = 0$ in (34), (34) becomes

$$(r_0 - q\xi_q[2n+2]_q)\gamma_{n+2} - q^2(r_0 - q\xi_q[2n-2]_q)\gamma_{n+1} = q^{n+1}c_0, \quad n \geq 1. \tag{55}$$

Next, let us suppose

$$r_0 - q\xi_q[2n]_q \neq 0, \quad n \geq 0. \tag{56}$$

On account of Lemma 1.6. and after some calculations, we get for (55),

$$\gamma_{n+2} = \frac{q^{2n}r_0(r_0 - q(q+1)\xi_q)\left\{\gamma_2 + q^{1-n}\,c_0\,[n]_q\,\frac{r_0 - q\xi_q[n+1]_q}{r_0(r_0 - q(q+1)\xi_q)}\right\}}{(r_0 - q\xi_q[2n]_q)(r_0 - q\xi_q[2n+2]_q)}, \quad n \geq 0. \tag{57}$$

Moreover, (35) gives

$$\gamma_2 = q\,\frac{\Big(r_0 - b_2q^3 + (q+1)c_2\Big)\gamma_1 + c_0}{r_0 - q(q+1)\xi_q}, \tag{58}$$

with constraint

$$\Big(r_0 - b_2q^3 + (q+1)c_2\Big)\gamma_1 + c_0 \neq 0. \tag{59}$$

The injection of (58) into (57) yields

$$\gamma_{n+2} = q^{2n+1}\,\frac{r_0\Big\{\Big(r_0 - b_2q^3 + (q+1)c_2\Big)\gamma_1 + c_0\Big\} + q^{-n}\,c_0\,[n]_q\,(r_0 - q\xi_q[n+1]_q)}{(r_0 - q\xi_q[2n]_q)(r_0 - q\xi_q[2n+2]_q)}, \quad n \geq 0. \tag{60}$$

Also, thanks to (27), the constraints (38) and (40) become successively

$$r_0 = \frac{c_0 - b_0}{\gamma_1}, \tag{61}$$

$$a_1 + b_2 = c_2 + r_0 - b_2q^2. \tag{62}$$

Before quoting the different canonical situations, let us proceed to the general transformation

$$\widehat{P}_n(x) = A^{-n}P_n(Ax), \quad n \geq 0, \tag{63}$$

$$\widehat{\gamma}_{n+1} = A^{-2}\,\gamma_{n+1}, \quad n \geq 0. \tag{64}$$

Then, the form $\widehat{u} = h_{A^{-1}}u$ fulfills the q-quadratic q-difference equation

$$H_q(A^{-\deg\Phi}\Phi(Ax)\widehat{u}) + A^{1-\deg\Phi}\Psi(Ax)\widehat{u} + A^{-\deg\Phi}B(Ax)\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big) = 0. \tag{65}$$

Any so-called canonical situation will be denoted by $\widehat{\gamma}_{n+1}$, $\widehat{u}$ and recall $q \in \widetilde{\mathbb{C}}$, except opposite mention. Moreover, in this case, according to (11), (30), (36)-(37) and (54), the structure relation (11) of the symmetrical

H_q-Laguerre-Hahn $\{P_n\}_{n\geq 0}$ of class zero may be written as
$(c_2x^2+c_0)(H_qP_{n+1})(x)-(q^2b_2x^2+b_0)P_n^{(1)}(qx)=$

$$\Big(c_2[n+1]_q-b_2q^{n+2}\Big)xP_{n+1}(x)+q^{-n}\Big(r_0-q\xi_q[2n]_q\Big)\gamma_{n+1}P_n(x),\ n\geq 0. \tag{66}$$

The change $x\leftarrow Ax$ in (66), after that multiplying the resulting by A^{-n} we get the structure relation corresponding to the shifted symmetrical H_q-Laguerre-Hahn $\{\widehat{P}_n\}_{n\geq 0}$ of class zero in the following way
$(c_2A^2x^2+c_0)(H_q\widehat{P}_{n+1})(x)-(q^2b_2A^2x^2+b_0)\widehat{P}_n^{(1)}(qx)=$

$$A^2\Big(c_2[n+1]_q-b_2q^{n+2}\Big)x\widehat{P}_{n+1}(x)+A^2q^{-n}\Big(r_0-q\xi_q[2n]_q\Big)\widehat{\gamma}_{n+1}\widehat{P}_n(x),\ n\geq 0. \tag{67}$$

3.1. $\Phi(x)=1$

In this case, $c_2=0$ and $c_0=1$. We get for (27), (56) and (60)-(62),

$$\xi_q=b_2q(1-q), \tag{68}$$

$$r_0-b_2q^2+b_2q^{2n+2}\neq 0,\quad n\geq 0, \tag{69}$$

$$\gamma_{n+1}=q^{2n-1}\ \frac{r_0\Big\{1+\Big(r_0-b_2q^3\Big)\gamma_1\Big\}+q^{-n+1}\,[n-1]_q\,(r_0-b_2q^2+b_2q^{n+2})}{(r_0-b_2q^2+b_2q^{2n})(r_0-b_2q^2+b_2q^{2n+2})},\quad n\geq 1, \tag{70}$$

$$r_0=\frac{1-b_0}{\gamma_1}, \tag{71}$$

and

$$a_1+b_2=r_0-b_2q^2. \tag{72}$$

3.1.1. $a_1+b_2=0$. Necessary $b_2\neq 0$ from regularity and by (72) we get $r_0=b_2q^2$. Consequently, (70) becomes

$$\gamma_{n+1}=\frac{b_2q^2(1-q)\gamma_1+[n]_q}{b_2q^{2n+1}},\quad n\geq 1, \tag{73}$$

since $q[n-1]_q=[n]_q-1,\ n\geq 1$. Choosing A in (63)-(65) such that $A^2b_2=2$ and putting $\gamma_1=\frac{\rho}{b_2q^2},\ \rho\neq 0$, then thanks to (71)-(73) we get the following situation

$$\begin{cases}\widehat{\gamma}_1=\frac{\rho}{2q^2},\ \rho\neq 0,\\ \widehat{\gamma}_{n+1}=q^{-2n-1}\ \frac{(1-q)\rho+[n]_q}{2},\ n\geq 1\ ;\ (1-q)\rho+[n]_q\neq 0,\ n\geq 1,\\ H_q(\widehat{u})-2x\widehat{u}+(2x^2+1-\rho)\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big)=0.\end{cases} \tag{74}$$

When $q\to 1$ in (74), we meet the singular symmetrical D-Laguerre-Hahn of class 0 of Hermite type [9] (see Table A.1. in the Appendix).

Now, choosing A in (63)-(65) such that $A^2b_2=2q^{\tau-2}$ and putting $\gamma_1=\frac{[\tau+1]_{q^{-1}}}{b_2q^3(1-q)},\ \tau\in\mathbb{N}$, then thanks to (71)-(73) we get the following situation

$$\begin{cases}\widehat{\gamma}_1=\frac{q^{-\tau-1}[\tau+1]_{q^{-1}}}{2(1-q)},\\ \widehat{\gamma}_{n+1}=q^{-n-\tau}\ \frac{[n+\tau+1]_{q^{-1}}}{2},\ n\geq 1,\\ H_q(\widehat{u})-2q^{\tau-2}x\widehat{u}+\Big(2q^{\tau-2}x^2+1-\frac{[\tau+1]_{q^{-1}}}{q(1-q)}\Big)\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big)=0.\end{cases} \tag{75}$$

In view of (75), we discover the co-dilates of the associated of order τ of the natural q^{-1}−analogue of Hermite [22].

3.1.2. $a_1 + b_2 \neq 0$. That is to say $r_0 \neq b_2 q^2$ since (72). Denoting

$$\rho := \frac{r_0}{r_0 - b_2 q^2} \neq 0. \tag{76}$$

Then,

$$\rho - 1 = \frac{b_2 q^2}{r_0 - b_2 q^2}. \tag{77}$$

On account of (76)-(77), (70) becomes

$$\gamma_{n+1} = q^{2n-1}\,\frac{\rho}{r_0}\,\frac{\rho + r_0\Big(\rho - q(\rho-1)\Big)\gamma_1 + q^{1-n}[n-1]_q\Big(1 + (\rho-1)q^n\Big)}{\Big(1 + (\rho-1)q^{2n-2}\Big)\Big(1 + (\rho-1)q^{2n}\Big)}, \quad n \geq 1. \tag{78}$$

Two cases arise:

3.1.2.1. $\rho = 1$. Therefore, $b_2 = 0$ and $a_1 = r_0 \neq 0$ on account of (77) and the item 2.2.1.2. Moreover, the formula in (78) becomes

$$\gamma_{n+1} = \frac{q^{2n-1}}{r_0}\Big(r_0\gamma_1 + [n]_{q^{-1}}\Big), \quad n \geq 1. \tag{79}$$

Choosing A in (63)-(65) such that $A^2 = 2r_0^{-1}q^{-\tau}$ and putting $r_0\gamma_1 = q[\tau+1]_q$ for $0 < q < 1$ or $q > 1$, by virtue of (71) and (79) we are led to

$$\begin{cases} \widehat{\gamma}_1 = q^{\tau+1}\,\frac{[\tau+1]_q}{2}, \\ \widehat{\gamma}_{n+1} = q^{n+\tau}\,\frac{[n+\tau+1]_q}{2},\ n \geq 1, \\ H_q(\widehat{u}) + 2q^{-\tau}x\widehat{u} + \Big(1 - q[\tau+1]_q\Big)\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big) = 0. \end{cases} \tag{80}$$

When $q \to 1$ in (80), we meet a restricted nonsingular symmetrical D-Laguerre-Hahn of class 0 of Hermite type [9] (see Table A.2. in the Appendix).

3.1.2.2. $\rho \neq 1$. In this case, (78) may be written as

$$\gamma_{n+1} = \frac{q^{2n-1}}{r_0}\,\frac{[n]_q + q\rho^{-1}r_0\gamma_1 + (1-q)\Big(r_0\gamma_1 + \rho^{-1}q^{1-n}[n]_q[n-1]_q\Big)}{\Big(\rho^{-1} + (1-\rho^{-1})q^{2n-2}\Big)\Big(\rho^{-1} + (1-\rho^{-1})q^{2n}\Big)}, \quad n \geq 1, \tag{81}$$

since $q[n-1]_q = [n]_q - 1,\ n \geq 1,\ \rho \neq 0$ and $\rho \neq 1$.
Choosing A in (63)-(65) such that $A^2 = 2r_0^{-1}q^{-2\tau-3}$ and putting $r_0\gamma_1 = \rho q^{-\tau-2}[\tau+1]_q$ for $0 < q < 1$ or $q > 1$, by virtue of (71)-(72), (76)-(77) and (81) we obtain

$$\begin{cases} \widehat{\gamma}_1 = \rho q^{\tau+1}\,\frac{[\tau+1]_q}{2},\ \tau \neq -1, \\ \widehat{\gamma}_{n+1} = \frac{1}{2}\,q^{2n+\tau+1}\,\frac{[n+\tau+1]_q + (1-q)\Big(\rho q^{-1}[\tau+1]_q + \rho^{-1}q^{\tau-n+2}[n]_q[n-1]_q\Big)}{\Big(\rho^{-1}+(1-\rho^{-1})q^{2n-2}\Big)\Big(\rho^{-1}+(1-\rho^{-1})q^{2n}\Big)},\ n \geq 1, \\ H_q(\widehat{u}) + 2q^{-2\tau-5}\Big(\rho^{-1}(1+q^2) - 1\Big)x\widehat{u} + \\ \qquad \Big(2q^{-2\tau-5}(1-\rho^{-1})x^2 + 1 - \rho q^{-\tau-2}[\tau+1]_q\Big)\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big) = 0. \end{cases} \tag{82}$$

The form $\widehat{u}$ is regular, if and only if,

$$\tau \neq -1, \quad [n+\tau+1]_q + (1-q)\Big(\rho q^{-1}[\tau+1]_q + \rho^{-1}q^{\tau-n+2}[n]_q[n-1]_q\Big) \neq 0,\ n \geq 1.$$

When $q \to 1$ in (82), we meet the general situation of the nonsingular symmetrical D-Laguerre-Hahn of class 0 of Hermite type [9] (see Table A. 2. in the Appendix).

3.2. $\Phi(x) = x^2$

In this case, $c_2 = 1$ and $c_0 = 0$. We get for (27) and (59)-(62),

$$r_0 - b_2q^3 + q + 1 \neq 0. \tag{83}$$

$$\xi_q = 1 + b_2q(1-q), \tag{84}$$

$$\gamma_{n+1} = q^{2n-1} \frac{r_0\big(r_0 - b_2q^3 + q + 1\big)\gamma_1}{(r_0 - q\xi_q[2n-2]_q)(r_0 - q\xi_q[2n]_q)}, \quad n \geq 1, \tag{85}$$

$$r_0 = -\frac{b_0}{\gamma_1}, \tag{86}$$

$$a_1 + b_2 = 1 + r_0 - b_2q^2. \tag{87}$$

with (56) is globally unchanged.

3.2.1. $\xi_q = 0$. Then, $b_2 = q^{-1}(q-1)^{-1}$ and $r_0 \neq (q-1)^{-1}$ thanks to (84) and (83). In addition, (85) yields

$$\gamma_{n+1} = q^{2n-1}\Big(1 - r_0^{-1}(q-1)^{-1}\Big)\gamma_1, \quad n \geq 1, \tag{88}$$

Choosing A in (63)-(65) such that $A^2 = \Big(1 - r_0^{-1}(q-1)^{-1}\Big)\gamma_1$, putting $\rho = \frac{1}{1-r_0^{-1}(q-1)^{-1}}$, with evidently $\rho \neq 0,\ \rho \neq 1$, and by virtue of (56), (87), (86) and (88) we get the following situation

$$\begin{cases} \widehat{\gamma_1} = \rho, \\ \widehat{\gamma}_{n+1} = q^{2n-1},\ n \geq 1, \\ H_q(x^2\widehat{u}) + q^{-1}(1-q)^{-1}(1-\rho)^{-1}\Big(1+q-\rho\Big)x\widehat{u} - \\ \qquad (1-q)^{-1}\Big(q^{-1}x^2 + \rho^2(1-\rho)^{-1}\Big)\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big) = 0. \end{cases} \tag{89}$$

3.2.2. $\xi_q \neq 0$. Then, $b_2 \neq q^{-1}(q-1)^{-1}$ and one may write for (85)

$$\gamma_{n+1} = q^{2n-1} \frac{r_0\big(r_0 - b_2q^3 + q + 1\big)\gamma_1}{q^2\xi_q^2} \frac{1}{\Big([2n-2]_q - q^{-1}\xi_q^{-1}r_0\Big)\Big([2n]_q - q^{-1}\xi_q^{-1}r_0\Big)}, \quad n \geq 1. \tag{90}$$

Chosing A in (63)-(65) such that $A^2 = -\frac{r_0\big(r_0 - b_2q^3+q+1\big)\gamma_1}{q^2\xi_q^2}$ and putting

$$q^{-1}\xi_q^{-1}r_0 = -2\tau - q - 2,\ \frac{q\xi_q}{r_0 - b_2q^3 + q + 1} = -\frac{\rho}{2\tau+1}, \tag{91}$$

with constraints

$$\tau \neq -\frac{1}{2},\ \tau \neq -1 - \frac{q}{2},\ \rho \neq 0,\ 1 + q - q^2 + (q-1)(\rho^{-1}-1)(2\tau+1) \neq 0, \tag{92}$$

(83) is then valid, (91) yields

$$r_0 = -\frac{2\tau + q + 2}{1 + q - q^2 + (q-1)(\rho^{-1}-1)(2\tau+1)},\ b_2 = \frac{1 - q^2 + q(\rho^{-1}-1)(2\tau+1)}{q^2\Big(1 + q - q^2 + (q-1)(\rho^{-1}-1)(2\tau+1)\Big)}, \tag{93}$$

and thanks to (86)-(87), (91)-(93) and (89) we obtain

$$\begin{cases} \widehat{\gamma}_1 = -\frac{\rho}{(2\tau+q+2)(2\tau+1)}, \\ \widehat{\gamma}_{n+1} = -\frac{q^{2n-1}}{\Big([2n-2]_q+2\tau+q+2\Big)\Big([2n]_q+2\tau+q+2\Big)},\ n\geq 1, \\ H_q(x^2\widehat{u}) - \frac{1+q(2\tau+1)\Big((q+1)\rho^{-1}-1\Big)}{q^2\Big(1+q-q^2+(q-1)(\rho^{-1}-1)(2\tau+1)\Big)}x\widehat{u} + \frac{1}{1+q-q^2+(q-1)(\rho^{-1}-1)(2\tau+1)}\times \\ \qquad \Big(q^{-2}\Big(1-q^2+q(\rho^{-1}-1)(2\tau+1)\Big)x^2 - \frac{\rho}{(2\tau+1)}\Big)\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big) = 0. \end{cases} \tag{94}$$

When $q \to 1$ in (94), we meet the general situation of the nonsingular symmetrical D-Laguerre-Hahn of class 0 of Bessel type [9] (see Table B. 1. in the Appendix).

3.3. $\Phi(x) = x^2 + c_0,\ c_0 \neq 0$

In this case, (60) becomes

$$\gamma_{n+1} = q^{2n-1}\ \frac{r_0\Big\{\Big(r_0 - b_2q^3 + q + 1\Big)\gamma_1 + c_0\Big\} + q^{1-n}\, c_0\, [n-1]_q\, (r_0 - q\xi_q[n]_q)}{(r_0 - q\xi_q[2n-2]_q)(r_0 - q\xi_q[2n]_q)},\quad n\geq 1. \tag{95}$$

Also, we get for (27), (59) and (62),

$$\xi_q = 1 + b_2q(1-q), \tag{96}$$

$$\Big(r_0 - b_2q^3 + q + 1\Big)\gamma_1 + c_0 \neq 0, \tag{97}$$

$$a_1 + (1+q^2)b_2 = 1 + r_0, \tag{98}$$

and (61) remains unchanged.

3.3.1. $\xi_q = 0$

Then $b_2 = q^{-1}(q-1)^{-1}$ by (96). In addition, the constraint (97) yields

$$\Big(r_0 - (q-1)^{-1}\Big)\gamma_1 + c_0 \neq 0. \tag{99}$$

By virtue of (99), we get for (95),

$$\gamma_{n+1} = \frac{q^{2n-1}}{r_0}\Big\{\Big(r_0 - (q-1)^{-1}\Big)\gamma_1 + c_0 + q^{1-n}\, c_0\, [n-1]_q\Big\},\quad n\geq 1. \tag{100}$$

It is possible to take $r_0 = (q-1)^{-1} \neq 0$ since $q \in \widetilde{\mathbb{C}}$, then (99) is valid since $c_0 \neq 0$ and (100) may be written as

$$\gamma_{n+1} = (qc_0)(1-q^{-n})q^{2n-1},\quad n\geq 1. \tag{101}$$

Putting $qc_0 = -c^2$ in (101), writing $\gamma_1 = \rho,\ \rho \neq 0$ and thanks to (98) and (61), we are led to the following situation ($\rho \neq 0$, $c \neq 0$, $q \in \widetilde{\mathbb{C}}$)

$$\begin{cases} \widehat{\gamma}_1 = \rho, \\ \widehat{\gamma}_{n+1} = -c^2(1-q^{-n})q^{2n-1},\ n\geq 1, \\ H_q((x^2 - q^{-1}c^2)u) - q^{-1}(q-1)^{-1}xu + q^{-1}(q-1)^{-1}\Big(x^2 - (q-1)c^2 - q\rho\Big)\Big(x^{-1}u(h_qu)\Big) = 0. \end{cases} \tag{102}$$

We meet the perturbed of order one of a certain symmetrical modified q^{-1}–classical q^{-1}–Jacobi kind (see (3.29) in [20] with $q \leftarrow q^{-1},\ n \leftarrow n-1,\ n \geq 1$).

3.3.2. $\xi_q \neq 0$

On account of (96), $b_2 \neq q^{-1}(q-1)^{-1}$. In view of (95) and the constraint (97), two cases appear: $r_0 - b_2q^3 + q + 1 = 0$ or $r_0 - b_2q^3 + q + 1 \neq 0$.

3.3.2.1. $r_0 - b_2q^3 + q + 1 = 0$

Then (97) remains valid since $c_0 \neq 0$ and

$$r_0 = b_2q^3 - q - 1, \tag{103}$$

with necessary

$$b_2 \neq q^{-3}(q+1), \tag{104}$$

since $r_0 \neq 0$. Now, putting

$$q^{-1}\xi_q^{-1}r_0 = -(\alpha+1), \quad \alpha \neq -1. \tag{105}$$

On account of (96), (103) and (105), necessary $\alpha \neq (q-1)^{-1}$, we get for b_2,

$$b_2 = q^{-2}\,\frac{1-q\alpha}{1+(1-q)\alpha}, \tag{106}$$

and the constraint (104) is valid. Consequently, we may write for (95),

$$\gamma_{n+1} = -\frac{c_0}{q\xi_q}\,q^{n-1}\,\frac{[n]_q\Big([n]_q + q\alpha + q - 1\Big)}{\Big([2n-2]_q + \alpha + 1\Big)\Big([2n]_q + \alpha + 1\Big)}, \quad n \geq 1. \tag{107}$$

Putting $\gamma_1 = -\frac{c_0}{q\xi_q}\,\rho,\ \rho \neq 0$, choosing A in (63)-(65) such that $A^2 = -\frac{c_0}{q\xi_q}$ and thanks to (61), (98), (103)-(107) we are led to the following situation

$$\begin{cases} \widehat{\gamma}_1 = \rho, \\ \widehat{\gamma}_{n+1} = q^{n-1}\,\frac{[n]_q\Big([n]_q+q\alpha+q-1\Big)}{\Big([2n-2]_q+\alpha+1\Big)\Big([2n]_q+\alpha+1\Big)},\ n \geq 1, \\ H_q\Big(\Big(x^2 - \frac{1}{1+(1-q)\alpha}\Big)\widehat{u}\Big) + \frac{q^{-1}\alpha-(1+q^{-2})}{1+(1-q)\alpha}\,x\widehat{u} + \\ \qquad \frac{1}{1+(1-q)\alpha}\Big(q^{-2}(1-q\alpha)x^2 + \rho(\alpha+1) - 1\Big)\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big) = 0. \end{cases} \tag{108}$$

The form $\widehat{u}$ is regular, if and only if,

$$\rho \neq 0, \quad \alpha \neq (q-1)^{-1}, \quad \alpha + 1 \neq -[2n]_q,\ n \geq 0, \quad q(\alpha+1) \neq 1 - [n]_q,\ n \geq 1.$$

When $q \to 1$ in (108), we meet the general situation of the singular symmetrical D-Laguerre-Hahn of class 0 of Jacobi type [9] (see Table C. 1. in the Appendix).
Meanwhile, there is another way to write (95) by replacing $[n]_q$ by its expression,

$$\gamma_{n+1} = -\frac{c_0}{q\xi_q}\,q^n\,\frac{(q^n-1)\Big(q^{n-1} - 1 - q^{-1}\xi_q^{-1}r_0(q-1)\Big)}{\Big(q^{2n-2} - 1 - q^{-1}\xi_q^{-1}r_0(q-1)\Big)\Big(q^{2n} - 1 - q^{-1}\xi_q^{-1}r_0(q-1)\Big)}, \quad n \geq 1. \tag{109}$$

Putting

$$q^{-1}\xi_q^{-1}r_0 = [-2\alpha-1]_q, \quad 0 < q < 1 \text{ or } q > 1, \quad \alpha \neq -\frac{1}{2}, \tag{110}$$

which is equivalent to $1+q^{-1}\xi_q^{-1}r_0(q-1)=q^{-2\alpha-1},\quad \alpha\neq -\frac{1}{2}$. Therefore, (109) becomes

$$\gamma_{n+1}=-\frac{c_0}{q\xi_q}\,\frac{(q^n-1)(q^{n+2\alpha}-1)}{(q^{2n+2\alpha-1}-1)(q^{2n+2\alpha+1}-1)}\,q^{n+2\alpha+1},\quad n\geq 1. \tag{111}$$

On account of (96), (103) and (110), necessary $1-q-q^{-2\alpha-1}\neq 0$, we get for b_2,

$$b_2=-q^{-2}\,\frac{1+q(1+[-2\alpha-1]_q)}{1-q-q^{-2\alpha-1}}, \tag{112}$$

and the constraint (104) is valid. Putting $\gamma_1=-\frac{c_0}{q\xi_q}\rho,\ \rho\neq 0$, choosing A in (63)-(65) such that $A^2=-\frac{c_0}{q\xi_q}$ and thanks to (61), (98), (103)-(104), (110)-(112) we are led to the following situation

$$\begin{cases} \widehat{\gamma}_1=\rho, \\ \widehat{\gamma}_{n+1}=\frac{(q^n-1)(q^{n+2\alpha}-1)}{(q^{2n+2\alpha-1}-1)(q^{2n+2\alpha+1}-1)}\,q^{n+2\alpha+1},\ n\geq 1, \\ H_q\Big(\Big(x^2+\frac{1}{1-q-q^{-2\alpha-1}}\Big)\widehat{u}(\alpha)\Big)+q^{-2}\,\frac{[3]_q+q[-2\alpha-1]_q}{1-q-q^{-2\alpha-1}}\,x\widehat{u}(\alpha)- \\ \frac{q^{-2}}{1-q-q^{-2\alpha-1}}\Big\{\Big(1+q\Big(1+[-2\alpha-1]_q\Big)\Big)x^2-q^2\Big(1+\rho[-2\alpha-1]_q\Big)\Big\}\Big(x^{-1}\widehat{u}(\alpha)(h_q\widehat{u}(\alpha))\Big)=0. \end{cases} \tag{113}$$

The form $\widehat{u}(\alpha)$ is regular, if and only if,

$$\rho\neq 0,\quad 1-q-q^{-2\alpha-1}\neq 0,\quad \alpha\neq -\frac{n}{2},\ n\geq 1. \tag{114}$$

We meet the perturbed of order one of a certain modified symmetrical $q-$classical $q-$Jacobi kind (see (8) in [25] with $n\leftarrow n-1,\ n\geq 1$). Moreover, taking $\alpha=\dfrac{1}{2}$ in (113), the constraints in (114) are valid for $0<q<1$ or $q>1$ and (113) becomes $\Big(\widehat{u}:=\widehat{u}\Big(\frac{1}{2}\Big)\Big)$,

$$\begin{cases} \widehat{\gamma}_1=\rho, \\ \widehat{\gamma}_{n+1}=q\,\frac{q^{n+1}}{(q^n+1)(q^{n+1}+1)},\ n\geq 1, \\ H_q\Big(\Big(x^2+\frac{1}{1-q-q^{-2}}\Big)\widehat{u}\Big)+q^{-2}\,\frac{[3]_q+q[-2]_q}{1-q-q^{-2}}\,x\widehat{u}- \\ \frac{q^{-2}}{1-q-q^{-2}}\Big\{\Big(1+q\Big(1+[-2]_q\Big)\Big)x^2-q^2\Big(1+\rho[-2]_q\Big)\Big\}\Big(x^{-1}\widehat{u}(h_q\widehat{u})\Big)=0. \end{cases} \tag{115}$$

Denoting $\widehat{\widehat{u}}=h_{q^{-\frac{1}{2}}}\widehat{u}$, it is also a symmetrical H_q-Laguerre-Hahn of class 0 fulfilling

$$\begin{cases} \widehat{\widehat{\gamma}}_1=q^{-1}\rho, \\ \widehat{\widehat{\gamma}}_{n+1}=\frac{q^{n+1}}{(q^n+1)(q^{n+1}+1)},\ n\geq 1, \\ H_q\Big(\Big(x^2+\frac{q^{-1}}{1-q-q^{-2}}\Big)\widehat{\widehat{u}}\Big)+q^{-2}\,\frac{[3]_q+q[-2]_q}{1-q-q^{-2}}\,x\widehat{\widehat{u}}- \\ \frac{q^{-2}}{1-q-q^{-2}}\Big\{\Big(1+q\Big(1+[-2]_q\Big)\Big)x^2-q\Big(1+\rho[-2]_q\Big)\Big\}\Big(x^{-1}\widehat{\widehat{u}}(h_q\widehat{\widehat{u}})\Big)=0. \end{cases} \tag{116}$$

The situation in (116) deals with the co-dilates of the perturbed of order one of the modified symmetrical q-classical q-Chebyshev form of the second kind (see (10) in [25] with $n\leftarrow n-1,\ n\geq 1$).

3.3.2.2. $r_0 - b_2q^3 + q + 1 \neq 0$

Then putting,

$$\begin{cases} q^{-1}\xi_q^{-1}r_0 = -(2\tau + 2\alpha + 3), \\ c_0^{-1}\gamma_1(r_0 - b_2q^3 + q + 1) = \frac{(\tau+1)(\tau+2\alpha+1)}{2\tau+2\alpha+3}, \\ -\frac{q\xi_q(2\tau+2\alpha+1)}{r_0-b_2q^3+q+1} = \rho, \end{cases} \tag{117}$$

with necessary the following constraints,

$$\begin{cases} 2\tau + 2\alpha + 3 \neq 0, \quad (\tau + 1)(\tau + 2\alpha + 1) \neq 0, \quad 2\tau + 2\alpha + 1 \neq 0, \quad \rho \neq 0, \\ \mho := (1 - \rho)(1 - q)(2\tau + 2\alpha + 1) + \rho(q - 2) \neq 0. \end{cases} \tag{118}$$

Consequently, we may write for (95),

$$\gamma_{n+1} = -\frac{c_0}{q\xi_q}\, q^{2n-1}\times$$

$$\frac{\left(q^{-n}[n]_q + \tau + 1\right)\left([n]_q + \tau + 2\alpha + 1\right) + q^{-n}(q - 1)[n]_q\left(2\alpha + 1 + (\tau + 1)(2 - [n]_q)\right)}{\left([2n - 2]_q + 2\tau + 2\alpha + 3\right)\left([2n]_q + 2\tau + 2\alpha + 3\right)},\ n \geq 1. \tag{119}$$

Moreover, on account of (96) and (117)-(118) we get for γ_1, b_2, $q\xi_q$, r_0,

$$\begin{cases} \gamma_1 = \frac{-c_0}{q\xi_q}\ \frac{(\tau+1)(\tau+2\alpha+1)}{(2\tau+2\alpha+3)(2\tau+2\alpha+1)}\ \rho, \\ b_2 = q^{-2}\ \frac{q(\rho-1)(2\tau+2\alpha+1)+\rho(q-1)}{\mho}, \\ q\xi_q = -\frac{\rho}{\mho}, \\ r_0 = \frac{\rho(2\tau+2\alpha+3)}{\mho}. \end{cases} \tag{120}$$

Now, choosing A in (63)-(65) such that $A^2 = -\frac{c_0}{q\xi_q}$ and thanks to (61), (98) and (117)-(120), we are led to the following situation

$$\begin{cases} \widehat{\gamma_1} = \frac{(\tau+1)(\tau+2\alpha+1)}{(2\tau+2\alpha+3)(2\tau+2\alpha+1)}\,\rho, \\ \widehat{\gamma}_{n+1} = q^{2n-1}\times \\ \frac{\left(q^{-n}[n]_q+\tau+1\right)\left([n]_q+\tau+2\alpha+1\right)+q^{-n}(q-1)[n]_q\left(2\alpha+1+(\tau+1)(2-[n]_q)\right)}{\left([2n-2]_q+2\tau+2\alpha+3\right)\left([2n]_q+2\tau+2\alpha+3\right)}, \quad n \geq 1, \\ H_q\left(\left(x^2 + \frac{\rho}{\mho}\right)\widehat{u}\right) + \frac{q^{-2}}{\mho}\left\{(1 - q + q^2)\rho + q(q + 1 - \rho)(2\tau + 2\alpha + 1)\right\}x\widehat{u} + \frac{q^{-2}}{\mho}\times \\ \quad \left\{\left(q(\rho - 1)(2\tau + 2\alpha + 1) + \rho(q - 1)\right)x^2 + \rho q^2\left(1 - \rho\,\frac{(\tau+1)(\tau+2\alpha+1)}{(2\tau+2\alpha+1)}\right)\right\}\left(x^{-1}\widehat{u}\left(h_q\widehat{u}\right)\right) = 0. \end{cases} \tag{121}$$

The form $\widehat{u}$ is regular, if and only if,

$$\begin{cases} \rho \neq 0,\ \tau \neq -1,\ \tau + 2\alpha + 1 \neq 0,\ 2\tau + 2\alpha + 1 \neq 0,\ 2\tau + 2\alpha + 3 \neq -[2n - 2]_q,\ n \geq 1, \\ \left(q^{-n}[n]_q + \tau + 1\right)\left([n]_q + \tau + 2\alpha + 1\right) + q^{-n}(q - 1)[n]_q\left(2\alpha + 1 + (\tau + 1)(2 - [n]_q)\right) \neq 0,\ n \geq 1. \end{cases}$$

When $q \to 1$ in (121), we meet the general situation of the nonsingular symmetrical D-Laguerre-Hahn of class 0 of Jacobi type [9] (see Table C. 2. in the Appendix).

Acknowledgements

The authors are very grateful to the referees for the constructive and valuable comments and recommendations.

References

[1] J. Alaya, P. Maroni, Symmetric Laguerre-Hahn forms of class $s = 1$, Integral Transforms and Spc. Funct., 2 (1996), 301-320.
[2] J. Alaya, P. Maroni, Some semi-classical and Laguerre-Hahn forms defined by pseudo-functions, Methods Appl. Anal., 3 (1) (1996), 12-30.
[3] G. Bangerezako, The Fourth Order Difference Equation for the Laguerre-Hahn Polynomials Orthogonal on Special Non-uniform Lattices, Ramanujan J., 5 (2) (2001), 167-181.
[4] K. Ben Gharbi, L. Khériji, M. Ihsen. Tounsi, Characterization of the symmetric D-Laguerre-Hahn orthogonal polynomial sequences of even class via the quadratic decomposition, J. Difference Equ. Appl., 24 (7) (2018), 1164-1184.
[5] K. Ben Gharbi, L. Khériji, On the Christoffel product of a D-Laguerre–Hahn form by a polynomial of one degree, Afr. Mat., 32 (2021), 1147-1157.
[6] B. Ben Hadj Ali, M. Mejri, Division Problem of the Perturbed Form of Order One of the Chebyshev Form of Second Kind, Filomat., 36 (10) (2022), 3491–3504.
[7] A. Branquinho, M.N. Rebocho, Characterization theorem for Laguerre–Hahn orthogonal polynomials on non-uniform lattices, J. Math. Anal. Appl., 427 (2015), 185–201.
[8] H. Bouakkaz, P. Maroni, Description des polynômes de Laguerre-Hahn de classe zero, Ann. Comp. Appl. Math., 9 (1991), 189-194.
[9] H. Bouakkaz. Les polynômes orthogonaux de Laguerre-Hahn de classe zéro, Thèse de doctorat, Univ. Pierre et Marie Curie, Paris (1990).
[10] K. Castillo, M. N. de Jesus, F. Marcellán, J. Petronilho, H_q-semiclassical orthogonal polynomials via polynomial mappings, Ramanujan J., 54 (1) (2021), 113-136.
[11] T. S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
[12] H. Dueñas, F. Marcellán, E. Prianes, Perturbations of Laguerre-Hahn functional. Modification by the derivative of a Dirac delta, Integral Transforms Spec. Funct., 20 (2009), 59-77.
[13] H. Dueñas, L. E. Garza, Perturbations of Laguerre-Hahn class linear functionals by Dirac delta derivatives, Bol. Mat., 19 (1) (2012), 65-90.
[14] M. Foupouagnigni, A. Ronveaux, Difference equation for the co-recursive rth associated orthogonal polynomials of the D_q-Laguerre-Hahn class, J. Comput. Appl. Math., 153 (2003), 213-223.
[15] M. Foupouagnigni, A. Ronveaux, W. Koepf, Fourth order q-difference equation for the first associated of the q-classical orthogonal polynomials, J. Comp. Appl. Math., 101 (1999), 231-236.
[16] A. Ghressi , L. Khériji, The symmetrical H_q-semiclassical orthogonal polynomials of class one, SIGMA 5 (2009), 076, 22 pages.
[17] A. Ghressi, L. Khériji, M. Ihsen Tounsi, An introduction to the q-Laguerre-Hahn orthogonal q-polynomials, SIGMA 7 (2011), 092, 20 pages.
[18] A. Ghressi, L. Khériji, Integral and discrete representations of certain Brenke type orthogonal q-polynomials. Ramanujan J., 30 (2) (2013), 163-171.
[19] S. Jbeli, L. Khériji, On some perturbed q-Laguerre–Hahn orthogonal q-polynomials, Period Math Hung., (2022). https://doi.org/10.1007/s10998-022-00463-9.
[20] L. Khériji, P. Maroni, The H_q- Classical Orthogonal Polynomials, Acta. Appl. Math., 71 (1) (2002), 49-115.
[21] L. Khériji, An introduction to the H_q-semiclassical orthogonal polynomials, Methods Appl. Anal., 10 (3) (2003), 387-411.
[22] L. Khériji, P. Maroni. On the natural q-analogues of the classical orthogonal polynomials, Eurasian Math. J., 4 (2) (2013), 82-103.
[23] A. Magnus, Riccati acceleration of Jacobi continued fractions and Laguerre-Hahn orthogonal polynomials. In padé approximation and its applications, Bad Honnef 1983 (Bad Honnef, 1983), 213-230, Lecture Notes in Math., 1071, Springer, Berlin, 1984.
[24] P. Maroni, Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques. In Orthogonal polynomials and their applications (*Erice, 1990*), C. Brezinski et al Editors, 95-130, IMACS Ann. Comput. Appl. Math., 9, Baltzer, Basel, 1991.
[25] M. Mejri, q-Chebyshev polynomials and their q-classical characters, , Probl. Anal. Issues Anal., Vol. 11 (29) No 1 (2022), 81–101.
[26] M. Sghaier, M. Zaatra and A. Khlifi, Laguerre-Freud equations associated with the D-Laguerre-Hahn forms of class one, Adv. Pure Appl. Math., 10 (4) (2019), 395-411.
[27] G. Yoon, Darboux transforms and orthogonal polynomials, Bull. Korean Math. Soc., 39 (2002), 359-376.
[28] M. Zaatra, Laguerre-Freud equations associated with the H_q-semi-classical forms of class one, Filomat., 32 (19) (2018), 6769-6787.
[29] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math., 85 (1997), 67-86.

Appendix

Table A.1.
The singular symmetrical D-Laguerre-Hahn of class 0 of Hermite type ($\rho \neq 0$) [9]

$$\gamma_1 = \frac{\rho}{2}, \quad \gamma_{n+1} = \frac{n}{2},\ n \geq 1.$$
$$u' - 2xu + (2x^2 + 1 - \rho)(x^{-1}u^2) = 0.$$

Table A.2.
The nonsingular symmetrical D-Laguerre-Hahn of class 0 of Hermite type($\rho \neq 0$, $\tau \neq -n$, $n \geq 1$) [9]

$$\gamma_1 = \rho\,\frac{1+\tau}{2}, \quad \gamma_{n+1} = \frac{n+\tau+1}{2},\ n \geq 1.$$
$$u' + 2\,\frac{2-\rho}{\rho}\,xu + \Big(2\,\frac{\rho-1}{\rho}\,x^2 + 1 - \rho(1+\tau)\Big)(x^{-1}u^2) = 0.$$

Table B.1.
The nonsingular symmetrical D-Laguerre-Hahn of class 0 of Bessel type ($\tau \neq -1 - \frac{n}{2}$, $n \geq -2$, $\tau \neq -n-1$, $n \geq 0$) [9]

$$\gamma_1 = -\frac{\rho}{(2\tau+1)(2\tau+3)}, \quad \gamma_{n+1} = -\frac{1}{(2n+2\tau+1)(2n+2\tau+3)},\ n \geq 1.$$
$$(x^2u)' + 2\Big((\tau+1)(1-\frac{2}{\rho}) + \frac{1}{\rho} - 1\Big)xu + \Big((\frac{1}{\rho}-1)(2\tau+1)\,x^2 - \frac{\rho}{2\tau+1}\Big)(x^{-1}u^2) = 0.$$

Table C.1.
The singular symmetrical D-Laguerre-Hahn of class 0 of Jacobi type ($\rho \neq 0$, $\alpha \neq -n$, $n \geq 1$) [9]

$$\gamma_1 = \rho, \quad \gamma_{n+1} = \frac{n(n+\alpha)}{(2n+\alpha-1)(2n+\alpha+1)},\ n \geq 1.$$
$$\Big((x^2-1)u\Big)' + (\alpha-2)\,xu + \Big((1-\alpha)\,x^2 + \rho\alpha + \rho - 1\Big)(x^{-1}u^2) = 0.$$

Table C.2.
The nonsingular symmetrical D-Laguerre-Hahn of class 0 of Jacobi type ($\rho \neq 0, \tau \neq -n-1,\ \tau + 2\alpha \neq -n-1,\ 2\tau + 2\alpha \neq -2n-1,\quad n \geq 0$) [9]

$$\gamma_1 = \rho\,\frac{(\tau+1)(\tau+2\alpha+1)}{(2\tau+2\alpha+1)(2\tau+2\alpha+3)},$$
$$\gamma_{n+1} = \frac{(n+\tau+1)(n+\tau+2\alpha+1)}{(2n+2\tau+2\alpha+1)(2n+2\tau+2\alpha+3)},\ n \geq 1.$$
$$\Big((x^2-1)u\Big)' + 2\Big(2(1-\frac{2}{\rho})(\tau+\alpha) - \frac{2}{\rho}\Big)xu +$$
$$\Big((\frac{1}{\rho}-1)(2\tau+2\alpha+1)\,x^2 - 1 + \rho\,\frac{(\tau+1)(\tau+2\alpha+1)}{2\tau+2\alpha+1}\Big)(x^{-1}u^2) = 0.$$