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Extremal solutions of fuzzy fractional differential equations with ψ-Caputo derivative via monotone iterative method

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Abstract. In this paper, using the monotone iterative technique combined with the method of upper and lower solutions, the authors investigate the existence of extremal solutions to a class of fuzzy fractional differential equations (FFDEs) with time-delays involving the ψ-Caputo derivative. Also, with aid of generalized Grönwall inequality, we investigative the Hyers-Ulam stability of solution for the system under consideration. Lately, two examples are provided to illustrate the theoretical results.

1. Introduction

In recent years, there has been a growing interest in the study of a class of dynamic systems described by fuzzy fractional differential equations with time-delays. These equations introduce a novel perspective in mathematical modeling by incorporating the ψ-Caputo derivative, a specialized fractional derivative, which allows for a more accurate representation of complex phenomena in various fields of science and engineering. A novel and broad fractional derivative known as the ψ-Caputo fractional derivative was presented by Almeida [1]. You may find more information and properties of this fractional derivative in [2–5, 13, 14, 22–24]. In summary, the theory of ψ-Caputo fuzzy fractional differential equations is a relatively new area of research and the existence and stability results for these equations are still being actively investigated.
An efficient tool that provides existence results in a closed set formed by the lower and upper solutions is the monotone iterative approach combined with the method of upper and lower solutions. This method has been extensively studied and a range of nonlinear problems have been solved (see [15–20, 25]).
In [21] Derbazi et al. considered the following initial value problem of fractional differential equations

$$\begin{cases} \mathcal{D}_{0^+}^{\gamma;\psi}\mathbf{z}(u) = \mathfrak{h}\Big(u, \mathbf{z}(u)\Big), & u \in [0, T], \\ \mathbf{z}(u) = u^* \in \mathbb{R}, \end{cases} \tag{1}$$

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where $\mathcal{D}_{0^+}^{\gamma;\psi}$ is the ψ-Caputo fractional derivative of order $0 < \gamma < 1$ and $\mathfrak{h} : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function. They studied the existence and uniqueness of extremal solutions.
Wang et al. [29] studied the existence and stability of solutions of Caputo type FFDEs with time-delays of the form

$$\begin{cases} {}^C\mathcal{D}_{0^+}^{\gamma}\mathbf{z}(u) = \mathfrak{g}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma)\Big), & u \in [0,T], \\ \mathbf{z}(u) = \varphi(u), & u \in [-\sigma, 0], \end{cases} \tag{2}$$

where ${}^C\mathcal{D}_{0^+}^{\gamma}$ is the Caputo fractional derivative of order $0 < \gamma < 1$ and $\mathfrak{g} : \mathbf{I} \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is continuous function, $\sigma \in \mathbb{R}^+$ represents the delay, $\varphi(u)$ is history function. They established existence results by Schauder's fixed point theorem and a hypothetical condition. Also they showed the uniqueness of the solution by using Banach contraction principle. In addition, with aid of generalized Grönwall inequality the Ulam-Hyers stability are discussed.
Vinh An et al. [26] investigated sufficient conditions for the existence of extremal solutions of the following fuzzy fractional Volterra integral equations involving the generalized kernel functions

$$\mathbf{z}(u) \ominus_{gH} f(u) = \frac{1}{\Gamma(\gamma)} \int_a^u \psi'(v)\Big(\psi'(u) - \psi(v)\Big)^{\gamma-1} h(v, \mathbf{z}(v))dv, \tag{3}$$

where $0 < \gamma < 1$, $f : [a,c] \longrightarrow \mathbf{E^n}$ and $h : [a,c] \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is a given function. the results obtained are based on the method of upper and lower solutions coupled with its associated monotone iteration scheme.
Arhrrabi et al. [6]-[12] studied different types of fuzzy stochastic and fuzzy fractional differential equations. To the best of our knowledge, no results have been published on the existence of extremal solutions to systems of fuzzy fractional differential equations with ψ-Caputo derivatives using the monotone iterative method. As a consequence, we want to fill the gap in the literature and advance this field of study.
Here, we are concerned with a novel class of fuzzy fractional differential equations with ψ-Caputo derivative that are motivated by the aforementioned studies:

$$\begin{cases} \mathcal{D}_{0^+}^{\gamma;\psi}\mathbf{z}(u) = \mathfrak{f}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma)\Big), & u \in \mathbf{I} := [0,d], \\ \mathbf{z}(u) = \varphi(u), & u \in [-\sigma, 0], \end{cases} \tag{4}$$

where $0 < \zeta < 1$ and $\mathcal{D}_{0^+}^{\gamma;\psi}$ is the ψ-Caputo fractional derivative of order $0 < \gamma < 1$ and $\mathfrak{f} : \mathbf{I} \times \mathbf{E^n} \times \mathbf{E^n} \longrightarrow \mathbf{E^n}$ is continuous function, $\sigma \in \mathbb{R}^+$ represents the delay, $\varphi(u)$ is history function.
The rest of the paper is organized as follows. In Section 2, We introduce some essential definitions and propositions. The existence of extremal solutions of FFDEs are given in Section 3. Afterwards, in Section 4 Ulam–Hyers stability result of system under consideration is established. Section 5 includes two examples to demonstrate the usefulness of our findings. The last section is where you come to a conclusion.

2. Preliminaries

The definitions and propositions that are utilized throughout this paper are introduced in this part.

Definition 2.1. *[29] The set of fuzzy subsets of $\mathbb{R}^n$ is denoted by $\mathbf{E^n} := \{\Upsilon : \mathbb{R}^n \longrightarrow [0,1]\}$ which satisfies:*

(i) *Υ is upper semicontinous on $\mathbb{R}^n$,*

(ii) *Υ is fuzzy convex, i.e, for $0 \leq \lambda \leq 1$*

$$\Upsilon\Big(\lambda z_1 + (1-\lambda)z_2\Big) \geq min\left\{\Upsilon(z_1), \Upsilon(z_2)\right\}, \ \forall z_1, z_2 \in \mathbb{R}^n,$$

(*iii*) $[\Upsilon]^0 = \overline{\{z \in \mathbb{R}^n : \Upsilon(z) > 0\}}$ *is compact,*

(*iv*) Υ *is normal, i.e,* $\exists z_0 \in \mathbb{R}^n$ *such that* $\Upsilon(z_0) = 1$.

Remark 2.2. $\mathbf{E^n}$ *is called the space of fuzzy number.*

Definition 2.3. *[29] The p-level set of* $\Upsilon \in \mathbf{E^n}$ *is defined by:*
For $p \in (0,1]$, *we have* $[\Upsilon]^p = \{z \in \mathbb{R}^n | \Upsilon(z) \geq p\}$ *and for* $p = 0$ *we have* $[\Upsilon]^0 = \overline{\{z \in \mathbb{R}^n | \Upsilon(z) > 0\}}$.

Remark 2.4. *From Definition 2.1, it follows that the p-level set* $[\Upsilon]^p$ *of* Υ, *is a nonempty compact interval and* $[\Upsilon]^p = [\underline{\Upsilon}(p), \overline{\Upsilon}(p)]$. *Moreover,* $len([\Upsilon]^p) = l([\Upsilon]^p) = \overline{\Upsilon}(p) - \underline{\Upsilon}(p)$.

Definition 2.5. *[29] For addition and scalar multiplication in fuzzy set space* $\mathbf{E^n}$, *we have*

$$[\Upsilon_1 + \Upsilon_2]^p = [\Upsilon_1]^p + [\Upsilon_2]^p = \{z_1 + z_2 \mid z_1 \in [\Upsilon_1]^p, z_2 \in [\Upsilon_2]^p\},$$

and

$$[\alpha\Upsilon]^p = \alpha[\Upsilon]^p = \{\alpha z \mid z \in [\Upsilon]^p\},$$

for all $p \in [0,1]$.

Definition 2.6. *[29] The Hausdorff distance is given by*

$$\begin{aligned}\mathbf{D}_\infty\Big(\Upsilon_1, \Upsilon_2\Big) &= \sup_{0\leq p\leq 1} \Big\{|\underline{\Upsilon}_1(p) - \underline{\Upsilon}_2(p)|, |\overline{\Upsilon}_1(p) - \overline{\Upsilon}_2(p)|\Big\},\\ &= \sup_{0\leq p\leq 1} \mathcal{D}_H\Big([\Upsilon_1]^p, [\Upsilon_2]^p\Big).\end{aligned}$$

Remark 2.7. $\mathbf{E^n}$ *is complete metric space with the above definition (see [29, 30]) and we have the following properties of* $\mathbf{D}_\infty$:

$$\mathbf{D}_\infty\Big(\Upsilon_1 + \Upsilon_3, \Upsilon_2 + \Upsilon_3\Big) = \mathbf{D}_\infty\Big(\Upsilon_1, \Upsilon_2\Big),$$

$$\mathbf{D}_\infty\Big(\lambda\Upsilon_1, \lambda\Upsilon_2\Big) = |\lambda|\mathbf{D}_\infty\Big(\Upsilon_1, \Upsilon_2\Big),$$

$$\mathbf{D}_\infty\Big(\Upsilon_1, \Upsilon_2\Big) \leq \mathbf{D}_\infty\Big(\Upsilon_1, \Upsilon_3\Big) + \mathbf{D}_\infty\Big(\Upsilon_3, \Upsilon_2\Big),$$

for all $\Upsilon_1, \Upsilon_2, \Upsilon_3 \in \mathbf{E^n}$ *and* $\lambda \in \mathbb{R}^n$.

Definition 2.8. *[29] Let* Υ_1, $\Upsilon_2 \in \mathbf{E^n}$, *if there exists* $\Upsilon_3 \in \mathbf{E^n}$ *such that* $\Upsilon_1 = \Upsilon_2 + \Upsilon_3$, *then* Υ_3 *is called the Hukuhara difference of* Υ_1 *and* Υ_2 *noted by* $\Upsilon_1 \ominus \Upsilon_2$.

Definition 2.9. *[27] The generalized Hukuhara difference* (*gH-difference*) *of* $\Upsilon_1, \Upsilon_2 \in \mathbf{E^n}$ *is defined as follows:*

$$\Upsilon_1 \ominus_{gH} \Upsilon_2 = \Upsilon_3 \Leftrightarrow \begin{cases} (i)\ \ \Upsilon_1 = \Upsilon_2 + \Upsilon_3, \ if \quad len([\Upsilon_1]^p) \geq len([\Upsilon_2]^p). \\ \\ (ii)\ \ \Upsilon_2 = \Upsilon_1 + (-1)\Upsilon_3, \ if \quad len([\Upsilon_2]^p) \geq len([\Upsilon_1]^p). \end{cases}$$

Definition 2.10. *[29] Let a fuzzy function* $\Upsilon : [a,b] \longrightarrow \mathbf{E^n}$. *If for every* $p \in [0,1]$, *the function* $u \longmapsto len[\Upsilon(u)]^p$ *is increasing* (*decreasing*) *on* $[a,b]$, *then* Υ *is called increasing* (*decreasing*) *on* $[a,b]$.

Remark 2.11. *If* Υ *is increasing or decreasing, then we say that* Υ *is monotone on* $[a,b]$.

Definition 2.12. *[28] Let* $\Upsilon, \Psi \in \mathbf{E^n}$. *The partial orders* $\leq$ *and* $\geq$ *in the fuzzy space can be defined as follows:*
- $\Upsilon \leq \Psi$ *if and only if* $\underline{\Upsilon}(p) \geq \underline{\Psi}(p)$ *and* $\overline{\Upsilon}(p) \leq \overline{\Psi}(p)$.
- $\Upsilon \geq \Psi$ *if and only if* $\underline{\Upsilon}(p) \leq \underline{\Psi}(p)$ *and* $\overline{\Upsilon}(p) \geq \overline{\Psi}(p)$.

Remark 2.13. *The last definition is equivalent to* $[\Upsilon]^p \subseteq [\Psi]^p$ $([\Upsilon]^p \supseteq [\Psi]^p)$, *for all* $p \in [0,1]$.

Proposition 2.14. *[28] For* $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4 \in \mathbf{E^n}$, *we have the following assertions:*
(1) $\Upsilon_1 = \Upsilon_2 \Longleftrightarrow \Upsilon_1 \preceq \Upsilon_2$ *and* $\Upsilon_1 \succeq \Upsilon_2$.
(2) $\Upsilon_1 \preceq \Upsilon_2 \Longrightarrow \Upsilon_1 + \Upsilon_3 \preceq \Upsilon_2 + \Upsilon_3$.
(3) $\Upsilon_1 \preceq \Upsilon_2$ *and* $\Upsilon_3 \preceq \Upsilon_4 \Longrightarrow \Upsilon_1 + \Upsilon_3 \preceq \Upsilon_2 + \Upsilon_4$.
(4) *Let* $(\Upsilon_n)_{n\in\mathbb{N}}$ *be an increasing sequence satisfying* $\Upsilon_n \longrightarrow \Upsilon$ *in* $\mathbf{E^n}$, *then* $\Upsilon_n \preceq \Upsilon$ $\forall n \in \mathbb{N}$.

Proposition 2.15. *[28] Let* $C\Big([c,d], \mathbf{E^n}\Big)$ *be the space of continuous fuzzy functions from* $[c,d]$ *into* $\mathbf{E^n}$ *and* $\Big(\mathbf{E^n}, \preceq\Big)$ *be a partially ordered space, then we have the following properties:*
(1) $\Big(C\Big([c,d], \mathbf{E^n}\Big), \preceq\Big)$ *is a partially ordered space.*
(2) *In the ordered space* $\Big(C\Big([c,d], \mathbf{E^n}\Big), \preceq\Big)$, *every pair of elements has an upper bound.*
(3) *Let* $(\mathbf{z}_n)_{n\in\mathbb{N}}$ *an increasing sequence in* $C\Big([c,d], \mathbf{E^n}\Big)$ *with the order* $\preceq$ *such that* $\mathbf{z}_n \longrightarrow \mathbf{z}$ *in* $C\Big([c,d], \mathbf{E^n}\Big)$, *then* $\mathbf{z}_n \preceq \mathbf{z}$ *for all* $n \in \mathbb{N}$.

Notation:
- $C\Big([c,d], \mathbf{E^n}\Big)$ denote the set of all continuous fuzzy functions.
- $AC\Big([c,d], \mathbf{E^n}\Big)$ denote the set of all absolutely continuous fuzzy functions on $[c,d]$ with value in $\mathbf{E^n}$.
- $C_{\gamma;\psi}\Big([c,d], \mathbf{E^n}\Big)$ denote the weighted space of the fuzzy function $\mathbf{z}$ on $[c,d]$ defined by

$$C_{\gamma;\psi}\Big([c,d], \mathbf{E^n}\Big) = \Big\{\mathbf{z} : [c,d] \longrightarrow \mathbf{E^n}, \Big((\psi(u) - \psi(c))\Big)^{\gamma} \mathbf{z}(u) \in C\Big([c,d], \mathbf{E^n}\Big)\Big\}.$$

Definition 2.16. *[27] The* ψ*-Riemann-Liouville fractional integral of order* $\gamma > 0$ *of function* $\mathbf{z} \in \mathbf{E^n}$ *on* $[c,d]$ *with respect to the nondecreasing differentiable function* $\psi : [c,d] \longrightarrow \mathbb{R}^+$ *with* $\psi'(u) \neq 0$ *is defined by*

$$\mathcal{I}^{\gamma;\psi}_{c+} \mathbf{z}(u) = \frac{1}{\Gamma(\gamma)} \int_c^u \psi'(v)\Big(\psi(u) - \psi(v)\Big)^{\gamma-1} \mathbf{z}(v)dv,$$

Definition 2.17. *[27] Let* $\mathbf{z}, \psi \in C^n\Big([c,d], \mathbf{E^n}\Big)$ *be two functions such that* ψ *is nondecreasing with* $\psi'(u) \neq 0$ *for all* $u \in [c,d]$. *The* ψ*-Caputo fractional derivative of order* $\gamma > 0$ *of a continuous function* $\mathbf{z}$ *is given by*

$$\mathcal{D}^{\gamma;\psi}_{c+} \mathbf{z}(u) := \mathcal{I}^{n-\gamma;\psi}_{c^+} \Big(\frac{1}{\psi'(v)} \frac{d}{du}\Big)^n \mathbf{z}(u),$$

where $n = [\gamma] + 1$ *for* $\gamma \notin \mathbb{N}$ *and* $\gamma = n$ *for* $\gamma \in \mathbb{N}$.

Definition 2.18. *A monotone fuzzy function* $\mathbf{z} \in C\Big([0,d], \mathbf{E^n}\Big)$ *is a solution of the system* (4) *if and only if* $\mathbf{z}$ *satisfies*

$$\mathbf{z}(u) \ominus_{gH} \varphi(0) = \frac{1}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u) - \psi(v)\Big)^{\gamma-1} \mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big)dv, \tag{5}$$

and $u \mapsto \mathcal{I}^{\gamma;\psi}_{c+} \mathfrak{f}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma)\Big)$ *is increasing on* $[0,d]$.

Remark 2.19. • *If* $\mathbf{z} \in C\Big([0,d], \mathbf{E^n}\Big)$ *such that* $len([\mathbf{z}(u)]^p) \geq len([\varphi(0)]^p)$, *then* (5) *becomes*

$$\mathbf{z}(u) = \varphi(0) + \frac{1}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u) - \psi(v)\Big)^{\gamma-1} \mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big)dv. \tag{6}$$

• *If* $\mathbf{z} \in C\Big([0,d], \mathbf{E^n}\Big)$ *such that* $len([\mathbf{z}(u)]^p) \leq len([\varphi(0)]^p)$, *then* (5) *becomes*

$$\mathbf{z}(u) = \varphi(0) \ominus \frac{(-1)}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u) - \psi(v)\Big)^{\gamma-1} \mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big)dv. \tag{7}$$

Remark 2.20. • *Let $\psi(u) = u$, then the equation* (5) *becomes the following Riemann–Liouville fuzzy fractional integral equations*

$$\mathbf{z}(u) \ominus_{gH} \varphi(0) = \frac{1}{\Gamma(\gamma)} \int_0^u \Big(u - v\Big)^{\gamma-1} \mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big) dv.$$

• *Let $\psi(u) = u^\rho$, then the equation* (5) *becomes the following Katugampola fuzzy fractional integral equations*

$$\mathbf{z}(u) \ominus_{gH} \varphi(0) = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_0^u v^{\rho-1}\Big(u^\rho - v^\rho\Big)^{\gamma-1} \mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big) dv.$$

• *Let $\psi(u) = ln(u)$, then the equation* (5) *becomes the following Hadamard fuzzy fractional integral equations*

$$\mathbf{z}(u) \ominus_{gH} \varphi(0) = \frac{1}{\Gamma(\gamma)} \int_0^u \Big(ln(u) - ln(v)\Big)^{\gamma-1} \mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big) \frac{dv}{v}.$$

Definition 2.21. *[28] • A monotone fuzzy function $\mathbf{z}^L \in\in C\Big([-\sigma, d], \mathbf{E^n}\Big)$ is said to be a lower solution for* (4) *if*

$$\begin{cases} \mathcal{D}^{\gamma;\psi}_{0^+} \mathbf{z}^L(u) \preceq \mathfrak{f}\Big(u, \mathbf{z}^L(u), \mathbf{z}^L(u-\sigma)\Big), & u \in \mathbf{I}, \\ \mathbf{z}^L(u) \preceq \varphi(u), & u \in [-\sigma, 0], \end{cases}$$

• *A monotone fuzzy function $\mathbf{z}^U \in\in C\Big([-\sigma, d], \mathbf{E^n}\Big)$ is said to be a upper solution for* (4) *if*

$$\begin{cases} \mathcal{D}^{\gamma;\psi}_{0^+} \mathbf{z}^U(u) \succeq \mathfrak{f}\Big(u, \mathbf{z}^U(u), \mathbf{z}^U(u-\sigma)\Big), & u \in \mathbf{I}, \\ \mathbf{z}^U(u) \succeq \varphi(u), & u \in [-\sigma, 0], \end{cases}$$

3. Existence and uniqueness

We make the following hypotheses concerning the coefficients of the system under consideration:

(**H**1) For all $\phi_1, \phi_2, \nu_1, \nu_2 \in \mathbf{E^n}$ and $u \in [-\sigma, d]$, there exist $N_1 > 0$ such that

$$\mathbf{D}_\infty\Big[\mathfrak{f}(u, \phi_1, \nu_1), \mathfrak{f}(u, \phi_2, \nu_2)\Big] \leq N_1\Big(\mathbf{D}_\infty[\phi_1, \phi_2] + \mathbf{D}_\infty[\nu_1, \nu_2]\Big),$$

(**H**2) For $u \in [-\sigma, d]$, there exist a positive constant N_2 such that

$$\mathbf{D}_\infty\Big[\varphi(u), \hat{0}\Big] \leq N_2.$$

We will now use the monotone iterative technique combined with the method of upper and lower solutions to demonstrate our results.

Theorem 3.1. *Assume that the hypotheses* (**H**1) *and* (**H**2) *are true, then there exists a unique monotone solution $\mathbf{z} \in [\mathbf{z}^L, \mathbf{z}^U]$ for the system* (4) *in $C\Big([-\sigma, d], \mathbf{E^n}\Big)$.*

Proof. We divide the subsequent proof into three steps.

Step 1: We'll demonstrate that system (4) has at least one solution. Consider the operator $\mathfrak{H} : C\Big([-\sigma, d], \mathbf{E^n}\Big) \longrightarrow C\Big([-\sigma, d], \mathbf{E^n}\Big)$ defined as follows

$$\mathfrak{H}(\mathbf{z}(u)) = \begin{cases} \varphi(0) \circledcirc \left(\frac{1}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u) - \psi(v)\Big)^{\gamma-1} \mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big) dv\right), u \in \mathbf{I}, \\ \varphi(u), \quad u \in [-\sigma, 0], \end{cases} \tag{8}$$

where $\circledcirc := \{+, \ominus(-1)\}$ and the fuzzy function $u \longmapsto \mathcal{I}_{0^+}^{\gamma;\psi}\mathfrak{f}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma)\Big)$ is increasing on $[0, d]$.

∘ Firstly, we will prove that $\mathfrak{H}$ is completely continuous. For this, let us prove that:

ⓐ- $\mathfrak{H}$ is continuous. Indeed, for any integer $n \geq 1$, define $\mathbf{z}_n(u) = \varphi(u)$ for all $u \in [-\sigma, 0]$. For all $u \in \mathbf{I}$

$$\mathfrak{H}(\mathbf{z}_n(u)) = \varphi(0) \circledcirc \left(\frac{1}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u)-\psi(v)\Big)^{\gamma-1}\mathfrak{f}\Big(v, \mathbf{z}_n(v), \mathbf{z}_n(v-\sigma)\Big)dv\right). \tag{9}$$

By using the properties of the metric $\mathbf{D}_\infty$ and hypothesis (**H**1), we have

$$\begin{aligned}
&\mathbf{D}_\infty\Big[\mathfrak{H}(\mathbf{z}_n(u)), \mathfrak{H}(\mathbf{z}(u))\Big] \\
&= \mathbf{D}_\infty\Bigg[\frac{1}{\Gamma(\gamma)}\int_0^u \frac{\psi'(v)\mathfrak{f}\Big(v, \mathbf{z}_n(v), \mathbf{z}_n(v-\sigma)\Big)}{\Big(\psi(u)-\psi(v)\Big)^{1-\gamma}}dv, \frac{1}{\Gamma(\gamma)}\int_0^u \frac{\psi'(v)\mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big)}{\Big(\psi(u)-\psi(v)\Big)^{1-\gamma}}dv\Bigg], \\
&\leq \frac{1}{\Gamma(\gamma)}\int_0^u \psi'(v)\Big(\psi(u)-\psi(v)\Big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathfrak{f}\Big(v, \mathbf{z}_n(v), \mathbf{z}_n(v-\sigma)\Big), \mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big)\Big]dv, \\
&\leq \frac{N_1\Big(\psi(u)-\psi(0)\Big)^\gamma}{\Gamma(\gamma+1)}\Big(\mathbf{D}_\infty\Big[\mathbf{z}_n(u), \mathbf{z}(u)\Big] + \mathbf{D}_\infty\Big[\mathbf{z}_n(u-\sigma), \mathbf{z}(u-\sigma)\Big]\Big),
\end{aligned}$$

or, by using the definition of $\mathbf{D}_\infty$, we have

$$\begin{aligned}
\mathbf{D}_\infty[\mathbf{z}_n(v-\sigma), \mathbf{z}(v-\sigma)] &= \sup_{0\leq p\leq 1}\max_{0\leq v\leq u}\{|\ \underline{\mathbf{z}_n}(v-\sigma, p) - \underline{\mathbf{z}}(v-\sigma, p)\ |, |\ \overline{\mathbf{z}_n}(v-\sigma, p) - \overline{\mathbf{z}}(v-\sigma, p)\ |\}, \\
&= \sup_{0\leq p\leq 1}\max_{-\sigma\leq\mu\leq u-\sigma}\{|\ \underline{\mathbf{z}_n}(\mu, p) - \underline{\mathbf{z}}(\mu, p)\ |, |\ \overline{\mathbf{z}_n}(\mu, p) - \overline{\mathbf{z}}(\mu, p)\ |\}, \\
&\leq \sup_{0\leq p\leq 1}\max_{-\sigma\leq\mu\leq 0}\{|\ \underline{\mathbf{z}_n}(\mu, p) - \underline{\mathbf{z}}(\mu, p)\ |, |\ \overline{\mathbf{z}_n}(\mu, p) - \overline{\mathbf{z}}(\mu, p)\ |\} \\
&\qquad + \sup_{0\leq p\leq 1}\max_{0\leq\mu\leq u-\sigma}\{|\ \underline{\mathbf{z}_n}(\mu, p) - \underline{\mathbf{z}}(\mu, p)\ |, |\ \overline{\mathbf{z}_n}(\mu, p) - \overline{\mathbf{z}}(\mu, p)\ |\}, \\
&\leq \sup_{0\leq p\leq 1}\max_{0\leq v\leq u}\{|\ \underline{\mathbf{z}_n}(v, p) - \underline{\mathbf{z}}(s, r)\ |, |\ \overline{\mathbf{z}_n}(v, p) - \overline{\mathbf{z}}(v, p)\ |\} = \mathbf{D}_\infty[\mathbf{z}_n(v), \mathbf{z}(v)].
\end{aligned}$$

Then, we get

$$\mathbf{D}_\infty\Big[\mathfrak{H}(\mathbf{z}_n(u)), \mathfrak{H}(\mathbf{z}(u))\Big] \leq \frac{2N_1\Big(\psi(u)-\psi(0)\Big)^\gamma}{\Gamma(\gamma+1)}\mathbf{D}_\infty\Big[\mathbf{z}_n(u), \mathbf{z}(u)\Big].$$

We can conclude that $\mathbf{D}_\infty\Big[\mathfrak{L}(\mathbf{z}_n(u)), \mathfrak{L}(\mathbf{z}(u))\Big] \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, $\mathfrak{H}$ is continuous.

ⓑ- We prove that there exists a positive constant ξ_1 and for all $\varsigma_1 > 0$ satisfying for all $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_1} := \Big\{\mathbf{z}(u) \in C\Big([-\sigma, d], \mathbf{E^n}\Big) | \mathbf{D}_\infty[\mathbf{z}(u), \hat{0}] \leq \varsigma_1\Big\}$ one has $\mathbf{D}_\infty\Big[\mathfrak{H}\Big(\mathbf{z}(u)\Big), \hat{0}\Big] \leq \xi_1$. In fact, for all $u \in \mathbf{I}$ and $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_1}$, we have

$$\begin{aligned}
\mathbf{D}_\infty\Big[\mathfrak{H}\Big(\mathbf{z}(u)\Big), \hat{0}\Big] &= \mathbf{D}_\infty\Big[\varphi(0) \circledcirc \Big(\frac{1}{\Gamma(\gamma)}\int_0^u \psi'(v)\Big(\psi(u)-\psi(v)\Big)^{\gamma-1}\mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big)dv\Big), \hat{0}\Big], \\
&\leq \mathbf{D}_\infty\Big[\varphi(0), \hat{0}\Big] + \frac{1}{\Gamma(\gamma)}\int_0^u \psi'(v)\Big(\psi(u)-\psi(v)\Big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big), \hat{0}\Big]dv, \\
&\leq N_2 + \frac{\Big(\psi(u)-\psi(0)\Big)^\gamma}{\Gamma(\gamma+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma)\Big), \hat{0}\Big].
\end{aligned}$$

Since the function $\mathfrak{f}$ is continuous, there is exist a constant $M_{\mathfrak{f}} > 0$ such that $\mathbf{D}_\infty\Big[\mathfrak{f}(u, \phi, \mu), \hat{0}\Big] \leq M_{\mathfrak{f}}$. Then

$$\mathbf{D}_\infty\Big[\mathfrak{H}\Big(\mathbf{z}(u)\Big), \hat{0}\Big] \leq N_2 + \frac{M_{\mathfrak{f}}\Big(\psi(d)-\psi(0)\Big)^\gamma}{\Gamma(\gamma+1)} := \xi_1.$$

Therefore, for every $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_1}$, we have $\mathbf{D}_\infty\Big[\mathfrak{H}\big(\mathbf{z}(u)\big), \hat{0}\Big] \leq \xi_1$, this implies that $\mathfrak{H}(\mathbf{B}_{\varsigma_1}) \subseteq \mathbf{B}_{\xi_1}$.
ⓒ- $\mathfrak{H}$ maps bounded set into equi-continuous set. Indeed, for each $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_2}$ and $u_1, u_2 \in \mathbf{I}$ such that $0 \leq u_1 < u_2 \leq d$, we have

$$\begin{aligned}
&\mathbf{D}_\infty\Big[\mathfrak{H}\big(\mathbf{z}(u_1)\big), \mathfrak{H}\big(\mathbf{z}(u_2)\big)\Big] \\
&= \mathbf{D}_\infty\Bigg[\frac{1}{\Gamma(\gamma)}\int_0^{u_1} \frac{\psi'(v)\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big)}{\big(\psi(u_1)-\psi(v)\big)^{1-\gamma}} dv, \frac{1}{\Gamma(\gamma)}\int_0^{u_2} \frac{\psi'(v)\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big)}{\big(\psi(u_2)-\psi(v)\big)^{1-\gamma}} dv\Bigg], \\
&\leq \mathbf{D}_\infty\Bigg[\frac{1}{\Gamma(\gamma)}\int_0^{u_1} \frac{\psi'(v)\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big)}{\big(\psi(u_1)-\psi(v)\big)^{1-\gamma}} dv, \frac{1}{\Gamma(\gamma)}\int_0^{u_1} \frac{\psi'(v)\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big)}{\big(\psi(u_2)-\psi(v)\big)^{1-\gamma}} dv \\
&\qquad + \frac{1}{\Gamma(\gamma)}\int_{u_1}^{u_2} \frac{\psi'(v)\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big)}{\big(\psi(u_2)-\psi(v)\big)^{1-\gamma}} dv\Bigg], \\
&\leq \frac{1}{\Gamma(\gamma)}\int_0^{u_1} \psi'(v)\Big|\big(\psi(u_1)-\psi(v)\big)^{\gamma-1} - \big(\psi(u_2)-\psi(v)\big)^{\gamma-1}\Big|\mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big), \hat{0}\Big] dv \\
&\qquad + \frac{1}{\Gamma(\gamma)}\int_{u_1}^{u_2} \psi'(v)\big(\psi(u_2)-\psi(v)\big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big), \hat{0}\Big] dv, \\
&\leq \frac{\big(\psi(u_1)-\psi(0)\big)^\gamma + \big(\psi(u_2)-\psi(u_1)\big)^\gamma - \big(\psi(u_2)-\psi(0)\big)^\gamma}{\Gamma(\gamma+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big), \hat{0}\Big] \\
&\qquad + \frac{\big(\psi(u_2)-\psi(u_1)\big)^\gamma}{\Gamma(\gamma+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big), \hat{0}\Big], \\
&\leq \frac{\big(\psi(u_1)-\psi(0)\big)^\gamma + 2\big(\psi(u_2)-\psi(u_1)\big)^\gamma - \big(\psi(u_2)-\psi(0)\big)^\gamma}{\Gamma(\gamma+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\big), \hat{0}\Big], \\
&\leq \frac{\big(\psi(u_1)-\psi(0)\big)^\gamma + 2\big(\psi(u_2)-\psi(u_1)\big)^\gamma - \big(\psi(u_2)-\psi(0)\big)^\gamma}{\Gamma(\gamma+1)} M_{\mathfrak{f}} := \Psi M_{\mathfrak{f}}.
\end{aligned}$$

We have Ψ is independent of $\mathbf{z}(u)$ and $\Psi \longrightarrow 0$ as $u_2 \longrightarrow u_1$. Then, we obtain

$$\mathbf{D}_\infty\Big[\mathfrak{H}\big(\mathbf{z}(u_1)\big), \mathfrak{H}\big(\mathbf{z}(u_2)\big)\Big] \longrightarrow 0.$$

It means that $\mathfrak{H}(\mathbf{B}_{\varsigma_2})$ is equi-continuous. Then, according to Ascoli-Arzelà Theorem, $\mathfrak{H}$ is completely continuous.
∘ Secondly, we will prove that there is a closed, convex and bounded subset $\mathbf{B}_\xi = \Big\{\mathbf{z}(u) \in C\big([-\sigma, d], \mathbf{E^n}\big) | \mathbf{D}_\infty[\mathbf{z}(u), \hat{0}] \leq \xi\Big\}$ such that $\mathfrak{H}(\mathbf{B}_\xi) \subseteq \mathbf{B}_\xi$. We know that $\mathbf{B}_\xi$ is a closed, convex and bounded subset of $C\big([-\sigma, d], \mathbf{E^n}\big)$ for all $\xi > 0$. Suppose that for all $\xi > 0$, $\exists \mathbf{z}_\xi(u) \in \mathbf{B}_\xi$ such that $\mathfrak{H}(\mathbf{z}_\xi(u)) \notin \mathcal{B}_\xi$, that is $\mathbf{D}_\infty\Big[\mathfrak{H}(\mathbf{z}_\xi(u)), \hat{0}\Big] > \xi$. Then

$$\begin{aligned}
\xi &< \mathbf{D}_\infty\Big[\mathfrak{H}(\mathbf{z}_\xi(u)), \hat{0}\Big] = \mathbf{D}_\infty\Bigg[\varphi(0) \circledcirc \bigg(\frac{1}{\Gamma(\gamma)}\int_0^u \psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathfrak{f}\big(v, \mathbf{z}_\xi(v), \mathbf{z}_\xi(v-\sigma)\big) dv\bigg), \hat{0}\Bigg], \\
&\leq \mathbf{D}_\infty\Big[\varphi(0), \hat{0}\Big] + \frac{1}{\Gamma(\gamma)}\int_0^u \psi'(v)\big(\psi(u)-\psi(v)\big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}_\xi(v), \mathbf{z}_\xi(v-\sigma)\big), \hat{0}\Big] dv, \\
&\leq N_2 + \frac{\big(\psi(u)-\psi(0)\big)^\gamma}{\Gamma(\gamma+1)}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}_\xi(v), \mathbf{z}_\xi(v-\sigma)\big), \hat{0}\Big], \\
&\leq N_2 + \frac{\big(\psi(u)-\psi(0)\big)^\gamma}{\Gamma(\gamma+1)} M_{\mathfrak{f}}.
\end{aligned}$$

Taking limit as $\xi \longrightarrow +\infty$, we obtain that $N_2 + \frac{\left(\psi(u)-\psi(0)\right)^{\gamma}}{\Gamma(\gamma+1)}M_{\mathfrak{f}} \longrightarrow +\infty$ which is in contradiction with $N_2 + \frac{\left(\psi(u)-\psi(0)\right)^{\gamma}}{\Gamma(\gamma+1)}M_{\mathfrak{f}}$ is bounded. Therefore, for every positive constant ξ, we obtain $\mathfrak{H}(\mathbf{B}_{\xi}) \subseteq \mathbf{B}_{\xi}$. By means of Schauder's fixed point Theorem implying that there is at least one solution to the system (4).

Step 2: In this step, we demonstrate that $\mathbf{z} \in [\mathbf{z}^L, \mathbf{z}^U]$. For this, we consider for all $\varepsilon > 0$

$$\mathbf{z}^L_{\varepsilon}(u) + \varepsilon(\sigma + u) = \mathbf{z}^L(u),$$

and

$$\mathbf{z}^U_{\varepsilon}(u) = \mathbf{z}^U(u) + \varepsilon(\sigma + u).$$

After that, we obtain

$$\mathbf{z}^L_{\varepsilon}(u) \prec \mathbf{z}^L(u), \quad u \in [-\sigma, d],$$

and

$$\mathbf{z}^U_{\varepsilon}(u) \succ \mathbf{z}^U(u), \quad u \in [-\sigma, d].$$

Consequently, it is simple to deduce

$$\begin{cases} \mathbf{z}^L_{\varepsilon}(u) \preceq \mathbf{z}^L(u) \preceq \mathbf{z}^U(u) \preceq \mathbf{z}^U_{\varepsilon}(u), & u \in \mathbf{I}, \\ \mathbf{z}^L_{\varepsilon}(u) \preceq \mathbf{z}^L(u) \preceq \mathbf{z}^U(u) \preceq \mathbf{z}^U_{\varepsilon}(u), & u \in [-\sigma, 0], \end{cases}$$

and

$$\mathbf{z}^L_{\varepsilon}(0) \preceq \mathbf{z}^L(0) \preceq \mathbf{z}^U(0) \preceq \mathbf{z}^U_{\varepsilon}(0),$$

where $\mathbf{z}^L(u)$, $\mathbf{z}^U(u)$ are the lower and upper solutions of the system (4).
Therefore, we get

$$\mathbf{z}^L_{\varepsilon}(u) \preceq \mathbf{z}^L(u) \preceq \mathbf{z}(u) \preceq \mathbf{z}^U(u) \preceq \mathbf{z}^U_{\varepsilon}(u), \quad u \in [-\sigma, 0],$$

and

$$\mathbf{z}^L_{\varepsilon}(0) \preceq \mathbf{z}(0) \preceq \mathbf{z}^U_{\varepsilon}(0),$$

where $\mathbf{z}(u)$ is a solutions of the system (4).
Now, we must demonstrate that

$$\mathbf{z}^L_{\varepsilon}(u) \prec \mathbf{z}(u) \prec \mathbf{z}^U_{\varepsilon}(u), \quad u \in \mathbf{I}.$$

If the previous claim is false, then there exists $u_1 \in \mathbf{I}$ such that

$$\mathbf{z}(u_1) = \mathbf{z}^L_{\varepsilon}(u_1),$$

and

$$\mathbf{z}^L_{\varepsilon}(u) \prec \mathbf{z}(u) \prec \mathbf{z}^U_{\varepsilon}(u), \quad u \in \mathbf{I} \backslash \{u_1\}. \tag{10}$$

Moreover, we have $\mathcal{D}^{\gamma;\psi}_{0^+}\mathbf{z}(u_1) \succeq \mathcal{D}^{\gamma;\psi}_{0^+}\mathbf{z}^U_{\varepsilon}(u_1)$, which give

$$\begin{aligned} \mathcal{D}^{\gamma;\psi}_{0^+}\mathbf{z}^U_{\varepsilon}(u_1) &= \mathfrak{f}\Big(u_1, \mathbf{z}^U_{\varepsilon}(u_1), \mathbf{z}^U_{\varepsilon}(u_1 - \sigma))\Big) \\ &\preceq \mathcal{D}^{\gamma;\psi}_{0^+}\mathbf{z}(u_1) = \mathfrak{f}\Big(u_1, \mathbf{z}(u_1), \mathbf{z}(u_1 - \sigma))\Big). \end{aligned}$$

Therefore, from $\mathbf{z}(u) \preceq \mathbf{z}^U_{\varepsilon}(u)$, $u \in [-\sigma, 0]$ and (10), we get

$$\mathbf{z}(u_1 + \lambda) \prec \mathbf{z}^U_{\varepsilon}(u_1 + \lambda), \quad \lambda \in [-\sigma, 0].$$

Given the increasing property of the function $\mathfrak{f}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma))\Big)$, it is simple to get

$$\mathfrak{f}\Big(u_1, \mathbf{z}_\varepsilon^U(u_1), \mathbf{z}_\varepsilon^U(u_1-\sigma))\Big) \geq \mathfrak{f}\Big(u_1, \mathbf{z}(u_1), \mathbf{z}(u_1-\sigma))\Big),$$

which is a contradiction. Hence we know that $\mathbf{z}(u) \prec \mathbf{z}_\varepsilon^U(u)$, $u \in \mathbf{I}$.
Similarly, we can prove that $\mathbf{z}_\varepsilon^L(u) \prec \mathbf{z}(u)$, $u \in \mathbf{I}$. Therefore, $\mathbf{z}_\varepsilon^L(u) \prec \mathbf{z}(u) \prec \mathbf{z}_\varepsilon^U(u)$, $u \in \mathbf{I}$ holds. Now as $\varepsilon \longrightarrow 0$, we get that $\mathbf{z}^L(u) \preceq \mathbf{z}(u) \preceq \mathbf{z}^U(u)$.

Step 3: In this last step, we shall prove the uniqueness. Let $\mathbf{z}$ and $\mathbf{w}$ two different solutions of (4). We have $\mathbf{z}(u) = \mathbf{w}(u) = \varphi(u)$ for all $u \in [-\sigma, 0]$, and for all $u \in \mathbf{I}$ we have

$$\begin{aligned}
&\mathbf{D}_\infty\Big[\mathbf{z}(u), \mathbf{w}(u)\Big] \\
&= \mathbf{D}_\infty\Big[\frac{1}{\Gamma(\gamma)}\int_0^u \frac{\psi'(v)\mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big)}{\Big(\psi(u)-\psi(v)\Big)^{1-\gamma}}dv, \frac{1}{\Gamma(\gamma)}\int_0^u \frac{\psi'(v)\mathfrak{f}\Big(v, \mathbf{w}(v), \mathbf{w}(v-\sigma)\Big)}{\Big(\psi(u)-\psi(v)\Big)^{1-\gamma}}ds\Big], \\
&\leq \frac{1}{\Gamma(\gamma)}\int_0^u \psi'(v)\Big(\psi(u)-\psi(v)\Big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathfrak{f}(s, \mathbf{z}(v), \mathbf{z}(v-\sigma)), \mathfrak{f}(v, \mathbf{w}(v), \mathbf{w}(v-\sigma))\Big]dv, \\
&\leq \frac{N_1}{\Gamma(\gamma)}\int_0^u \psi'(v)\Big(\psi(u)-\psi(v)\Big)^{\gamma-1}\Big(\mathbf{D}_\infty\Big[\mathbf{z}(v), \mathbf{w}(v)\Big] + \mathbf{D}_\infty\Big[\mathbf{z}(v-\sigma), \mathbf{w}(v-\sigma)\Big]\Big)dv, \\
&\leq \frac{2N_1}{\Gamma(\gamma)}\int_0^u \psi'(v)\Big(\psi(u)-\psi(v)\Big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathbf{z}(v), \mathbf{w}(v)\Big]dv.
\end{aligned}$$

Therefore, the generalized Grönwall inequality implies that

$$\mathbf{D}_\infty\Big[\mathbf{z}(u), \mathbf{w}(u)\Big] = 0.$$

As a result, the solution of (4) is unique. □

4. Stability result

In this section, Ulam–Hyers stability is demonstrated for the system (4).

Definition 4.1. *[29] For all $\varepsilon > 0$, if $\mathbf{z} \in C\Big([-\sigma, d], \mathbf{E^n}\Big)$ satisfies*

$$\begin{cases} \mathbf{D}_\infty\Big[\mathbf{z}(u), \varphi(0) \circledcirc \Big(\frac{1}{\Gamma(\gamma)}\int_0^u \psi'(v)\Big(\psi(u)-\psi(v)\Big)^{\gamma-1}\mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big)dv\Big)\Big] \leq \varepsilon, & u \in \mathbf{I}, \\ \mathbf{z}(u) = \varphi(u), \quad u \in [-\sigma, 0], \end{cases} \tag{11}$$

then there is a real number $C_{\mathfrak{f}} > 0$ and a solution $\mathbf{w} \in C\Big([-\sigma, d], \mathbf{E^n}\Big)$ of system (4) *with*

$$\mathbf{D}_\infty\Big[\mathbf{w}(u), \mathbf{z}(u)\Big] \leq \varepsilon C_{\mathfrak{f}},$$

which implies the solution to system (4) *is Ulam-Hyers stable.*

Remark 4.2. $\mathbf{z} \in C\Big([-\sigma, d], \mathbf{E^n}\Big)$ *is a solution of* (11) *if and only if* $\exists \phi \in C\Big([-\sigma, d], \mathbf{E^n}\Big)$ *such that*

(*i*) $\mathbf{D}_\infty\Big[\phi(u), \hat{0}\Big] \leq \varepsilon$,

(*ii*) $\begin{cases} \mathscr{D}_{0^+}^{\gamma;\psi}\mathbf{z}(u) = \mathfrak{f}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma)\Big) + \phi(u), & u \in \mathbf{I}, \\ \mathbf{z}(u) = \varphi(u), & u \in [-\sigma, 0], \end{cases}$

Theorem 4.3. *Under the hypotheses* (**H**1) *and* (**H**2), *the system* (4) *is Ulam–Hyers stable.*

Proof. Let $\mathbf{z}(u)$ be the solution of the system (11) and $\mathbf{w}(u)$ be the solution of the proposed system (4). In light of Definition 2.18 and Remark 2.19, we have

$$\mathbf{z}(u) = \varphi(0) \circledcirc \Big[\frac{1}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u) - \psi(v)\Big)^{\gamma-1}\Big(\mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big) + \phi(v)\Big)dv\Big], \tag{12}$$

and $u \mapsto \mathcal{I}_{c+}^{\gamma;\psi}\Big(\mathfrak{f}\Big(u, \mathbf{z}(u), \mathbf{z}(u-\sigma)\Big) + \phi(u)\Big)$ is increasing on $[0, d]$, where $\circledcirc := \{+, \ominus(-1)\}$. Note that for $u \in [-\sigma, 0]$, we have $\mathbf{D}_\infty\Big[\mathbf{w}(u), \mathbf{z}(u)\Big] = 0$.
For $u \in \mathbf{I}$, we have

$$\begin{aligned}
&\mathbf{D}_\infty\Big[\mathbf{w}(u), \mathbf{z}(u)\Big]\\
&\leq \frac{1}{\Gamma(\gamma)} \int_0^u \psi'(s)\Big(\psi(u) - \psi(v)\Big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathfrak{f}\Big(v, \mathbf{w}(v), \mathbf{w}(v-\sigma)\Big), \mathfrak{f}\Big(v, \mathbf{z}(v), \mathbf{z}(v-\sigma)\Big)\Big]dv\\
&\qquad + \frac{1}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u) - \psi(s)\Big)^{\gamma-1}\mathbf{D}_\infty\Big[\phi(v), \hat{0}\Big]dv,\\
&\leq \frac{N_1}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u) - \psi(v)\Big)^{\gamma-1}\Big(\mathbf{D}_\infty\Big[\mathbf{w}(v), \mathbf{z}(v)\Big] + \mathbf{D}_\infty\Big[\mathbf{w}(v-\sigma), \mathbf{z}(v-\sigma)\Big]\Big)dv + \frac{\Big(\psi(u) - \psi(0)\Big)^{\gamma}}{\Gamma(\gamma+1)}\varepsilon,\\
&\leq \frac{2N_1}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u) - \psi(v)\Big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathbf{w}(v), \mathbf{z}(v)\Big]dv + \frac{\Big(\psi(u) - \psi(0)\Big)^{\gamma}}{\Gamma(\gamma+1)}\varepsilon,\\
&\leq \frac{2N_1}{\Gamma(\gamma)} \int_0^u \psi'(v)\Big(\psi(u) - \psi(s)\Big)^{\gamma-1}\mathbf{D}_\infty\Big[\mathbf{w}(v), \mathbf{z}(v)\Big]dv + \frac{\Big(\psi(u) - \psi(0)\Big)^{\gamma}}{\Gamma(\gamma+1)}\varepsilon.
\end{aligned}$$

So, the generalized Grönwall inequality implies that

$$\mathbf{D}_\infty\Big[\mathbf{w}(u), \mathbf{z}(u)\Big] \leq \frac{\Big(\psi(u) - \psi(0)\Big)^{\gamma}}{\Gamma(\gamma+1)}\varepsilon \mathbf{E}_\gamma\Big(2N_1\Big(\psi(u) - \psi(0)\Big)^{\gamma}\Big).$$

Let

$$C_{\mathfrak{f}} = \frac{\Big(\psi(u) - \psi(0)\Big)^{\gamma}}{\Gamma(\gamma+1)}\mathbf{E}_\gamma\Big(2N_1\Big(\psi(u) - \psi(0)\Big)^{\gamma}\Big),$$

then

$$\mathbf{D}_\infty\Big[\mathbf{w}(u), \mathbf{z}(u)\Big] \leq \varepsilon C_{\mathfrak{f}}.$$

Therefore, from Definition 4.1, the system (4) is Ulam–Hyers stable. □

5. Example

In this section, we provide an illustration of the results from the previous part.

5.1. Example 1

Consider the following fuzzy fractional differential system

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{3};u^2}\mathbf{z}(u) = \frac{1}{12}\mathbf{z}(u) + \frac{1}{20}\mathbf{z}(u-1) + (-3, 0, 3), & u \in (0, 4],\\ \mathbf{z}(u) = (-u-2, 0, u+2), \quad u \in [-1, 0], \end{cases} \tag{13}$$

where $\gamma = \frac{1}{3}$ and $\mathfrak{f}(u, \mathbf{z}(u), \mathbf{z}(u-1)) = \frac{1}{12}\mathbf{z}(u) + \frac{1}{20}\mathbf{z}(u-1) + (-3, 0, 3)$. The verification demonstrates that all assumptions in Theorem 3.1 are met in full. Then, the problem (13) has an extremal solution $\mathbf{z} \in [\mathbf{z}^L, \mathbf{z}^U]$ on $[-1, 4]$. Also, we can verify that system (13) satisfies all assumptions in Theorem 4.3. Then, system (13) is Ulam–Hyers stable.

5.2. Example 2

Let

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{2};u}\mathbf{z}(u) = \frac{1}{\pi^2}\mathbf{z}(u)\log\left(\frac{1}{u}+1\right) + \frac{1}{\pi^2+1}\sin\mathbf{z}(u-2), & u \in (0,3], \\ \mathbf{z}(u) = (u+2, 0, -u-2), \quad u \in [-2,0], \end{cases} \tag{14}$$

where $\mathfrak{f}(u,\mathbf{z}(u),\mathbf{z}(u-2)) = \frac{1}{\pi^2}\mathbf{z}(u)\log\left(\frac{1}{u}+1\right) + \frac{1}{\pi^2+1}\sin\mathbf{z}(u-2)$ and $\gamma = \frac{1}{2}$.
For $\mathbf{z},\mathbf{w} \in \mathbf{E^n}$, we have

$$\mathbf{D}_\infty\Big[\mathfrak{f}(u,\mathbf{z}(u),\mathbf{z}(u-2)),\mathfrak{f}(u,\mathbf{w}(u),\mathbf{w}(u-2))\Big] \le N_1\Big(\mathbf{D}_\infty[\mathbf{z}(u),\mathbf{w}(u)] + \mathbf{D}_\infty[\mathbf{z}(u-2),\mathbf{w}(u-2)]\Big),$$

where $N_1 = \max\{\frac{1}{\pi^2},\frac{1}{\pi^2+1}\}$. Then, the problem (14) has an extremal solution $\mathbf{z} \in [\mathbf{z}^L,\mathbf{z}^U]$ on $[-1,4]$. Also, we can verify that system (14) satisfies all assumptions in Theorem 4.3. Then, system (14) is Ulam–Hyers stable.

6. Conclusion

This research has examined a class of ψ-Caputo fuzzy fractional differential equations with time delay in the sense of generalized Hukuhara differentiability. The monotone iterative technique combined with the method of upper and lower solutions are employed under Lipschitz conditions to demonstrate the existence of extremal solutions. Finally, by using the generalized Grönwall inequality, Ulam–Hyers stability result for the main system is provided.

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