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Geometry of almost ∗-η-Ricci-Yamabe soliton on Kenmotsu manifolds

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Abstract. The goal of the present object is to study almost ∗-η-Ricci-Yamabe soliton within the framework of Kenmotsu manifolds. It is shown that if a Kenmotsu manifold admits a ∗-η-Ricci-Yamabe soliton, then it is η-Einstein. Next, we prove that if a $(\kappa, -2)'$-nullity distribution, where $\kappa < -1$ acknowledges a ∗-η-Ricci-Yamabe soliton, then the manifold is Ricci flat. Later, if g represents a gradient almost ∗-η-Ricci-Yamabe soliton and ξ leaves the scalar curvature r invariant on a Kenmotsu manifold, then the manifold is an η-Einstein. Further, we have studied on a Kenmotsu manifold if g represents an almost ∗-η-Ricci-Yamabe soliton with potential vector field V is pointwise collinear with ξ, then the manifold is an η-Einstein. Lastly, we give an example of a gradient almost ∗-η-Ricci-Yamabe soliton on a 5-dimensional Kenmotsu manifold..

1. Introduction

In modern mathematics, contact geometry methods be involved in important role to the field of differential geometry. Contact geometry has enlarged from the mathematical formalism of classical mechanics. The concept of Ricci flow, which is an evolution equation for metrics defined over the connected almost contact metric manifolds whose automorphism groups have maximal dimensions.
In 1972, Kenmotsu [21] obtained some tensor equations to characterize the manifolds of the third class. Since then the manifolds of the third class were called Kenmotsu manifolds.

Very recently in 2019, Güler and Crasmareanu introduced a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map [15]. This flow is also known as Ricci-Yamabe flow of the type (ρ, q) [15].

A soliton to the Ricci-Yamabe flow is called Ricci-Yamabe soliton(RYS in short) if it moves only by one parameter group of diffeomorphism and scaling. The metric of the Riemannain manifold (M^n, g), $n > 2$ is said to admit (ρ, q)-Ricci-Yamabe soliton or simply Ricci-Yamabe soliton (RYS) (g, V, Ω, ρ, q) if it satisfies the equation:

$$\mathcal{L}_V g + 2\rho S + [2\Omega - qr]g = 0,$$

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where $\mathcal{L}_V g$ denotes the Lie derivative of the metric g along the vector field V, S is the Ricci tensor, r is the scalar curvature and Ω, ρ, q are real scalars. If Ω is a smooth function, then RYS becomes almost RYS.
In the above equation if the vector field V is the gradient of a smooth function f then the above equation is called gradient Ricci-Yamabe soliton (GRYS) and it is defined as:

$$Hess_f + \rho S + \Big[\Omega - \frac{1}{2}qr\Big]g = 0,$$

where $Hess_f$ is the Hessian of the smooth function f.
Moreover the RYS and GRYS are said to be expanding, steady or shrinking according as Ω is positive, zero, negative respectively.

In 2020, Mohd. Danish Siddiqi et al. [34] introduced a new generalization of RYS, namely η-RYS, which is given by,

$$\mathcal{L}_V g + 2\rho S + [2\Omega - qr]g + 2\mu\eta \otimes \eta = 0,$$

where μ is a constant and η is a 1-form on M.
In, 2021, Dey et al.[32] developed the notion $*$-η-Ricci-Yamabe soliton(in short $*$-η-RYS) as

$$\mathcal{L}_\xi g + 2\rho S^* + [2\Omega - qr^*]g + 2\mu\eta \otimes \eta = 0,$$

where μ and Ω are constants and η is a 1-form on M of dimension $(2n + 1)$.

As per the authors knowledge, the results concerning $*$-η-RYS were studied when the potential vector field V is the characteristic vector field ξ. Motivated from this, we generalize the definition by considering the potential vector field as arbitrary vector field V and define as:

$$\mathcal{L}_V g + 2\rho S^* + [2\Omega - qr^*]g + 2\mu\eta \otimes \eta = 0. \tag{1}$$

Now, if the scalars Ω and μ are smooth functions, then $*$-η-RYS becomes an almost $*$-η-RYS. Now, if we consider the potential vector field V as the gradient of a smooth function f, then the $*$-η-RYS equation can be rewritten as

$$Hess_f + \rho S^* + (\Omega - \frac{qr^*}{2})g + \mu\eta \otimes \eta = 0, \tag{2}$$

by gradient almost $*$-η-RYS, we mean a gradient $*$-η-RYS, where we consider Ω and μ are smooth functions.

Recently, many authors have been worked on RYS and η-RYS on contact geometry. D. Dey [11, 12] examined $*$-RYS on contact geometry and demonstrated RYS on almost kenmotsu manifolds. Besides, in [12], he has displayed that a $(\kappa, \mu)'$ almost Kenmotsu manifolds admitting a RYS or GRYS is locally isometric to the Riemannian product $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$. Now a natural **question** arises:
Does the above result is true for a $(2n + 1)$-dimensional $(\kappa, \mu)'$ almost Kenmotsu manifold admitting $*$-η-RYS?

We will answer this question affirmatively in the section 4. However, Siddiqi et al. [34] initiated the study of η-RYS on Riemannian submersions with the potential field. In recent years, Yoldaş [40] characterized Kenmotsu metric admitting η-RBS. Khatri et al. [22] investigated almost RYS on Kenmotsu manifold. Recently, in [6], authors have studied conformal η-Ricci soliton and gradient conformal η-Ricci soliton within the framework of Kenmotsu manifolds. Also, some authors in (see for details [3–5, 7–10, 16, 17, 23–26, 29–31, 33]) have evaluated Ricci soliton, conformal Ricci soliton and their generalizations on contact and complex manifolds. In [19, 27, 38], one can study further research on soliton geometry.

The paper is categorized is as follows. In section 2, after a brief introduction, we have discussed some preliminaries of contact metric manifolds. In section 3, if we consider a Kenmotsu manifold admits a $*$-η-RYS, then the manifold is η-Einstein. In section 4, we consider the metric of $(\kappa, \mu)'$-almost Kenmotsu manifold to represent $*$-η-RYS along with a special condition and obtain a Ricci flat manifold and is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. Section 5 deals with almost $*$-η-RYS and gradient almost $*$-η-RYS on Kenmotsu manifold and show that the manifold is η-Einstein. In last section, we have developed an example of $*$-η-RYS on 5-dimensional Kenmotsu manifold to prove our findings.

2. Notes on contact metric manifolds

By [1], a $(2n+1)$-dimensional smooth Riemannian manifold (M, g) is said to be an almost contact metric manifold if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ, a global 1-form η and an indefinite metric g on M satisfying the following relations.

$$\phi^2(X) = -X + \eta(X)\xi, \tag{3}$$

$$\eta(\xi) = 1 \tag{4}$$

for all vector field X on M. Generally, ξ and η are called characteristic vector field or Reeb vector field and almost 1-form respectively.

A Riemannian metric g is said to be an associated (or compatible) metric if it satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{5}$$

for all vector fields X, Y on M. An almost contact manifold $M^{2n+1}(\phi, \xi, \eta, g)$ together with a compatible metric g is known as almost contact metric manifold (see Blair [1])

In an almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ the following conditions are satisfied:

$$\phi\xi = 0, \tag{6}$$

$$\eta(\phi X) = 0, \tag{7}$$

$$g(X, \xi) = \eta(X), \tag{8}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{9}$$

for arbitrary $X\ Y \in \chi(M)$. The normality of an almost contact structure is equivalent with the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ (for more details see [1]).

Definition 2.1. *On an almost contact metric manifold M, a vector field X is said to be contact vector field if there exist a smooth function f such that* $\mathcal{L}_X\xi = f\xi$.

Definition 2.2. *On an almost contact metric manifold M, a vector field X is said to be infinitesimal contact transformation if* $\mathcal{L}_X\eta = f\eta$ *for some function f. In particular, we call X as a strict infinitesimal contact transformation if* $\mathcal{L}_X\eta = 0$.

By [1], an almost Kenmotsu manifold is defined as an almost contact metric manifold if it satisfies η is closed i.e. $d\eta = 0$ and $\Phi = 2\eta \wedge \phi$, where the fundamental 2-form ϕ of the almost contact metric manifold is defined by $\phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y on M (see [20]). On the product $M^{2n+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$, there exists an almost complex structure J defined by

$$J(X, f\tfrac{d}{dt}) = (\phi X - f\xi, \eta(X)\tfrac{d}{dt}),$$

where X denotes a vector field tangent to M^{2n+1}, t is the co-ordinate of $\mathbb{R}$ and f is C^∞-function on $M^{2n+1}\times\mathbb{R}$. If J is integrable, then almost contact metric structure on M^{2n+1} is said to be normal. A normal almost Kenmotsu manifold is called a Kenmotsu manifold (see [21]). An almost Kenmotsu manifold is a Kenmotsu manifold if and only if

$$(\nabla_X\phi)Y = g(\phi X, Y) - \eta(Y)\phi X \tag{10}$$

for any vector fields X, Y on M^{2n+1}. On a Kenmotsu manifold the following holds [21]:

$$\nabla_X \xi = X - \eta(X)\xi \tag{11}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{12}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{13}$$

$$QX = -2nX, \tag{14}$$

$$(\mathcal{L}_\xi g)(X, Y) = 2g(X, Y) - 2\eta(X)\eta(Y) \tag{15}$$

for any vector fields X, Y on M^{2n+1}, where $\mathcal{L}$ is the Lie derivative operator, R is the curvature tensor of g and Q the Ricci operator associated with $(1, 2)$ Ricci tensor S given by $S(X, Y) = g(QX, Y)$ for all vector fields X, Y on M^{2n+1}. It is shown that a Kenmotsu manifold is locally a wraped product $I \times_f N^{2n}$, where I is an open interval with coordinate t, $f = ce^t$ is the warping function for some positive constant c and N^{2n} is a K$\ddot{a}$hlerian manifold [21].

A $(2n + 1)$-dimensional Kenmotsu manifold is said to be a η-Einstein Kenmotsu manifold if there exists two smooth functions a and b such that the following relation holds

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{16}$$

for all X and $Y \in \chi(M)$. Clearly if $b = 0$ then η-Einstein manifold reduces to Einstein manifold. Now considering $X = \xi$ in the last equation and using (14) we have $a + b = -2n$. Contracting (16) over X and Y we get, $r = (2n + 1)a + b$, where r denotes the scalar curvature of the manifold. Solving these two we get, $a = \frac{1}{2n}(2n + r)$ and $b = -\frac{1}{2n}\{2n(2n + 1) + r\}$. Using these values (16) rewrite as

$$S(X, Y) = \frac{1}{2n}(2n + r)g(X, Y) - \frac{1}{2n}\{2n(2n + 1) + r\}\eta(X)\eta(Y). \tag{17}$$

On an almost Kenmotsu manifold we consider two $(1, 1)$-type tensor field $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ and open operator $\ell = R(., \xi)\xi$, where $\mathcal{L}_\xi \phi$ is the Lie derivative of ϕ along the direction ξ. The tensor field h and h' plays an important role in an almost Kenmotsu manifold. Both of them are symmetric and satisfies the following relations [28]:

$$\nabla_X \xi = X - \eta(X)\xi - \phi hX(\nabla_\xi \xi = 0), \tag{18}$$

$$h\xi = h'\xi = 0, \tag{19}$$

$$h\phi + \phi h = 0, tr(h) = tr(h') = 0 \tag{20}$$

for any X, $Y \in \chi(M)$, where ∇ is the Levi-Civita connection of the metric g. In addition the following curvature property is also satisfied:

$$R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y \tag{21}$$

for any vector fields X, Y on M and R is the Riemannian curvature tensor of (M, g). The $(1, 1)$-type symmetric tensor field $h' = h\phi$ is anti-commuting with ϕ and $h'\xi = 0$.

By $(\kappa, \mu)'$-almost Kenmotsu manifold we mean almost Kenmotsu manifold where the characteristic vector field ξ satisfies the $(\kappa, \mu)'$-nullity distribution. It is clear in [2], i.e.,

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)h'X - \eta(X)h'Y\} \tag{22}$$

for any vector fields X, Y on M, where κ and μ are real constant. On a $(\kappa, \mu)'$-almost Kenmotsu manifold M, we have (for details see [2])

$$h = 0 \Leftrightarrow h' = 0, h'^2 = (k + 1)\phi^2, \tag{23}$$

$$h^2(X) = -(\kappa+1)[X - \eta(X)\xi] \tag{24}$$

for $X \in \chi(M)$. From previous relation it follows that $h' = 0$ if and only if $\kappa = -1$ and $h \neq 0$ otherwise. Let $X \in Ker(\eta)$ be an eigenvector field of h' orthogonal to ξ with respect to the eigenvalue α. Then from (23) we get $\alpha^2 = -(\kappa+1)$ which implies $\kappa \leq -1$. In [2] since the same symbol μ is used in the coefficient of $\mu \otimes \mu$ in the definition of $*$-η-Ricci Yamabe soliton and in $(\kappa,\mu)'$-almost Kenmotsu manifold, so to reduce the complications in notations we use $(\kappa,-2)'$-almost Kenmotsu manifold throughout this paper as $\mu = -2$ used in proposition 4.1 of [2].

We recall some useful results on a $(2n+1)$ dimensional $(\kappa, 2)'$-almost Kenmotsu manifold M with $\kappa \leq -1$ as follows:

$$R(\xi,X)Y = \kappa\{g(X,Y)\xi - \eta(Y)X\} - 2\{g(h'X,Y)\xi - \eta(Y)h'X\}, \tag{25}$$

$$QX = -2nX + 2n(\kappa+1)\eta(X)\xi - 2nh'(X), \tag{26}$$

$$r = 2n(\kappa - 2n), \tag{27}$$

$$(\nabla_X\eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y), \tag{28}$$

where X and $Y \in \chi(M)$, Q, r are the Ricci operator and scalar curvature of M respectively.

The notion (κ,μ)-nullity distribution on a contact metric manifold M was introduced by Blair et al. [1], which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu) &= \{Z \in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ &+ \mu[g(Y,Z)hX - g(X,Z)hY]\} \end{aligned} \tag{29}$$

for any vector fields X, Y on $T_p(M)$, where $T_p(M)$ denotes the tangent space on M at any point $p \in M$ and R is the Riemannian tensor. In [2], Dileo and Pastore introduced the notion of $(\kappa,\mu)'$-nullity distribution, on an almost Kenmotsu manifold (M,ϕ,ξ,η,g), which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' &= \{Z \in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ &+ \mu[g(Y,Z)h'X - g(X,Z)h'Y]\} \end{aligned} \tag{30}$$

for any vector fields X, Y on $T_p(M)$. For further details on almost Kenmotsu manifolds, we refer the reader to go through the references [11, 34].

3. $*$-η-RYS on Kenmotsu manifold

In this section, we investigate that the metric g of a $(2n+1)$-dimensional Kenmotsu manifold admitting a $*$-η-RYS. We recall some important lemmas relevant to our works.

Lemma 3.1. *[36] The Ricci operator Q on a $(2n+1)$-dimensional Kenmotsu manifold satisfies*

$$(\nabla_X Q)\xi = -QX - 2nX, \tag{31}$$

$$(\nabla_\xi Q)X = -2QX - 4nX \tag{32}$$

for arbitrary vector field X on the manifold.

Lemma 3.2. *[36] The $*$-Ricci tensor S^* on a $(2n+1)$-dimensional Kenmotsu manifold is given by*

$$S^*(X,Y) = S(X,Y) + (2n-1)g(X,Y) + \eta(X)\eta(Y) \tag{33}$$

for arbitrary vector fields X and $Y \in \chi(M)$ and the corresponding $*$-scalar curvature is given by the expression $r^* = r + 4n^2$.

Lemma 3.3. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold admits a $*$-η-RYS. Then the curvature tensor R with the soliton vector V are given by the expression*

$$\begin{aligned}(\mathcal{L}_V R)(X,\xi)\xi &= 4n(1-\rho)\{\eta(X)\xi + X\} + q\{X(r)\xi - X(Dr)\}\\ &+ q\{\xi(Dr) - \xi(r)\xi - Dr\}.\end{aligned}$$

Proof. Suppose the metric g of Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ represents a $*$-η-RYS. So both of the equations (1) and (33) are satisfied. Combining these two equations, we get

$$\begin{aligned}(\mathcal{L}_V g)(X,Y) &= -2\rho S(X,Y) - \{2\rho(2n-1) + 2\Omega - q(r+4n^2)\}g(X,Y)\\ &-2(\mu+\rho)\eta(X)\eta(Y). \end{aligned} \tag{34}$$

We take a covariant derivative with respect to arbitrary vector field Z and using the identity (12) to obtain

$$\begin{aligned}&(\nabla_Z \mathcal{L}_V g)(X,Y) = -2\rho(\nabla_Z S)(X,Y) + qZ(r)g(X,Y)\\ &-2(\mu+\rho)\{g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}\end{aligned} \tag{35}$$

for all $X\ Y\ Z \in \chi(M)$. Again from Yano [39] we have the following communication formula

$$(\mathcal{L}_V \nabla_Z g - \nabla_Z \mathcal{L}_V g - \nabla_{[V,Z]} g)(X,Y) = -g((\mathcal{L}_V \nabla)(X,Z),Y) - g((\mathcal{L}_V \nabla)(Y,Z),X),$$

where g is the metric connection i.e., $\nabla g = 0$. So the above equation reduces to

$$(\nabla_Z \mathcal{L}_V g)(X,Y) = g((\mathcal{L}_V \nabla)(X,Z),Y) + g((\mathcal{L}_V \nabla)(Y,Z),X) \tag{36}$$

for all vector fields $X,\ Y,\ Z \in \chi(M)$. Combining (35) and (36) and by a straightforward combinatorial computation and applying the symmetry of $(\mathcal{L}_V \nabla)$, (36) implies

$$\begin{aligned}g((\mathcal{L}_V \nabla)(X,Y),Z) &= \rho\{(\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(Z,X)\}\\ &- qZ(r)g(X,Y) + qX(r)g(Y,Z) + qY(r)g(X,Z)\\ &- 2(\mu+\rho)\{g(X,Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\}\end{aligned} \tag{37}$$

for arbitrary vector fields X, Y and Z on M. Using (31) and (32), the foregoing equation yields

$$(\mathcal{L}_V \nabla)(X,\xi) = 2\rho QX + \{4n + q\xi(r)\}X + qX(r)\xi - q\eta(X)Dr \tag{38}$$

for all $X \in \chi(M)$. Now, we differentiate covariantly this with respect to arbitrary vector field Y and using (11), we achieve

$$\begin{aligned}(\nabla_Y \mathcal{L}_V \nabla)(X,\xi) &= 2\rho(\nabla_Y Q)X - (\mathcal{L}_V \nabla)(X,Y) + \eta(Y)(\mathcal{L}_V \nabla)(X,\xi).\\ &+ qY(\xi(r))X + qg(X,\nabla_Y Dr) - qX(r)\phi^2 Y\\ &- q\eta(X)\nabla_Y Dr - qg(X,Y)Dr + \eta(X)\eta(Y)Dr.\end{aligned} \tag{39}$$

We know that, $(\mathcal{L}_V R)(X,Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z)$. In view of (39) in the previous relation we acquire

$$\begin{aligned}(\mathcal{L}_V R)(X,Y)\xi &= 2\rho(\nabla_X Q)Y - 2\rho(\nabla_Y Q)X + \eta(X)(\mathcal{L}_V \nabla)(Y,\xi)\\ &- \eta(Y)(\mathcal{L}_V \nabla)(X,\xi) + q\{X(\xi(r))Y - Y(\xi(r))X\}\\ &+ q\{X(r)\phi^2 Y - Y(r)\phi^2 X\}\\ &+ q\{\eta(X)\nabla_Y Dr - \eta(Y)\nabla_X Dr\},\end{aligned} \tag{40}$$

for arbitrary vector fields X and Y on M and where we have used that $\mathcal{L}_V \nabla$ $Hess_r$ are symmetric. In view of (11) Putting $Y = \xi$ in (40) and using (14), (31) and (32) we get

$$\begin{aligned}(\mathcal{L}_V R)(X,\xi)\xi &= 4n(1-\rho)\{\eta(X)\xi + X\} + q\{X(r)\xi - X(Dr)\}\\ &+ q\{\xi(Dr) - \xi(r)\xi - Dr\}.\end{aligned} \tag{41}$$

Thus, we end our proof. □

Theorem 3.4. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold represents a $*$-η-RYS. Then the manifold is an η-Einstein.*

Proof. We take a Lie derivative of $g(\xi, \xi) = 1$ along the potential vector field V, in account of (34) to obtain

$$\eta(\mathcal{L}_V \xi) = \Omega + \mu - \frac{q}{2}(r + 4n^2). \tag{42}$$

Putting $Y = \xi$ and following (4) and (8), the equation (34) provides

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = -\{2\Omega + 2\mu - q(r + 4n^2)\}\eta(X) \tag{43}$$

for arbitrary vector field X on M. From (13) we get $R(X, \xi)\xi = \eta(X)\xi - X$. Taking Lie derivative along the potential vector field V and using (42) and (43), this reduces to

$$(\mathcal{L}_V R)(X, \xi)\xi = \{2\Omega + 2\mu - q(r + 4n^2)\}(X - \eta(X)\xi) \tag{44}$$

for all $X \in \chi(M)$. Then from (41), we get

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where $a = \frac{1}{2(\rho-1)}\{2\Omega + 2\mu - q(r + 4n^2) - 4n(\rho - 1)\}$ and $b = -\frac{1}{2(\rho-1)}\{2\Omega + 2\mu - q(r + 4n^2)\}$ for all $X\ Y \in \chi(M)$. Thus, we complete the theorem. □

4. $*$-η-RYS on $(\kappa, \mu)'$-almost Kenmotsu manifold with $\kappa < -1$

In this section, we consider a $(2n + 1)$-dimensional almost Kenmotsu manifold, where the characteristic vector field ξ satisfies $(\kappa, -2)'$-nullity distribution. Then we take the metric g to represent a $*$-η-RYS. Here, we write a lemma and used this for our work.

Lemma 4.1. *[13] On a $(\kappa, \mu)'$-almost Kenmotsu manifold with $\kappa < -1$ the $*$-Ricci tensor is given by*

$$S^*(X, Y) = -(\kappa + 2)\{g(X, Y) - \eta(X)\eta(Y)\} \tag{45}$$

for any vector fields X and Y on M.

Theorem 4.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to $(\kappa, -2)'$-nullity distribution where $\kappa < -1$. If the metric g represents a $*$-η-RYS satisfying*

$$\Omega + \mu \neq \tfrac{q}{2}(r + 4n^2) + \tfrac{q}{2}\{\xi(\xi(r)) - \xi(Dr)\},$$

then M is Ricci-flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Proof. Comparing (1) and (45) and using $r^* = r + 4n^2$, we get

$$(\mathcal{L}_V g)(X, Y) = \{2\rho(\kappa + 2) - 2\Omega + q(r + 4n^2)\}g(X, Y) - 2\{\rho(\kappa + 2) + \mu\}\eta(X)\eta(Y) \tag{46}$$

for all vector fields X and Y on M. Now taking covariant derivative of (46) along the arbitrary vector field Z and using (8) we have

$$\begin{aligned}(\nabla_Z \mathcal{L}_V g)(X, Y) &= qZ(r)g(X, Y) - 2\{\rho(\kappa + 2) + \mu\}\\ &[\eta(Y)g(X, Z) + \eta(X)g(Y, Z) + \eta(Y)g(h'Z, X) + \eta(X)g(h'Z, Y)\\ &-2\eta(X)\eta(Y)\eta(Z)].\end{aligned} \tag{47}$$

Then using (36) and by the symmetry of $(\mathcal{L}_V \nabla)$ from the above equation (47) we get

$$\begin{aligned}(\mathcal{L}_V \nabla)(X, Y) &= -2\{\rho(\kappa + 2) + \mu\}[g(X, Y) + g(h'X, Y) - \eta(X)\eta(Y)]\\ &- q\{g(X, Y)Dr - X(r)Y - Y(r)X\},\end{aligned} \tag{48}$$

for all $X, Y \in \chi(M)$. Putting $Y = \xi$ and using (4), (8) and (19), we acquire

$$(\mathcal{L}_V\nabla)(X,\xi) = -q\{\eta(X)Dr - X(r)\xi - \xi(r)X\} \tag{49}$$

for arbitrary vector X on M. Now taking differentiation (49) covariantly along arbitrary vector field Y and using (18) and (48) into account we can get

$$\begin{aligned}(\nabla_Y\mathcal{L}_V\nabla)(X,\xi) &= 2\{\rho(\kappa+2)+\mu\}[g(X,Y)+g(h'X,Y)-\eta(X)\eta(Y)]\xi \\ &- q\{(\nabla_Y\eta)XDr+\eta(X)Y(Dr)-X(r)\nabla_Y\xi - Y(\xi(r))X\}\end{aligned} \tag{50}$$

for any vector fields X and Y on M. Again Yano shows that the well-known curvature property, $(\mathcal{L}_VR)(X,Y)Z = (\nabla_X\mathcal{L}_V\nabla)(Y,Z) - (\nabla_Y\mathcal{L}_V\nabla)(X,Z)$. Using (18), (28) and setting $Z = \xi$ then using (50), we obtain

$$(\mathcal{L}_VR)(X,Y)\xi = q\{\eta(Y)X(Dr) - \eta(X)Y(Dr) + Y(\xi(r))X - X(\xi(r)Y\} \tag{51}$$

for any abitrary vector field X and Y on M. Now taking Lie derivative of (22) along the potential vector field V and also using (4) and (19), we get

$$\begin{aligned}&(\mathcal{L}_VR)(X,\xi)\xi = \kappa[g(X,\mathcal{L}_V\xi)\xi - 2\eta(\mathcal{L}_V\xi)X - ((\mathcal{L}_V\eta)X)\xi] \\ &+2[2\eta(\mathcal{L}_V\xi)h'X - \eta(X)(h'(\mathcal{L}_V\xi)) - g(h'X,\mathcal{L}_V\xi)\xi - (\mathcal{L}_Vh')X]\end{aligned} \tag{52}$$

for any $X \in \chi(M)$. Putting $Y = \xi$ in (46) we infer

$$(\mathcal{L}_V\eta)X - g(X,\mathcal{L}_V\xi) = \{-2\Omega + q(r+4n^2) - 2\mu\}\eta(X) \tag{53}$$

for any $X \in \chi(M)$. Now we put $X = \xi$ into (53) to yield

$$\eta(\mathcal{L}_V\xi) = \Omega - \frac{q}{2}(r+4n^2) + \mu. \tag{54}$$

By the help of (51), (53) and (54), we can write the equation (52) as

$$\begin{aligned}&\kappa\{-2\Omega + q(r+4n^2) - 2\mu + q\xi(\xi(r)) - q\xi(Dr)\}(X - \eta(X)\xi) + 2\{2\Omega - \\ &q(r+4n^2) + 2\mu\}h'X - 2\eta(X)h'(\mathcal{L}_V\xi) - 2g(h'X,\mathcal{L}_V\xi)\xi - 2(\mathcal{L}_Vh')X \\ &= 0.\end{aligned} \tag{55}$$

We take an inner product of (55) with respect to the arbitrary vector field Y on M to obtain

$$\begin{aligned}&\{-2\Omega + q(r+4n^2) - 2\mu + q\xi(\xi(r)) - q\xi(Dr)\}[\kappa\{g(X,Y) - \eta(X)\eta(Y)\} \\ &-2g(h'X,Y)] - 2\eta(X)g(h'(\mathcal{L}_V\xi),Y) - 2g(h'X,\mathcal{L}_V\xi)\eta(Y) \\ &-2g((\mathcal{L}_Vh')X,Y) = 0.\end{aligned} \tag{56}$$

As the above equation (56) is true for any vector fields X and Y on M. We replacing X by $\phi(X)$ and Y by $\phi(Y)$ and taking (7)into account we get as

$$\begin{aligned}&\{-2\Omega + q(r+4n^2) - 2\mu + q\xi(\xi(r)) - q\xi(Dr)\}[\kappa g(\phi X,\phi Y) \\ &-2g(h'\phi X,\phi Y)] - 2g((\mathcal{L}_Vh')\phi X,\phi Y) = 0\end{aligned} \tag{57}$$

for all vector fields X and Y on M. Since $spec(h') = \{0,\alpha,-\alpha\}$, let X and V belong to the eigenspaces of $-\alpha$ and α denoted by $[-\alpha]'$ and $[\alpha]'$ respectively. Then $\phi X \in [\alpha]'$ (see [2]). Then (57) can be rewritten as

$$\begin{aligned}&\{-2\Omega + q(r+4n^2) - 2\mu + q\xi(\xi(r)) - q\xi(Dr)\}(\kappa - 2)g(\phi X,\phi Y) \\ &-2g((\mathcal{L}_Vh')\phi X,\phi Y) = 0\end{aligned} \tag{58}$$

for all vector fields on M. Now, we want to find the value of $g((\mathcal{L}_Vh')\phi X,\phi Y)$. To find this value, we prove a more generalized results: In a $(\kappa,\mu)'$-almost Kenmotsu manifold $(\mathcal{L}_Xh')Y = 0$, where the vector fields X and Y belong to same eigenspaces.

Without doubt we assume that $X, Y \in [\alpha]'$, where $spec(h') = \{0,-\alpha,\alpha\}$. We consider a local orthonormal ϕ-basis as $\{\xi, e_i, \phi e_i\}$, $i = 1,2,3,...,n$ then

$$\nabla_X Y = \sum_{i=1}^{n} g(\nabla_X Y, e_i)e_i - (\alpha+1)g(X,Y)\xi.$$

and

$$\begin{aligned}(\mathcal{L}_X h')Y &= \mathcal{L}_X(h'Y) - h'(\mathcal{L}_X Y)\\ &= \alpha(\mathcal{L}_X Y) - h'(\mathcal{L}_X Y)\\ &= \alpha(\nabla_X Y - \nabla_Y X) - h'(\nabla_X Y - \nabla_Y X)\\ &= \alpha(\alpha+1)g(X,Y)\xi - \alpha(\alpha+1)g(X,Y)\xi\\ &= 0.\end{aligned}$$

Similarly, we can prove that the above results hold good if $X, Y \in [-\alpha]'$. For more details see [2]. Now from (58), we get

$$\{-2\Omega + q(r+4n^2) - 2\mu + q\xi(\xi(r)) - q\xi(Dr)\}(\kappa-2)g(\phi X, \phi Y) = 0 \tag{59}$$

for all vector fields X and Y on M. As $g(\phi X, \phi Y) \neq 0$ then from the for going equation we have either $\Omega + \mu = \frac{q}{2}(r+4n^2) + \frac{q}{2}\{\xi(\xi(r)) - \xi(Dr)\}$ or $\kappa = 2$.

Now, for $\Omega + \mu \neq \frac{q}{2}(r+4n^2) + \frac{q}{2}\{\xi(\xi(r)) - \xi(Dr)\}$ from the equation (59) we infer that $\kappa = 2$, (45) implies

$$S^*(X,Y) = -4\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{60}$$

Thus the $*$-Ricci tensor is η-Einstein manifold.

Again from (60) and the proposition 4.1 of [2]. we finally conclude that M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, where $\mathbb{H}^{n+1}(-4)$ is the hyperbolic space of constant curvature -4.

Also, for any vector fields X and Y on M. By hypothesis $\Omega + \mu \neq \frac{q}{2}(r+4n^2) + \frac{q}{2}\{\xi(\xi(r)) - \xi(Dr)\}$, from the equation (59) we infer that $\kappa = 2\alpha$. Again from $\alpha^2 = -(\kappa+1)$ we get $\alpha = -1$ and $\kappa = -2$, putting the value of κ in (45) $S^*(X,Y) = 0$ i.e. $*$-Ricci flat. This completes the proof. □

5. Almost $*$-η-RYS

In this section, we study an almost $*$-η-RYS and gradient almost $*$-η-RYS on a Kenmotsu manifold. We know that an almost $*$-η-RYS satisfying (1) for some smooth functions Ω and μ, is a generalization of RYS. In [18], Ghosh considered Ricci almost soliton on a Kenmotsu manifold and proved that if a Kenmotsu metric is a gradient Ricci almost soliton and the Reeb vector field ξ leaves the scalar curvature r invariant, then it is an Einstein manifold. To generalize the above results, we consider gradient almost $*$-η-RYS on Kenmotsu manifold and prove the following theorem.

Theorem 5.1. *If a Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ endows a gradient almost $*$-η-RYS and ξ leaves the scalar curvature r invariant, then (M, g) is an η-Einstein manifold with constant scalar curvature $r = -2n(2n+1)$.*

Proof. The gradient of the soliton equation (2) can be written for any X belongs to $\chi(M)$ as

$$\nabla_X Df + \rho QX + \{\rho(2n-1) + \Omega - \frac{q}{2}(r+4n^2)\}X + (\mu+\rho)\eta(X)\xi = 0. \tag{61}$$

Then applying the expression of Riemannian curvature tensor $R(X,Y)Df = \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df$, we get

$$\begin{aligned}R(X,Y)Df &= \rho(\nabla_Y Q)X - \rho(\nabla_X Q)Y + Y(\sigma)X - X(\sigma)Y\\ &+Y(\mu)\eta(X)\xi - X(\mu)\eta(Y)\xi + (\mu+\rho)\{\eta(Y)X - \eta(X)Y\}\end{aligned} \tag{62}$$

for all $X, Y \in \chi(M)$ and $\sigma = \rho(2n+1) + \Omega - \frac{q}{2}(r + 4n^2)$, a smooth function as Ω is a smooth function. Now putting $Y = \xi$ in (62) and using (31) and (32), we get

$$\begin{aligned} R(X,\xi)Df &= -\rho QX - 2n\rho X + \xi(\sigma)X - X(\sigma)\xi + \xi(\mu)\eta(X)\xi \\ &\quad -X(\mu)\xi + (\mu+\rho)(X - \eta(X)\xi) \end{aligned} \tag{63}$$

for any $X \in \chi(M)$. By virtue of (13), equation (63) reduces to

$$\begin{aligned} X(\sigma+\mu+f)\xi &= -\rho QX + \{\xi(\sigma+f) + \mu + \rho - 2\rho n\}X \\ &\quad +\{\xi(\mu) - \mu - \rho\}\eta(X)\xi \end{aligned} \tag{64}$$

for any $X \in \chi(M)$. Now, taking an inner product of (64) with ξ and using (13), we get $X(\sigma + \mu + f) = \xi(\sigma + \mu + f)\eta(X)$. Putting this into (64), we obtain

$$QX = \frac{1}{\rho}\{\xi(\sigma+f) + \mu + \rho - 2\rho n\}X - \frac{1}{\rho}\{\xi(\sigma+f) + \mu + \rho\}\eta(X)\xi \tag{65}$$

for any $X \in \chi(M)$. This shows that the manifold (M, g) is an η-Einstein manifold. Now contracting (62) over X with respect to an orthonormal basis $\{e_i\}$, $1 \le i \le 2n+1$, we compute

$$\begin{aligned} S(Y,Df) &= -\rho\sum_{i=1}^{2n+1} g((\nabla_{e_i}Q)Y, e_i) + Y(r) + 2nY(\sigma) \\ &\quad +Y(\mu) - \eta(Y)\xi(\mu) + 2n(\mu+\rho)\eta(Y). \end{aligned} \tag{66}$$

Now, using the formula for the Riemannian manifold which is well known:

$$trac_g\{X \to (\nabla_X Q)Y\} = \tfrac{1}{2}Y(r)$$

and

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i}Q)Y, e_i) = 1$$

then from (66), we get

$$S(Y,Df) = \frac{1}{2}Y(r) + 2nY(\sigma) + Y(\mu) - \eta(Y)\xi(\mu) + 2n(\mu+\rho)\eta(Y) \tag{67}$$

for any $X \in \chi(M)$. From (13), we can compute $S(\xi, Df) = -2n\xi(f)$, putting this into (67) to achieve $\xi(r) = -4n\{\xi(\sigma + f) + \mu + \rho\}$. Using this in the trace of (32), we get $\xi(\sigma + f) = (2n+1) - \mu - \rho + \frac{r}{2n}$. By this result, equation (65) reduces to

$$QX = \frac{1}{2n\rho}\{r + 4n^2 + +2n - 4n^2\rho\}X - \frac{1}{2n\rho}\{r + 4n^2 + 2n\}\eta(X)\xi \tag{68}$$

for any $X \in \chi(M)$. By our assumption, $\xi(r) = 0$, the trace of (32) gives $r = -2n(2n+1)$. Thus, from (68) the required result follows. □

Next part is a Kenmotsu metric as an almost $*$-η-RYS, whose non-zero potential vector field V is pointwise collinear to the Reeb vector field ξ, we extend the previous theorem by the following theorem as

Theorem 5.2. *If a Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ admits an almost $*$-η-RYS with non-zero potential vector field V collinear to the Reeb vector field ξ, the manifold becomes η-Eistein. Also, if ξ leaves the scalar curvature r invariant, then (M, g) is an Einstein manifold with $\tau + \Omega = \rho + \frac{q(r+4n^2)}{2}$.*

Proof. Since $V = \tau\xi$, for some smooth function τ on M, it follows that

$$(\mathcal{L}_V g)(X, Y) = X(r)\eta(Y) + Y(r)\eta(X) + 2\tau\{g(X, Y) - \eta(X)\eta(Y)\} \tag{69}$$

for any vector fields X and $Y \in \chi(M)$. For this condition the soliton equation (1) transforms into

$$\begin{aligned}&2\rho S(X, Y) + X(\tau)\eta(Y) + Y(\tau)\eta(X) + \{2\Omega - q(r + 4n^2)\\&+2\rho(2n - 1) + 2\Omega\}g(X, Y) = 2(\tau - \mu - \rho)\eta(X)\eta(Y)\end{aligned} \tag{70}$$

for any vector field X, $Y \in \chi(M)$. Now putting $X = \xi$ and $Y = \xi$ in (70) and using (14), we obtain $\xi(\tau) = \frac{q}{2}(r + 4n^2) - \mu - \Omega$. Thus putting in (70) yields $X(\tau) = \{\frac{q}{2}(r + 4n^2) - \mu - \Omega\}\eta(X)$, similarly $Y(\tau) = \{\frac{q}{2}(r + 4n^2) - \mu - \Omega\}\eta(Y)$. Using these two values, (70) implies that

$$\begin{aligned}S(X, Y) &= \frac{1}{\rho}\{\frac{q}{2}(r + 4n^2) - \Omega - \rho(2n - 1) - \tau\}g(X, Y)\\&+ \frac{1}{\rho}\{\tau + \Omega - \rho - \frac{q}{2}(r + 4n^2)\}\eta(X)\eta(Y).\end{aligned} \tag{71}$$

Hence, (71) implies that (M, g) is η-Einstein manifold. Moreover, if ξ leaves the scalar curvature r invariant, i.e., $\xi(r) = 0$, again tracing (32) gives $r = -2n(2n + 1)$. Using this in the trace of (71) yields $\tau + \Omega = \rho + \frac{q(r+4n^2)}{2}$. By (71), $S(X, Y) = -2ng(X, Y)$. Thus (M, g) is an Einstein manifold, which completes the proof. □

Theorem 5.3. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold satisfies a gradient almost $*$-η-RYS. Then either M is an η-Einstein when $r = -\frac{1}{\rho}\{4n(1 - \rho) + 2n\rho\} - 4n^2$ or there exists an open set where the potential vector field V is pointwise collinear with ξ when $r \neq -\frac{1}{\rho}\{4n(1 - \rho) + 2n\rho\} - 4n^2$.*

Proof. In view of (33) and from the definition of gradient almost $*$-η-RYS given by the equation (2), we get

$$\nabla_X Df = -\rho QX - \{\Omega - \frac{q}{2}(r + 4n^2) + \rho(2n - 1)\}X - (\mu + \rho)\eta(X)\xi \tag{72}$$

for any vector field X on M. Taking covariant derivative with the arbitrary vector field Y and then using (11) and (12) we have

$$\begin{aligned}\nabla_Y\nabla_X Df &= -\rho(\nabla_Y Q)X - \rho Q(\nabla_Y X) - Y(\sigma)X - \sigma(\nabla_Y X)\\&-Y(\mu)\eta(X)\xi - (\mu + \rho)\{g(X, Y)\xi - 2\eta(X)\eta(Y)\xi\\&+\eta(X)Y + \eta(\nabla_Y X)\xi\},\end{aligned} \tag{73}$$

where $\sigma = \Omega - \frac{q}{2}(r + 4n^2) + \rho(2n - 1)$. Now, we interchange X and Y in the previous equation to find

$$\begin{aligned}\nabla_X\nabla_Y Df &= -\rho(\nabla_X Q)Y - \rho Q(\nabla_X Y) - X(\sigma)Y - \sigma(\nabla_X Y)\\&-X(\mu)\eta(Y)\xi - (\mu + \rho)\{g(X, Y)\xi - 2\eta(X)\eta(Y)\xi\\&+\eta(Y)X + \eta(\nabla_X Y)\xi\}.\end{aligned} \tag{74}$$

Then applying the expression of Riemannian curvature tensor in

$$R(X, Y)Df = \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]Df},$$

we obtain

$$\begin{aligned}R(X, Y)Df &= \rho(\nabla_Y Q)X - \rho(\nabla_X Q)Y + Y(\sigma)X - X(\sigma)Y + Y(\mu)\eta(X)\xi\\&-X(\mu)\eta(Y)\xi - (\mu + \rho)\{\eta(Y)X - \eta(X)Y\}.\end{aligned} \tag{75}$$

Then we take an inner product w.r.t ξ and use of (31) and (32) to yield

$$g(R(X, Y)Df, \xi) = Y(\sigma)\eta(X) - X(\sigma)\eta(Y) + Y(\mu)\eta(X) - X(\mu)\eta(Y) \tag{76}$$

for any vector fields X and $Y \in \chi(M)$. Moreover taking inner product of (13) with the potential vector field Df provides

$$g(R(X,Y)\xi, Df) = \eta(X)g(Y,Df) - \eta(Y)g(X,Df) \tag{77}$$

for arbitrary vector fields X and Y on M. Comparing (76) and (77) and putting $Y = \xi$ in (75), we get $X(\sigma + f + \mu) = \xi(f + \sigma + \mu)\eta(X)$, where $\sigma = \Omega - \frac{q}{2}(r + 4n^2) + \rho(2n-1)$. From this we get

$$d(\sigma + f + \mu) = \xi(f + \sigma + \mu)\eta. \tag{78}$$

This shows that $\sigma + f + \mu$ is invariant along the distribution $Ker(\eta)$ that means if $X \in \chi(M)$ then $X(\sigma + f + \mu) = d(\sigma + f + \mu)X = 0$.

Now, we taking the inner product with respect to arbitrary vector field Z after putting $X = \xi$ in (75) and using (31) and (32), we get

$$\begin{aligned} g(R(\xi, Y)Df, Z) &= \rho S(Y,Z) + \{2n\rho - \xi(\sigma) + \mu + \rho\}g(Y,Z) \\ &+ Y(\sigma + \mu)\eta(Z) - \{\xi(\mu) + \mu + \rho\}\eta(Y)\eta(Z). \end{aligned} \tag{79}$$

Also, from (13), we can easily deduce for arbitrary vector fields Y and Z on M

$$g(R(\xi, Y)Df, Z) = \xi(f)g(Y,z) - Y(f)\eta(Z). \tag{80}$$

Comparing the equations (79) and (80) and applying (78), we can write

$$\begin{aligned} S(Y,Z) &= \frac{1}{\rho}\{\xi(\sigma + f) - 2n\rho - \mu - \rho\}g(Y,Z) \\ &\quad + \frac{1}{\rho}\{\mu + \rho - \xi(\sigma + f)\}\eta(Y)\eta(Z). \end{aligned} \tag{81}$$

As the equation (81) holds good for arbitrary vector fields Y and Z, so the manifold is an η-Einstein. Now contracting (81), we get

$$\xi(f + \sigma) = \frac{\rho r}{2n} + 2n\rho + \mu + 2\rho. \tag{82}$$

Putting this value in (81), we acquire

$$S(Y,Z) = \tfrac{1}{2n}(r + 2n)g(Y,Z) - \tfrac{1}{2n}(r + 4n^2 + 2n)\eta(Y)\eta(Z)$$

this shows that for arbitrary vector fields Y and Z on M which is same as (17). Now contracting (75) with respect to X reduces to

$$S(Y, Df) = \frac{\rho}{2}Y(r) + 2nY(\sigma) - 2n(\mu + \rho)\eta(Y), \tag{83}$$

which is hold for any $Y \in \chi(M)$. Now, taking into with (17), we compute

$$\begin{aligned} &(r + 2n)Y(f) - (r + 2n + 4n^2)\eta(Y)\xi(f) - n\rho Y(r) \\ &-4n^2Y(\sigma) - 4n^2(\mu + \rho)\eta(Y) = 0 \end{aligned} \tag{84}$$

for all $Y \in \chi(M)$. Now, putting $Y = \xi$ and then from (82), we can easily find the relation

$$\xi(r) = -2\rho(r + 2n + 4n^2). \tag{85}$$

As $d^2 = 0$ and $d\eta = 0$ from (78) we have $dr \wedge \eta = 0$ i.e. $dr(X)\eta(Y) - dr(Y)\eta(X) = 0$ for arbitrary $X, Y \in \chi(M)$. After putting $Y = \xi$ and then applying (85) this reduces to $X(r) = -2\rho(r + 2n + 4n^2)\xi$. Since X is an arbitrary vector field so, we can say that

$$Dr = -2\rho(r + 2n + 4n^2)\xi. \tag{86}$$

Let Y be a vector field of the distribution $Ker(\eta)$. Then (84) provides

$$\rho(r+2n)Y(f) - 4n^2 Y(\sigma) = 0.$$

Now using (78) and (82) we obtain $\{\rho(r+2n)+4n^2\}Y(f) = 0,$. From this and using the relation $r^* = r + 4n^2$ we can write

$$\{(r+4n^2)\rho + 4n(1-\rho) + 2n\rho\}(Df - \xi(f)\xi) = 0.$$

If $r = -\frac{1}{\rho}\{4n(1-\rho)+2n\rho\} - 4n^2$, then from (17) we obtain that the manifold is η-Einstein.

If $r \neq -\frac{1}{\rho}\{4n(1-\rho)+2n\rho\} - 4n^2$, on some open set Q of M, then $Df = \xi(f)\xi$ on that open set that is, the potential vector field is pointwise collinear with the characteristic vector field ξ. This completes the proof. □

6. Example of a 5-dimensional almost Kenmotsu manifold admitting a gradient almost ∗-η-RYS

Let us consider the set $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$ as our manifold where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. The vector fields defined below:

$$e_1 = e^{-v}\frac{\partial}{\partial x}, \qquad e_2 = e^{-v}\frac{\partial}{\partial y}, \qquad e_3 = e^{-v}\frac{\partial}{\partial z}, \qquad e_4 = e^{-v}\frac{\partial}{\partial u}, \qquad e_5 = \frac{\partial}{\partial v}$$

are linearly independent at each point of M. We define the metric g as

$$g(e_i,e_j) = \begin{cases} 1, & \text{if } i = j \textit{ and } i,j \in \{1,2,3,4,5\} \\ 0, & \text{otherwise.} \end{cases}$$

Let η be a 1-form defined by $\eta(X) = g(X,e_5)$, for arbitrary $X \in \chi(M)$. Let us define (1,1)-tensor field ϕ as:

$$\phi(e_1) = e_3, \qquad \phi(e_2) = e_4, \qquad \phi(e_3) = -e_1, \qquad \phi(e_4) = -e_2, \qquad \phi(e_5) = 0.$$

Then it satisfy the relations $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, where $\xi = e_5$ and X, Y is arbitrary vector field on M. So, (M,ϕ,ξ,η,g) defines an almost contact structure on M.

We can now deduce that,

$$\begin{array}{llll}
[e_1,e_2] = 0 & [e_1,e_3] = 0 & [e_1,e_4] = 0 & [e_1,e_5] = e_1 \\
[e_2,e_1] = 0 & [e_2,e_3] = 0 & [e_2,e_4] = 0 & [e_2,e_5] = e_2 \\
[e_3,e_1] = 0 & [e_3,e_2] = 0 & [e_3,e_4] = 0 & [e_3,e_5] = e_3 \\
[e_4,e_1] = 0 & [e_4,e_2] = 0 & [e_4,e_3] = 0 & [e_4,e_5] = e_4 \\
[e_5,e_1] = -e_1 & [e_5,e_2] = -e_2 & [e_5,e_3] = -e_3 & [e_5,e_4] = -e_4.
\end{array}$$

Let ∇ be the Levi-Civita connection of g. Then from *Koszul's formula* for arbitrary $X, Y, Z \in \chi(M)$ given by:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) \\ &- g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

we can have:

$$\begin{array}{lllll}
\nabla_{e_1}e_1 = -e_5 & \nabla_{e_1}e_2 = 0 & \nabla_{e_1}e_3 = 0 & \nabla_{e_1}e_4 = 0 & \nabla_{e_1}e_5 = e_1 \\
\nabla_{e_2}e_1 = 0 & \nabla_{e_2}e_2 = -e_5 & \nabla_{e_2}e_3 = 0 & \nabla_{e_2}e_4 = 0 & \nabla_{e_2}e_5 = e_2 \\
\nabla_{e_3}e_1 = 0 & \nabla_{e_3}e_2 = 0 & \nabla_{e_3}e_3 = -e_5 & \nabla_{e_3}e_4 = 0 & \nabla_{e_3}e_5 = e_3 \\
\nabla_{e_4}e_1 = 0 & \nabla_{e_4}e_2 = 0 & \nabla_{e_4}e_3 = 0 & \nabla_{e_4}e_4 = -e_5 & \nabla_{e_4}e_5 = e_4 \\
\nabla_{e_5}e_1 = 0 & \nabla_{e_5}e_2 = 0 & \nabla_{e_5}e_3 = 0 & \nabla_{e_5}e_4 = 0 & \nabla_{e_5}e_5 = 0.
\end{array}$$

Therefore $(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ is satisfied for arbitrary $X, Y \in \chi(M)$. So (M, ϕ, ξ, η, g) becomes a Kenmotsu manifold.
The non-vanishing components of curvature tensor are:

$$
\begin{array}{lll}
R(e_1, e_2)e_2 = -e_1 & R(e_1, e_3)e_3 = -e_1 & R(e_1, e_4)e_4 = -e_1 \\
R(e_1, e_5)e_5 = -e_1 & R(e_1, e_2)e_1 = e_2 & R(e_1, e_3)e_1 = e_3 \\
R(e_1, e_4)e_1 = e_4 & R(e_1, e_5)e_1 = e_5 & R(e_2, e_3)e_2 = e_3 \\
R(e_2, e_4)e_2 = e_4 & R(e_2, e_5)e_2 = e_5 & R(e_2, e_3)e_3 = -e_2 \\
R(e_2, e_4)e_4 = -e_2 & R(e_2, e_5)e_5 = -e_2 & R(e_3, e_4)e_3 = e_4 \\
R(e_3, e_5)e_3 = e_5 & R(e_3, e_4)e_4 = -e_3 & R(e_4, e_5)e_4 = e_5 \\
R(e_5, e_3)e_5 = e_3 & R(e_5, e_4)e_5 = e_4. &
\end{array}
$$

Now from the above results we have, $S(e_i, e_i) = -4$ for $i = 1, 2, 3, 4, 5$ and

$$S(X, Y) = -4g(X, Y) \ \ \forall X, Y \in \chi(M). \tag{87}$$

Contracting this we have $r = \sum_{i=1}^{5} S(e_i, e_i) = -20 = -2n(2n + 1)$ where dimension of the manifold $2n + 1 = 5$. Also, we have

$$S^*(e_i, e_i) = \begin{cases} -1, & \text{if } i = 1, 2, 3, 4 \\ 0, & \text{if } i = 5. \end{cases}$$

and $r^* = r + 4n^2 = -20 + 16 = -4$. So

$$S^*(X, Y) = -g(X, Y) + \eta(X)\eta(Y) \ \ \forall X, Y \in \chi(M). \tag{88}$$

Now, we consider a vector field V as

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{89}$$

Then from the above results we can justify that

$$(\mathcal{L}_V g)(X, Y) = 4\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{90}$$

which holds for all $X, Y \in \chi(M)$. From (88) and (90), we can conclude that g represents a $*$-η-RYS i.e., it satisfies (1) for potential vector field V defined by (89) for $\Omega = \rho - 2q - 2$ and $\mu = 2$. Here, we see that $\xi r = 0$ implies $\xi r^* = 0$ i. e., ξ leaves the scalar curvature invariant and from the identity (88), we get that the manifold becomes η-Einstein and $r^* = -4 = -2n$ i. e., $r = -20 = -2n(2n + 1) =$. Therefore, Theorem 5.1 are verified from this example.

7. Conclusion

In this manuscript to analyse solutions of (1) and signalize Einstein and quasi-Einstein metrics, we have employed the methods of local Riemannian or semi-Riemannian geometry in a large class of metrics of $*$-η-RYS on contact geometry, specially on Kenmotsu manifolds. Besides, our results creates a requisite and persuasion mantle in the field of differential geometry. Also, $*$-η-RYS has significant and motivational contribution in the area of mathematical physics, general relativity and quantum cosmology, further research of complex geometry. The physical characteristics and idiosyncrasy of $*$-η-RYS can be thought from references [14, 37]. There are some questions arise from our article to study further research.

1. Is the Theorem 5.1 true without assuming an invariant condition of scalar curvature?
2. If we consider non-zero vector field V is not collinear with ξ, then whether Theorem 5.2 is true?
3. Whether the results of the this paper are also true for nearly Kenmotsu, paracontact manifolds and f-cosymplectic manifolds?

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References

[1] D. E. Blair, *Riemannian geometry of contact and sympletic manifolds*, Birkhauser, Second Edition, 2010.
[2] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds and nullity distributions*, J. Geom. **93** (2009), 46–61.
[3] S. Dey and S. Roy, *Characterization of general relativistic spacetime equipped with η-Ricci-Bourguignon soliton*, Journal of Geometry and Physics, **178** (2022), 104578.
[4] S. Dey and N. B. Turki, *$*$-η-Ricci Soliton and Gradient Almost $*$-η-Ricci Soliton Within the Framework of Para-Kenmotsu Manifolds*, Front. Phys. **10** (2022), 809405, https://doi.org/10.3389/fphy.2022.809405.
[5] S. Dey, S. Sarkar and A. Bhattacharyya, *$*$-η Ricci soliton and contact geometry*, appear to Ricerche di Matematica (2021), https://doi.org/10.1007/s11587-021-00667-0.
[6] S. Dey and S. Uddin, *Conformal η-Ricci almost solitons on Kenmotsu manifolds*, International Journal of Geometric Methods in Modern Physics, **19**(8) (2022), 2250121.
[7] S. Dey, *Conformal Ricci soliton and almost conformal Ricci soliton in paracontact geometry*, International Journal of Geometric Methods in Modern Physics, **20**, No. 3 (2023), 2350041.
[8] S. Dey, *Certain results of κ-almost gradient Ricci-Bourguignon soliton on pseudo-Riemannian manifolds*, Journal of Geometry and Physics, **184** (2022), 104725.
[9] S. Dey, *Certain results on gradient almost η-Ricci-Bourguignon soliton*, Quaestiones Mathematicae, **47(2)** (2024), 263–284.
[10] S .Dey and Y. J. Suh, *Geometry of almost contact metrics as an almost $*$-η-Ricci-Bourguignon solitons*, Reviews in Mathematical Physics, **35**(7) (2023), 2350012.
[11] D. Dey, *Almost Kenmotsu metric as Ricci-Yamabe soliton*, arXiv preprint arXiv:2005.02322 [math.DG] (2020).
[12] D. Dey, *$*$-Ricci-Yamabe Soliton and Contact Geometry*, arXiv preprint arXiv:2109.04220v1 [math.DG](2021).
[13] X. Dai, Y. Zhao, U. C. De, *$*$-Ricci soliton on (κ, μ)-almost Kenmotsu manifolds*, Open Math. **17** (2019), 874–882.
[14] K. L. Duggal, *Almost Ricci solitons and physical applications*, International Electronic Journal of Geometry, **10**(2) (2017), 1–10.
[15] S. Güler and M. Crasmareanu, *Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy*, Turk. J. Math. **43** (2019), 2361–2641.
[16] D. Ganguly, S. Dey, A. Ali and A. Bhattacharyya, *Conformal Ricci soliton and Quasi-Yamabe soliton on generalized Sasakian space form*, Journal of Geometry and Physics, **169** (2021), 104339.
[17] D. Ganguly, S. Dey and A. Bhattacharyya, *On trans-Sasakian 3-manifolds as η-Einstein solitons*, Carpathian Mathematical Publications, **13**(2) (2021), 460–474.
[18] A. Ghosh, *Ricci soliton and Ricci almost soliton within the framework of Kenmotsu manifols*, Carpathian Math. Publ. **11**(1) (2019), 59–69.
[19] S. K. Hui and D. Chakraborty, *Generalized Sasakian-space-forms and Ricci almost solitons with a conformal killing vector field*, New Trends in Mathematical Sciences, **4**(3) (2016), 263–269.
[20] D. Janssens and L. Vanheckr, *Almost contact structures and curvature tensors*, Kodai Math. J. **4** (1981), 1–27.
[21] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, The Tôhoku Mathematical Journal, **24** (1972), 93–103.
[22] M. Khatri and J. P. Singh, *Almost Ricci-Yamabe soliton on almost Kenmotsu manifolds*, arXiv:2110.12867v1 [math.DG](2021).
[23] Y. L. Li, D. Ganguly, S. Dey and A. Bhattacharyya, *Conformal η-Ricci solitons within the framework of indefinite Kenmotsu manifolds*, AIMS Math. **7** (2022), 5408–5430.
[24] Y. L. Li, S. Dey, S. Pahan and A. Ali, *Geometry of conformal η-Ricci solitons and conformal η-Ricci almost solitons on paracontact geometry*, Open Mathematics, **20(1)** (2022), 574–589.
[25] S. Mondal, S. Dey and A. Bhattacharyya, *Characterization of Almost η-Ricci Yamabe Soliton and Gradient Almost η-Ricci Yamabe Soliton on Almost Kenmotsu Manifolds*, Acta. Math. Sin.-English Ser. **39** (2023), 728–748.
[26] S. Mondal, S. Dey, Y. J. Suh and A. Bhattacharyya, *The Geometry of δ-Ricci-Yamabe Almost Solitons on Paracontact Metric Manifolds*, KYUNGPOOK Math. J. **63** (2023), 623–638.
[27] P. Mandal and S. K. Hui, *Submanifolds of cosymplectic statistical-space-forms*, Palestine Journal of Mathematics, **11**(4) (2022), 347–359.
[28] A. M. Pastore and V. Saltarelli, *Generalized nullity distribution on almost Kenmotsu manifolds*, Int. Elec. J. Geom. **4** (2011), 168–183.
[29] S. Roy, S. Dey, A. Bhattacharyya and S. K. Hui, *$*$-Conformal η-Ricci Soliton on Sasakian manifold*, Asian-European Journal of Mathematics, **15**, No. 2 (2022), 2250035.
[30] S. Roy, S. Dey and A. Bhattacharyya, *Conformal Einstein soliton within the framework of para-Kähler manifold*, Differential Geometry-Dynamical Systems(DGDS), Vol.**23** (2021), 235–243.
[31] S. Roy, S. Dey and A. Bhattacharyya, *A Kenmotsu metric as a conformal η-Einstein soliton*, Carpathian Mathematical Publications, **13**(1) (2021), 110–118.
[32] S. Roy, S. Dey, A. Bhattacharyya and Md. D. Siddiqi, *$*$-η-Ricci-Yamabe solitons on α-Cosymplectic manifolds with a quarter-symmetric metric connection*, arXiv:2109.04700v1 [math.DG](2021).
[33] S. Sarkar, S. Dey and X. Chen, *Certain results of conformal and $*$-conformal Ricci soliton on para-cosymplectic and para-Kenmotsu manifolds*, Filomat **35**:15 (2021), 5001–5015.

[34] M. D. Siddiqi and M. A. Akyol, *η-Ricci-Yamabe soliton on Riemannian submanifolds from Riemannian manifold*, preprint (2020), arXiv:2004.14124 [math.DG].

[35] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds*, Töhoku Math.J. **21** (1969), 21–38.

[36] V. Venkatesha, D. M. Naik and H. A. Kumara, *$*$-Ricci solitons and gradient almost $*$-Ricci solitons on Kenmotsu manifolds*, Math. Slovaca, **69** (6) (2019), 1447–1458.

[37] E. Woolgar, *Some applications of Ricci flow in physics*, Canadian Journal of Physics, **86**(4) (2008), 645–651.

[38] S. K. Yadav, S. K. Hui, M. Iqbal, P. Mandal and M. Aslam, *CR-submanifolds of SQ-Sasakian manifold*, Tamkang Journal of Mathematics, **54**(2) (2023), 121–138.

[39] K. Yano, *Integral Formulas in Riemannian Geometry*, Pure and Applied Mathematics, No.1 Marcel Dekker Inc, New York (1970).

[40] I. H. Yoldaş, *On Kenmotsu manifolds admitting η-Ricci-Yamabe solitons*, Int. J. geom. Methods Mod. Phys. **18**(12) (2021), 2150189.