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$\frac{1}{2}$-superderivation and transposed Poisson structures on the super Heisenberg-Virasoro algebra

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Abstract. We describe transposed Poisson structures on the super Heisenberg-Virasoro algebra. We show that the super Heisenberg-Virasoro algebra does not admit non-trivial $\frac{1}{2}$-superderivations, and consequently it does not possess non-trivial transposed Poisson structures.

1. Introduction

Poisson algebras originated from the study of Poisson geometry in the 1970s and have found applications in an extremely wide range of areas in mathematics and physics, such as Poisson manifolds, algebraic geometry, operads, quantization theory, quantum groups, and classical and quantum mechanics. The study of Poisson algebras also gave rise to other related algebraic structures, such as noncommutative Poisson algebras [18], generic Poisson algebras [14], Poisson bialgebras[16], etc. Recently, a dual notion of the Poisson algebra (transposed Poisson algebra), has been introduced in the paper [2] of Bai, Bai, Guo and Wu.

More recently, relations between $\frac{1}{2}$-derivations ($\frac{1}{2}$-biderivations) of Lie algebras and transposed Poisson algebras have been established [8, 20]. These ideas were used to describe all transposed Poisson structures on the Witt algebra which is one of the first examples of non-trivial transposed Poisson algebras [8], the Virasoro algebra [8], the algebra $\mathcal{W}(a, b)$ [8], the thin Lie algebra [8], super Virasoro algebra [8], $N = 2$ superconformal algebra [8], the twisted Heisenberg-Virasoro algebra [20], the Schrödinger-Virasoro algebra [20], the extended Schrödinger-Virasoro algebra [20], the 3-dimensional Heisenberg Lie algebra [20], Block Lie algebras and superalgebras [9], Witt type algebras [12], oscillator Lie algebras [3], Galilean and solvable Lie algebras [13], generalized Witt algebras and Block Lie algebras [11], Schrödinger algebra in $(n + 1)$-dimensional space-time [19] and the Lie algebra of upper triangular matrices [10]. A list of actual open questions on transposed Poisson algebras was given in [3].

Throughout this paper, we denote by $\mathbb{C}$ and $\mathbb{Z}$ the sets of complex numbers and integers, respectively. As an important infinite-dimensional Lie algebra, the twisted Heisenberg-Virasoro algebra $\mathcal{HV}$ is the

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universal central extension of the Lie algebra

$$\overline{\mathcal{HV}} := \left\{ f(t)\frac{d}{dt} + g(t) \mid f(t), g(t) \in \mathbb{C}[t, t^{-1}] \right\}$$

of differential operators of order at most one, which was studied in reference [1]. The structure and representation theories of the twisted Heisenberg-Virasoro algebra and its various extended Lie algebras have been extensively investigated (see, e.g. [4, 6, 7, 17]). Recently, transposed Poisson structures on the super Virasoro algebra were researched in [8], which inspires us to study the super case of the twisted Heisenberg-Virasoro algebra.

Now, let us recall the definition of the super Heisenberg-Virasoro algebra given by [5, 15].

Definition 1.1. *The super Heisenberg-Virasoro algebra $s\mathcal{HV}$ is an infinite-dimensional Lie superalgebra generated by even elements $\{L_m, I_m, c\}_{m\in\mathbb{Z}}$ and odd elements $\{G_r\}_{r\in i+\mathbb{Z}}$, where $i = 0$ (the Ramond case), or $i = \frac{1}{2}$ (the Neveu-Schwarz case). In both cases, c is central in the superalgebra, and super-brackets are given by*

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c, \tag{1}$$

$$[L_m, I_n] = nI_{m+n}, \tag{2}$$

$$[L_m, G_r] = rG_{m+r}, \quad [G_r, G_s] = 2I_{r+s}, \quad [I_m, I_n] = [I_m, G_r] = 0,$$

for $m, n \in \mathbb{Z}, r, s \in i + \mathbb{Z}$.

By the definition, we have the following decomposition:

$$s\mathcal{HV} = s\mathcal{HV}_{\bar{0}} \bigoplus s\mathcal{HV}_{\bar{1}},$$

where $s\mathcal{HV}_{\bar{0}} = span_{\mathbb{C}}\{L_m, I_m, c \mid m \in \mathbb{Z}\}$, $s\mathcal{HV}_{\bar{1}} = span_{\mathbb{C}}\{G_r \mid r \in i + \mathbb{Z}\}$. Notice that the even part $s\mathcal{HV}_{\bar{0}}$ is isomorphic to the twisted Heisenberg-Virasoro algebra $\mathcal{HV}$ with some trivial center elements. Recall that a Lie superalgebra $\mathcal{L}$ is *perfect* if $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$. Note that the super Heisenberg-Virasoro algebra is perfect, which can be easily checked using the above definition.

In this paper, we will study $\frac{1}{2}$-superderivations of the super Heisenberg-Virasoro algebra. We find that there are no non-trivial transposed Poisson structures defined on the super Heisenberg-Virasoro algebra.

2. Preliminaries

In this section, we recall some definitions and known results for studying transposed Poisson structures. Although all algebras and vector spaces are considered over the complex field, many results can be proven over other fields without modification of the proofs.

Definition 2.1. *(see [21]) Let $\mathcal{L}$ be a superalgebra and δ an element of the ground field. A homogeneous endomorphism ψ of the superspace of endomorphisms is called a δ-superderivation if*

$$\psi([x, y]) = \delta([\psi(x), y] + (-1)^{deg(\psi)deg(x)}[x, \psi(y)]).$$

The main example of $\frac{1}{2}$-superderivations is the multiplication by an element from the ground field. Let us call such $\frac{1}{2}$-superderivations as *trivial $\frac{1}{2}$-superderivations*.

Lemma 2.2. *Let ψ_1 and ψ_2 be δ_1- and δ_2-superderivations of a superalgebra $\mathcal{L}$, respectively. Then the supercommutator*

$$[\![\psi_1, \psi_2]\!]_s = \psi_1\psi_2 - (-1)^{deg(\psi_1)deg(\psi_2)}\psi_2\psi_1$$

is a $\delta_1\delta_2$-superderivation. Similarly, the commutator $[\![\psi_1, \psi_2]\!]$ of δ_1- and δ_2-derivations of an algebra is a $\delta_1\delta_2$-derivation.

Proof. For arbitrary $x, y \in \mathcal{L}$, we have

$$\begin{aligned}
[\![\psi_1, \psi_2]\!]_s([x, y]) &= (\psi_1\psi_2 - (-1)^{deg(\psi_1)deg(\psi_2)}\psi_2\psi_1)([x, y]) \\
&= \psi_1\psi_2([x, y]) - (-1)^{deg(\psi_1)deg(\psi_2)}\psi_2\psi_1([x, y]) \\
&= \delta_2\psi_1([\psi_2(x), y] + (-1)^{deg(x)deg(\psi_2)}[x, \psi_2(y)]) \\
&\quad - (-1)^{deg(\psi_1)deg(\psi_2)}\delta_1\psi_2([\psi_1(x), y] + (-1)^{deg(x)deg(\psi_1)}[x, \psi_1(y)]) \\
&= \delta_2\psi_1([\psi_2(x), y]) + \delta_2(-1)^{deg(x)deg(\psi_2)}\psi_1([x, \psi_2(y)]) \\
&\quad - \delta_1(-1)^{deg(\psi_1)deg(\psi_2)}\psi_2([\psi_1(x), y]) \\
&\quad - \delta_1(-1)^{deg(\psi_1)(deg(\psi_2)+deg(x))}\psi_2([x, \psi_1(y)]) \\
&= \delta_1\delta_2([\psi_1\psi_2(x), y] + (-1)^{deg(\psi_1)deg(\psi_2(x))}[\psi_2(x), \psi_1(y)]) \\
&\quad + \delta_1\delta_2(-1)^{deg(x)deg(\psi_2)}([\psi_1(x), \psi_2(y)] + (-1)^{deg(x)deg(\psi_1)}[x, \psi_1\psi_2(y)]) \\
&\quad - \delta_1\delta_2(-1)^{deg(\psi_1)deg(\psi_2)}([\psi_2\psi_1(x), y] \\
&\quad + (-1)^{deg(\psi_1(x))deg(\psi_2)}[\psi_1(x), \psi_2(y)]) \\
&\quad - \delta_1\delta_2(-1)^{deg(\psi_1)(deg(\psi_2)+deg(x))}([\psi_2(x), \psi_1(y)] \\
&\quad + (-1)^{deg(x)deg(\psi_2)}[x, \psi_2\psi_1(y)]) \\
&= \delta_1\delta_2([\psi_1\psi_2(x), y] + (-1)^{deg(\psi_1)(deg(\psi_2)+deg(x))}[\psi_2(x), \psi_1(y)] \\
&\quad + (-1)^{deg(x)deg(\psi_2)}[\psi_1(x), \psi_2(y)] \\
&\quad + (-1)^{deg(x)(deg(\psi_1)+deg(\psi_2))}[x, \psi_1\psi_2(y)] - (-1)^{deg(\psi_1)deg(\psi_2)}[\psi_2\psi_1(x), y] \\
&\quad - (-1)^{deg(x)deg(\psi_2)}[\psi_1(x), \psi_2(y)] \\
&\quad - (-1)^{deg(\psi_1)(deg(\psi_2)+deg(x))}[\psi_2(x), \psi_1(y)] \\
&\quad - (-1)^{deg(\psi_1)deg(\psi_2)+deg(x)(deg(\psi_1)+deg(\psi_2))}[x, \psi_2\psi_1(y)]) \\
&= \delta_1\delta_2([(\psi_1\psi_2 - (-1)^{deg(\psi_1)deg(\psi_2)}\psi_2\psi_1)(x), y] \\
&\quad + (-1)^{deg(x)(deg(\psi_1)+deg(\psi_2))}[x, (\psi_1\psi_2 - (-1)^{deg(\psi_1)deg(\psi_2)}\psi_2\psi_1)(y)]) \\
&= \delta_1\delta_2([[\![\psi_1, \psi_2]\!]_s(x), y] + (-1)^{deg(x)deg([\![\psi_1,\psi_2]\!]_s)}[x, [\![\psi_1, \psi_2]\!]_s(y)]).
\end{aligned}$$

By Definition 2.1, we know $[\![\psi_1, \psi_2]\!]_s$ is a $\delta_1\delta_2$-superderivation. And by similar calculations, we can obtain that the commutator $[\![\psi_1, \psi_2]\!]$ of δ_1- and δ_2-derivations of an algebra is a $\delta_1\delta_2$-derivation. □

Transposed Poisson algebras were first introduced by Bai, Bai, Guo and Wu in [2].

Definition 2.3. *Let $\mathcal{L}$ be a vector space equipped with two nonzero bilinear operations $\cdot$ and $[\cdot,\cdot]$. The triple $(\mathcal{L}, \cdot, [\cdot,\cdot])$ is called a transposed Poisson algebra if $(\mathcal{L}, \cdot)$ is a commutative associative algebra and $(\mathcal{L}, [\cdot,\cdot])$ is a Lie algebra that satisfies the following compatibility condition*

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y].$$

One naturally defines a transposed Poisson superalgebra as a superization of the notion of a transposed Poisson algebra.

Definition 2.4. *Let $\mathcal{L} = \mathcal{L}_{\bar{0}} \bigoplus \mathcal{L}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space equipped with two nonzero bilinear super-operations $\cdot$ and $[\cdot,\cdot]$. The triple $(\mathcal{L}, \cdot, [\cdot,\cdot])$ is called a transposed Poisson superalgebra if $(\mathcal{L}, \cdot)$ is a supercommutative associative algebra and $(\mathcal{L}, [\cdot,\cdot])$ is a Lie superalgebra that satisfies the following compatibility condition*

$$2z \cdot [x, y] = [z \cdot x, y] + (-1)^{deg(x)deg(z)}[x, z \cdot y],\ x, y, z \in \mathcal{L}_{\bar{0}} \cup \mathcal{L}_{\bar{1}}.$$

Definition 2.5. *Let $(\mathcal{L}, [\cdot,\cdot])$ be a Lie superalgebra. A transposed Poisson superalgebra structure on $(\mathcal{L}, [\cdot,\cdot])$ is a supercommutative associative multiplication $\cdot$ on $\mathcal{L}$ which makes $(\mathcal{L}, \cdot, [\cdot,\cdot])$ a transposed Poisson superalgebra.*

It is easy to see that Definitions 2.1 and 2.3 imply the following key lemma.

Lemma 2.6. *(see* [8]*) Let* $(\mathcal{L}, \cdot, [\cdot, \cdot])$ *be a transposed Poisson algebra and* z *an arbitrary element from* $\mathcal{L}$*. Then the left multiplication* l_z *in the associative commutative algebra* $(\mathcal{L}, \cdot)$ *gives a* $\frac{1}{2}$*-derivation of the Lie algebra* $(\mathcal{L}, [\cdot, \cdot])$*.*

The super case is the following lemma.

Lemma 2.7. *Let* $(\mathcal{L}, \cdot, [\cdot, \cdot])$ *be a transposed Poisson superalgebra and* $z \in \mathcal{L}_{\bar{0}} \cup \mathcal{L}_{\bar{1}}$*. Then the left multiplication* l_z *in the supercommutative associative algebra* $(\mathcal{L}, \cdot)$ *gives a* $\frac{1}{2}$*-superderivation of the Lie superalgebra* $(\mathcal{L}, [\cdot, \cdot])$ *and* $deg(l_z) = deg(z)$*.*

Proof. For arbitrary $x, y \in \mathcal{L}_{\bar{0}} \cup \mathcal{L}_{\bar{1}}$, we have

$$\begin{aligned} l_z([x, y]) &= z \cdot [x, y] \\ &= \tfrac{1}{2}([z \cdot x, y] + (-1)^{deg(x)deg(z)}[x, z \cdot y]) \\ &= \tfrac{1}{2}([z \cdot x, y] + (-1)^{deg(x)deg(l_z)}[x, z \cdot y]). \end{aligned}$$

By Definition 2.1, we know l_z is a $\frac{1}{2}$-superderivation of the Lie superalgebra $(\mathcal{L}, [\cdot, \cdot])$. □

By Lemmas 2.6 and 2.7, it is easy to prove the following lemma.

Lemma 2.8. *(see* [8]*) Let* $\mathcal{L}$ *be a Lie algebra (or superalgebra) of dimension* > 1 *without non-trivial* $\frac{1}{2}$*-derivations. Then every transposed Poisson structure defined on* $\mathcal{L}$ *is trivial.*

3. TP-structures on the super Heisenberg-Virasoro algebra

In this section, we describe transposed Poisson superalgebra structures on the super Heisenberg-Virasoro algebra $s\mathcal{HV}$. To obtain this result, we first have to prove a few lemmas.

Set $(s\mathcal{HV}_{\bar{0}})_j = span_{\mathbb{C}}\{L_j, I_j\}$ for $0 \neq j \in \mathbb{Z}$ and $(s\mathcal{HV}_{\bar{0}})_0 = span_{\mathbb{C}}\{L_0, I_0, c\}$. Then $s\mathcal{HV}_{\bar{0}} = \bigoplus_{j\in\mathbb{Z}}(s\mathcal{HV}_{\bar{0}})_j$ is a $\mathbb{Z}$-graded algebra. It is easy to see that $s\mathcal{HV}_{\bar{0}}$ is finitely generated. Let φ be a $\frac{1}{2}$-derivation on $s\mathcal{HV}_{\bar{0}}$. Then, by Lemma 2.4 in [19], the $\mathbb{Z}$-grading of $s\mathcal{HV}_{\bar{0}}$ induces the decomposition

$$\varphi = \sum_{j\in\mathbb{Z}} \varphi_j,$$

where φ_j is a $\frac{1}{2}$-derivation on $s\mathcal{HV}_{\bar{0}}$ of degree j, i.e., $\varphi_j((s\mathcal{HV}_{\bar{0}})_k) \subseteq (s\mathcal{HV}_{\bar{0}})_{j+k}, \forall k \in \mathbb{Z}$.

Lemma 3.1. *Every* $\frac{1}{2}$*-derivation on* $s\mathcal{HV}_{\bar{0}}$ *is trivial.*

Proof. Let φ_j be a $\frac{1}{2}$-derivation on $s\mathcal{HV}_{\bar{0}}$ of degree j. We can assume that

$$\varphi_j(L_m) = \alpha_{j,m}L_{j+m} + \alpha'_{j,m}I_{j+m} + \delta_{j+m,0}a_{j,m}c, \tag{3}$$

$$\varphi_j(I_m) = \beta_{j,m}L_{j+m} + \beta'_{j,m}I_{j+m} + \delta_{j+m,0}b_{j,m}c, \tag{4}$$

$$\varphi_j(c) = \gamma_j L_j + \gamma'_j I_j + \delta_{j,0}d_j c,$$

where the coefficients are elements in $\mathbb{C}$. Firstly, we apply φ_j to both sides of the relation $[L_m, c] = 0$ and we obtain

$$\begin{aligned} 0 &= 2\varphi_j([L_m, c]) = [\varphi_j(L_m), c] + [L_m, \varphi_j(c)] \\ &= [L_m, \varphi_j(c)] = [L_m, \gamma_j L_j + \gamma'_j I_j + \delta_{j,0}d_j c] \\ &= (j - m)\gamma_j L_{j+m} + j\gamma'_j I_{j+m} + \tfrac{m^3-m}{12}\gamma_j\delta_{j+m,0}c, \end{aligned}$$

which gives

$$(j-m)\gamma_j = 0, \quad j\gamma'_j = 0.$$

Choosing $j \neq m$, we have $\gamma_j = 0$ for all $j \in \mathbb{Z}$. In addition, one has $\gamma'_j = 0$ for $0 \neq j \in \mathbb{Z}$. It follows that

$$\varphi_j(c) = 0,\ j \neq 0. \tag{5}$$

$$\varphi_0(c) = \gamma'_0 I_0 + d_0 c. \tag{6}$$

To compute the other coefficients, we have two cases to consider.

Case 1. $j \neq 0$

Applying φ_j to both sides of (1) and by (3) and (5), we can get

$$\begin{aligned} 2\varphi_j([L_m, L_n]) &= 2(n-m)\varphi_j(L_{m+n}) + \frac{m^3-m}{6}\delta_{m+n,0}\varphi_j(c) = 2(n-m)\varphi_j(L_{m+n}) \\ &= 2(n-m)\alpha_{j,m+n}L_{j+m+n} + 2(n-m)\alpha'_{j,m+n}I_{j+m+n} \\ &\quad + 2(n-m)\delta_{j+m+n,0}a_{j,m+n}c. \end{aligned} \tag{7}$$

On the other hand, we have

$$\begin{aligned} 2\varphi_j([L_m, L_n]) &= [\varphi_j(L_m), L_n] + [L_m, \varphi_j(L_n)] \\ &= [\alpha_{j,m}L_{j+m} + \alpha'_{j,m}I_{j+m} + \delta_{j+m,0}a_{j,m}c, L_n] \\ &\quad + [L_m, \alpha_{j,n}L_{j+n} + \alpha'_{j,n}I_{j+n} + \delta_{j+n,0}a_{j,n}c] \\ &= ((n-j-m)\alpha_{j,m} + (j+n-m)\alpha_{j,n})L_{j+m+n} \\ &\quad + ((j+n)\alpha'_{j,n} - (j+m)\alpha'_{j,m})I_{j+m+n} \\ &\quad + \delta_{j+m+n,0}\left(\frac{(j+m)^3-(j+m)}{12}\alpha_{j,m} + \frac{m^3-m}{12}\alpha_{j,n}\right)c. \end{aligned} \tag{8}$$

Comparing (7) with (8), we have

$$2(n-m)\alpha_{j,m+n} = (n-j-m)\alpha_{j,m} + (j+n-m)\alpha_{j,n}, \tag{9}$$

$$2(n-m)\alpha'_{j,m+n} = (j+n)\alpha'_{j,n} - (j+m)\alpha'_{j,m}, \tag{10}$$

and

$$2(n-m)\delta_{j+m+n,0}a_{j,m+n} = \delta_{j+m+n,0}\left(\frac{(j+m)^3-(j+m)}{12}\alpha_{j,m} + \frac{m^3-m}{12}\alpha_{j,n}\right). \tag{11}$$

Applying φ_j to both sides of (2) and by (4), we obtain

$$\begin{aligned} 2\varphi_j([L_m, I_n]) &= 2n\varphi_j(I_{m+n}) \\ &= 2n\beta_{j,m+n}L_{j+m+n} + 2n\beta'_{j,m+n}I_{j+m+n} + 2n\delta_{j+m+n,0}b_{j,m+n}c. \end{aligned} \tag{12}$$

On the other hand, we have

$$\begin{aligned} 2\varphi_j([L_m, I_n]) &= [\varphi_j(L_m), I_n] + [L_m, \varphi_j(I_n)] \\ &= [\alpha_{j,m}L_{j+m} + \alpha'_{j,m}I_{j+m} + \delta_{j+m,0}a_{j,m}c, I_n] \\ &\quad + [L_m, \beta_{j,n}L_{j+n} + \beta'_{j,n}I_{j+n} + \delta_{j+n,0}b_{j,n}c] \\ &= (j+n-m)\beta_{j,n}L_{j+m+n} + (n\alpha_{j,m} + (j+n)\beta'_{j,n})I_{j+m+n} \\ &\quad + \frac{m^3-m}{12}\delta_{j+m+n,0}\beta_{j,n}c. \end{aligned} \tag{13}$$

Comparing (12) with (13), we have

$$2n\beta_{j,m+n} = (j + n - m)\beta_{j,n}, \tag{14}$$

$$2n\beta'_{j,m+n} = n\alpha_{j,m} + (j + n)\beta'_{j,n}, \tag{15}$$

and

$$2n\delta_{j+m+n,0}b_{j,m+n} = \frac{m^3 - m}{12}\delta_{j+m+n,0}\beta_{j,n}. \tag{16}$$

Firstly, we see (9), (10), (14) and (15). By taking $n = 0$ in (9), then we have $(j - m)\alpha_{j,m} = (j - m)\alpha_{j,0}$. Thus

$$\alpha_{j,m} = \alpha_{j,0},\ m \neq j. \tag{17}$$

Furthermore, by letting $n = -m = j$ in (9) we have $4j\alpha_{j,0} = j\alpha_{j,-j} + 3j\alpha_{j,j}$. According to (17), we have $\alpha_{j,-j} = \alpha_{j,0}$. It follows that $\alpha_{j,j} = \alpha_{j,0}$. This, together with (17), gives

$$\alpha_{j,n} = \alpha_{j,0},\ \forall n \in \mathbb{Z}. \tag{18}$$

Letting $n = 0$ in (10), we have $(j - m)\alpha'_{j,m} = j\alpha'_{j,0}$, which yields

$$\alpha'_{j,m} = \frac{j}{j - m}\alpha'_{j,0},\ m \neq j. \tag{19}$$

Letting $m = -n$ with $n \notin \{j, -j, 0\}$ in (10), we can get

$$4n\alpha'_{j,0} = (j + n)\alpha'_{j,n} - (j - n)\alpha'_{j,-n}. \tag{20}$$

By (19), we know

$$\alpha'_{j,n} = \tfrac{j}{j-n}\alpha'_{j,0}, \quad \alpha'_{j,-n} = \tfrac{j}{j+n}\alpha'_{j,0}.$$

Substituting this into (20), we can get $4n\alpha'_{j,0} = \frac{4nj^2}{j^2-n^2}\alpha'_{j,0}$, which implies

$$\alpha'_{j,0} = 0. \tag{21}$$

Substituting this into (19), we can get $\alpha'_{j,m} = 0,\ m \neq j$. Letting $n = -m = j$ in (10) and by (21), we obtain $\alpha'_{j,j} = 0$. Hence

$$\alpha'_{j,n} = 0,\ \forall n \in \mathbb{Z}. \tag{22}$$

Setting $m = 0$ in (14), we have $(n - j)\beta_{j,n} = 0$, which implies $\beta_{j,n} = 0,\ n \neq j$. With this, by taking $n = -m = j$ in (14) one has $\beta_{j,j} = 0$. This proves

$$\beta_{j,n} = 0,\ \forall n \in \mathbb{Z}. \tag{23}$$

Setting $n = 0$ in (15), we obtain $\beta'_{j,0} = 0$. With this, taking $m = -n$ in (15) gives $0 = n\alpha_{j,-n} + (j + n)\beta'_{j,n}$. This, together with (18) gives

$$(j + n)\beta'_{j,n} = -n\alpha_{j,0}. \tag{24}$$

Letting $m = 0$ in (15), we can get $(n - j)\beta'_{j,n} = n\alpha_{j,0}$. Combining with (24), we have $\beta'_{j,n} = 0,\ n \neq 0$. Hence

$$\beta'_{j,n} = 0,\ \forall n \in \mathbb{Z}. \tag{25}$$

With this, taking $n = 1$ in (24) gives $\alpha_{j,0} = 0$. Substituting this into (18), we can get

$$\alpha_{j,n} = 0,\ \forall n \in \mathbb{Z}. \tag{26}$$

Now, taking $n = 0$ and $m = -j$ in (11) and by (26), we obtain

$$a_{j,-j} = 0. \tag{27}$$

Letting $n = 1$ and $m = -1 - j$ in (16) and by (23), we have

$$b_{j,-j} = 0. \tag{28}$$

By (26), (22), (23), (25), (27), (28) and (5), we can get

$$\varphi_j = 0, j \neq 0.$$

Case 2. $j = 0$.
Applying φ_0 to both sides of (1) and by (3) and (6), we can get

$$\begin{aligned}2\varphi_0([L_m, L_n]) &= 2(n-m)\varphi_0(L_{m+n}) + \frac{m^3-m}{6}\delta_{m+n,0}\varphi_0(c)\\ &= 2(n-m)\alpha_{0,m+n}L_{m+n} + 2(n-m)\alpha'_{0,m+n}I_{m+n} + \frac{m^3-m}{6}\delta_{m+n,0}\gamma'_0 I_0\\ &\quad + \delta_{m+n,0}(2(n-m)a_{0,m+n} + \frac{m^3-m}{6}d_0)c.\end{aligned} \tag{29}$$

On the other hand, when $j = 0$, (8) takes the form of

$$\begin{aligned}2\varphi_0([L_m, L_n]) &= (n-m)(\alpha_{0,m} + \alpha_{0,n})L_{m+n} + (n\alpha'_{0,n} - m\alpha'_{0,m})I_{m+n}\\ &\quad + \frac{m^3-m}{12}\delta_{m+n,0}(\alpha_{0,m} + \alpha_{0,n})c.\end{aligned} \tag{30}$$

Comparing (29) with (30), we have

$$2(n-m)\alpha_{0,m+n} = (n-m)(\alpha_{0,m} + \alpha_{0,n}), \tag{31}$$

$$2(n-m)\alpha'_{0,m+n} = n\alpha'_{0,n} - m\alpha'_{0,m},\ m + n \neq 0, \tag{32}$$

$$4m\alpha'_{0,0} - \frac{m^3-m}{6}\gamma'_0 = m(\alpha'_{0,-m} + \alpha'_{0,m}), \tag{33}$$

and

$$\delta_{m+n,0}(2(n-m)a_{0,m+n} + \frac{m^3-m}{6}d_0) = \delta_{m+n,0}\frac{m^3-m}{12}(\alpha_{0,m} + \alpha_{0,n}). \tag{34}$$

When $j = 0$, (14), (15) and (16) take the form of, respectively

$$2n\beta_{0,m+n} = (n-m)\beta_{0,n}, \tag{35}$$

$$2n\beta'_{0,m+n} = n(\alpha_{0,m} + \beta'_{0,n}), \tag{36}$$

and

$$2n\delta_{m+n,0}b_{0,m+n} = \frac{m^3-m}{12}\delta_{m+n,0}\beta_{0,n}. \tag{37}$$

Setting $n = 0$ in (31) and (32), it follows that respectively

$$\alpha_{0,m} = \alpha_{0,0}, \forall m \in \mathbb{Z}, \tag{38}$$

and

$$\alpha'_{0,m} = 0, m \neq 0. \tag{39}$$

According to (39) and taking $m = 1$ in (33), we obtain $\alpha'_{0,0} = 0$. This, together with (39), gives

$$\alpha'_{0,n} = 0, \forall n \in \mathbb{Z}. \tag{40}$$

According to (40) and by letting $m = 2$ in (33), we can get $\gamma'_0 = 0$. Taking $m = 0$ in (35) gives $\beta_{0,n} = 0$, $n \neq 0$. With this, letting $n = -m = 1$ in (35), we have $\beta_{0,0} = 0$. Hence $\beta_{0,n} = 0$, $\forall n \in \mathbb{Z}$. Taking $m = 0$ in (36), we have

$$\beta'_{0,n} = \alpha_{0,0}, n \neq 0. \tag{41}$$

By (38) and (41), letting $n = -m = 1$ in (36), we can get $\beta'_{0,0} = \alpha_{0,0}$. This, together with (41), gives $\beta'_{0,n} = \alpha_{0,0}$, $\forall n \in \mathbb{Z}$. In the following, we denote $\alpha_{0,0}$ as λ, i.e., $\alpha_{0,n} = \beta'_{0,n} = \lambda$, $\forall n \in \mathbb{Z}$. Taking $m = -n = 1$ in (34), we obtain $a_{0,0} = 0$. According to this and (38), by setting $m = -n = 2$ in (34) one has $d_0 = \lambda$.

Finally, it follows by setting $m = -n = 1$ in (37) that $b_{0,0} = 0$. As a conclusion, we obtain

$$\varphi_0 = \lambda Id_{s\mathcal{HV}_{\bar{0}}}.$$

Combining the two cases above, we get the desired result. □

Lemma 3.2. *Let $\psi_{\bar{0}}$ be an even $\frac{1}{2}$-superderivation of $s\mathcal{HV}$, then $\psi_{\bar{0}}$ is trivial.*

Proof. It is easy to see that the restriction $\psi_{\bar{0}} \mid_{s\mathcal{HV}_{\bar{0}}}$ is a $\frac{1}{2}$-derivation of $s\mathcal{HV}_{\bar{0}}$. By Lemma 3.1, we know $\psi_{\bar{0}} \mid_{s\mathcal{HV}_{\bar{0}}}$ is trivial. Hence, by subtracting a multiplication transformation we can suppose that $\psi_{\bar{0}} \mid_{s\mathcal{HV}_{\bar{0}}} = 0$. Next, we assume that

$$\psi_{\bar{0}}(G_r) = \sum_{t \in i+\mathbb{Z}} \Gamma_t^r G_t,$$

where $\Gamma_t^r \in \mathbb{C}$. Then we have

$$2\psi_{\bar{0}}([L_0, G_r]) = 2r\psi_{\bar{0}}(G_r) = 2r \sum_{t \in i+\mathbb{Z}} \Gamma_t^r G_t. \tag{42}$$

On the other hand, we have

$$\begin{aligned} 2\psi_{\bar{0}}([L_0, G_r]) &= [\psi_{\bar{0}}(L_0), G_r] + [L_0, \psi_{\bar{0}}(G_r)] = [L_0, \psi_{\bar{0}}(G_r)] \\ &= [L_0, \sum_{t \in i+\mathbb{Z}} \Gamma_t^r G_t] = \sum_{t \in i+\mathbb{Z}} t\Gamma_t^r G_t. \end{aligned} \tag{43}$$

Comparing (42) with (43), we obtain

$$2r \sum_{t \in i+\mathbb{Z}} \Gamma_t^r G_t = \sum_{t \in i+\mathbb{Z}} t\Gamma_t^r G_t,$$

which gives

$$\Gamma_t^r = 0, (t \neq 2r).$$

It follows that

$$\psi_{\bar{0}}(G_r) = 0, \forall r \in \frac{1}{2} + \mathbb{Z}, \tag{44}$$

and

$$\psi_{\bar{0}}(G_r) = \Gamma_{2r}^r G_{2r}, \ \forall r \in \mathbb{Z}.$$

Now, we need only to consider the case of $i = 0$.

$$\begin{aligned} 0 &= 4\psi_{\bar{0}}(I_{r+s}) = 2\psi_{\bar{0}}([G_r, G_s]) = [\psi_{\bar{0}}(G_r), G_s] + [G_r, \psi_{\bar{0}}(G_s)] \\ &= [\Gamma_{2r}^r G_{2r}, G_s] + [G_r, \Gamma_{2s}^s G_{2s}] \\ &= 2\Gamma_{2r}^r I_{2r+s} + 2\Gamma_{2s}^s I_{r+2s}. \end{aligned}$$

For $r \neq s$, we deduce

$$\Gamma_{2r}^r = 0, \ \forall r \in \mathbb{Z}.$$

This means

$$\psi_{\bar{0}}(G_r) = 0, \ \forall r \in \mathbb{Z}. \tag{45}$$

Summarizing (44) and (45), we have

$$\psi_{\bar{0}} = 0.$$

Hence, every even $\frac{1}{2}$-superderivation of $s\mathcal{HV}$ is trivial. □

Lemma 3.3. *Let $\psi_{\bar{1}}$ be an odd $\frac{1}{2}$-superderivation of $s\mathcal{HV}$, then $\psi_{\bar{1}} = 0$.*

Proof. Let ad_{G_r} be an inner odd superderivation of $s\mathcal{HV}$. Then, according to Lemma 2.2, we know that $[\![\psi_{\bar{1}}, ad_{G_r}]\!]_s$ is an even $\frac{1}{2}$-superderivation of $s\mathcal{HV}$. Furthermore, based on Lemma 3.2, we have that $[\![\psi_{\bar{1}}, ad_{G_r}]\!]_s$ is trivial, i.e., $[\![\psi_{\bar{1}}, ad_{G_r}]\!]_s = \beta_r Id_{s\mathcal{HV}}$, $\beta_r \in \mathbb{C}$ related to r. Next, we assume that

$$\psi_{\bar{1}}(c) = \sum_{t \in i+\mathbb{Z}} \alpha_t G_t,$$

where α_t are elements in $\mathbb{C}$. Whereupon we have

$$\begin{aligned} \beta_r c &= [\![\psi_{\bar{1}}, ad_{G_r}]\!]_s(c) = (\psi_{\bar{1}} ad_{G_r} + ad_{G_r}\psi_{\bar{1}})(c) = [G_r, \psi_{\bar{1}}(c)] \\ &= [G_r, \sum_{t \in i+\mathbb{Z}} \alpha_t G_t] = 2\sum_{t \in i+\mathbb{Z}} \alpha_t I_{r+t}, \end{aligned}$$

which gives

$$\alpha_t = 0, \ \forall t \in i + \mathbb{Z},$$

and

$$\beta_r = 0, \ \forall r \in i + \mathbb{Z}.$$

It follows that

$$\psi_{\bar{1}}(c) = 0, \tag{46}$$

and

$$[\![\psi_{\bar{1}}, ad_{G_r}]\!]_s = 0, \ \forall r \in i + \mathbb{Z}.$$

Hence, we see that

$$\begin{aligned} 0 &= 2[\![\psi_{\bar{1}}, ad_{G_r}]\!]_s(L_m) = 2(\psi_{\bar{1}} ad_{G_r} + ad_{G_r}\psi_{\bar{1}})(L_m) \\ &= 2\psi_{\bar{1}}([G_r, L_m]) + 2[G_r, \psi_{\bar{1}}(L_m)] \\ &= [\psi_{\bar{1}}(G_r), L_m] - [G_r, \psi_{\bar{1}}(L_m)] + 2[G_r, \psi_{\bar{1}}(L_m)] \\ &= [\psi_{\bar{1}}(G_r), L_m] + [G_r, \psi_{\bar{1}}(L_m)] \\ &= -[L_m, \psi_{\bar{1}}(G_r)] + [\psi_{\bar{1}}(L_m), G_r], \end{aligned}$$

which yields

$$[L_m, \psi_{\bar{1}}(G_r)] = [\psi_{\bar{1}}(L_m), G_r]. \tag{47}$$

Furthermore, one has

$$2\psi_{\bar{1}}([L_m, G_r]) = [\psi_{\bar{1}}(L_m), G_r] + [L_m, \psi_{\bar{1}}(G_r)]. \tag{48}$$

Substituting (47) into (48), we can get

$$\psi_{\bar{1}}([L_m, G_r]) = [\psi_{\bar{1}}(L_m), G_r] = [L_m, \psi_{\bar{1}}(G_r)]. \tag{49}$$

Afterwards, we can assume that

$$\psi_{\bar{1}}(G_r) = \sum_{k\in\mathbb{Z}} \theta_k^r L_k + \sum_{k\in\mathbb{Z}} \epsilon_k^r I_k + \sigma^r c,$$

where $\theta_k^r, \epsilon_k^r, \sigma^r$ are complex numbers. Whereupon we have

$$\begin{aligned} [L_0, \psi_{\bar{1}}(G_r)] &= [L_0, \sum_{k\in\mathbb{Z}} \theta_k^r L_k + \sum_{k\in\mathbb{Z}} \epsilon_k^r I_k + \sigma^r c] \\ &= \sum_{k\in\mathbb{Z}} \theta_k^r [L_0, L_k] + \sum_{k\in\mathbb{Z}} \epsilon_k^r [L_0, I_k] \\ &= \sum_{k\in\mathbb{Z}} k\theta_k^r L_k + \sum_{k\in\mathbb{Z}} k\epsilon_k^r I_k. \end{aligned} \tag{50}$$

On the other hand, we have

$$\begin{aligned} \psi_{\bar{1}}([L_0, G_r]) &= r\psi_{\bar{1}}(G_r) \\ &= r\sum_{k\in\mathbb{Z}} \theta_k^r L_k + r\sum_{k\in\mathbb{Z}} \epsilon_k^r I_k + r\sigma^r c. \end{aligned} \tag{51}$$

Comparing (50) with (51), we have

$$\theta_k^r = 0, \ (k \neq r), \tag{52}$$

$$\epsilon_k^r = 0, \ (k \neq r), \tag{53}$$

and

$$\sigma^r = 0, \ (r \neq 0). \tag{54}$$

By (52), (53) and (54), we can obtain

$$\psi_{\bar{1}}(G_r) = 0, \ \forall r \in \frac{1}{2} + \mathbb{Z}, \tag{55}$$

and

$$\psi_{\bar{1}}(G_r) = \theta_r^r L_r + \epsilon_r^r I_r, \ 0 \neq r \in \mathbb{Z}, \tag{56}$$

$$\psi_{\bar{1}}(G_0) = \theta_0^0 L_0 + \epsilon_0^0 I_0 + \sigma^0 c. \tag{57}$$

Hereafter, we can assume that

$$\psi_{\bar{1}}(L_m) = \sum_{t\in i+\mathbb{Z}} \omega_t^m G_t, \tag{58}$$

where $\omega_t^m \in \mathbb{C}$. Let us now discuss this scenario case by case.

Case 1. $i = \frac{1}{2}$, i.e., the Neveu-Schwarz case.

By (49), (55) and (58), we have

$$\begin{aligned} 0 &= r\psi_{\bar{1}}(G_{m+r}) = \psi_{\bar{1}}([L_m, G_r]) = [\psi_{\bar{1}}(L_m), G_r] \\ &= [\sum_{t\in\frac{1}{2}+\mathbb{Z}} \omega_t^m G_t, G_r] = 2\sum_{t\in\frac{1}{2}+\mathbb{Z}} \omega_t^m I_{t+r}, \end{aligned}$$

which gives

$$\omega_t^m = 0, \ \forall t \in \frac{1}{2} + \mathbb{Z}.$$

This means that

$$\psi_{\bar{1}}(L_m) = 0. \tag{59}$$

At last, we have

$$4\psi_{\bar{1}}(I_{r+s}) = 2\psi_{\bar{1}}([G_r, G_s]) = [\psi_{\bar{1}}(G_r), G_s] - [G_r, \psi_{\bar{1}}(G_s)] = 0. \tag{60}$$

Since r, s are arbitrary, we have

$$\psi_{\bar{1}}(I_m) = 0. \tag{61}$$

Hence, by (46), (55), (59) and (61), we obtain

$$\psi_{\bar{1}} = 0.$$

Case 2. $i = 0$, i.e., the Ramond case.

By (49) and (57), we have

$$\begin{aligned} 0 &= \psi_{\bar{1}}([L_m, G_0]) = [L_m, \psi_{\bar{1}}(G_0)] \\ &= [L_m, \theta_0^0 L_0 + \epsilon_0^0 I_0 + \sigma^0 c] \\ &= -m\theta_0^0 L_m + \frac{\theta_0^0(m^3-m)}{12}\delta_{m,0}c. \end{aligned}$$

Taking $m \neq 0$, we can deduce that

$$\theta_0^0 = 0.$$

This means that

$$\psi_{\bar{1}}(G_0) = \epsilon_0^0 I_0 + \sigma^0 c.$$

Setting $r \neq 0$ and according to (56), we have

$$[L_{-r}, \psi_{\bar{1}}(G_r)] = [L_{-r}, \theta_r^r L_r + \epsilon_r^r I_r] = 2r\theta_r^r L_0 + \frac{\theta_r^r(r - r^3)}{12}c + r\epsilon_r^r I_0. \tag{62}$$

On the other hand, we have

$$[L_{-r}, \psi_{\bar{1}}(G_r)] = \psi_{\bar{1}}([L_{-r}, G_r]) = r\psi_{\bar{1}}(G_0) = r\epsilon_0^0 I_0 + r\sigma^0 c. \tag{63}$$

Comparing (62) with (63), we can obtain

$$\theta_r^r = 0, \ (r \neq 0),$$

$$\epsilon_r^r = \epsilon_0^0, \ (r \neq 0),$$

and

$$\sigma^0 = 0.$$

So we have

$$\psi_{\bar{1}}(G_r) = \epsilon_0^0 I_r, \ \forall r \in \mathbb{Z}. \tag{64}$$

By (49) and (58), we have

$$\begin{aligned} r\epsilon_0^0 I_{m+r} &= r\psi_{\bar{1}}(G_{m+r}) = \psi_{\bar{1}}([L_m, G_r]) = [\psi_{\bar{1}}(L_m), G_r] \\ &= [\sum_{t\in\mathbb{Z}} \omega_t^m G_t, G_r] = 2\sum_{t\in\mathbb{Z}} \omega_t^m I_{t+r}, \end{aligned}$$

which implies

$$\omega_t^m = 0, \ (t \neq m), \tag{65}$$

and

$$2\omega_m^m = r\epsilon_0^0. \tag{66}$$

Taking $r = 0$ in (66), we can deduce

$$\omega_m^m = 0. \tag{67}$$

Then setting $r \neq 0$ in (66), we can deduce

$$\epsilon_0^0 = 0. \tag{68}$$

According to (65) and (67), we can deduce

$$\psi_{\bar{1}}(L_m) = 0. \tag{69}$$

Based on (64) and (68), we have

$$\psi_{\bar{1}}(G_r) = 0. \tag{70}$$

Finally, by performing a calculation similar to that in (60), we have

$$\psi_{\bar{1}}(I_m) = 0. \tag{71}$$

Hence, according to (46), (69), (70) and (71), we have

$$\psi_{\bar{1}} = 0.$$

The proof is now completed. □

Theorem 3.4. *There are no non-trivial $\frac{1}{2}$-superderivations of the super Heisenberg-Virasoro algebra $s\mathcal{HV}$.*

Proof. Let ψ be a $\frac{1}{2}$-superderivation of $s\mathcal{HV}$. Then the even part $\psi_{\bar{0}}$ and the odd part $\psi_{\bar{1}}$ of ψ are, respectively, an even $\frac{1}{2}$-superderivation and an odd $\frac{1}{2}$-superderivation of $s\mathcal{HV}$. According to Lemmas 3.2 and 3.3, we know that $\psi_{\bar{0}}$ is trivial and $\psi_{\bar{1}} = 0$, respectively. Since $s\mathcal{HV} = s\mathcal{HV}_{\bar{0}} \oplus s\mathcal{HV}_{\bar{1}}$ is a $\mathbb{Z}_2$-graded algebra and finitely generated, following a similar argument as in the proof of Lemma 2.4 in [19], we can deduce that

$$\psi = \psi_{\bar{0}} + \psi_{\bar{1}} = \psi_{\bar{0}},$$

which means ψ is trivial. The proof is completed. □

Theorem 3.5. *There exist no non-trivial transposed Poisson superalgebra structures defined on the super Heisenberg-Virasoro algebra $s\mathcal{HV}$.*

Proof. This statement follows from Theorem 3.4 and Lemma 2.8. □

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