Filomat 38:24 (2024), 8553–8564
https://doi.org/10.2298/FIL2424553G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Reproducing kernel Hilbert space method to solve fuzzy partial Volterra integro-differential equations

Ghaleb Gumah[a]

[a] *Department of Scientific Basic Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11134, Jordan*

Abstract. In this paper, a reproducing kernel Hilbert space method for the numerical solution of fuzzy partial Volterra integro-differential equations has been presented. The reproducing Hilbert space, kernel function properties, Gram-Schmidt orthogonalization process and the bounded linear operator in the same space have been developed, which helps this method to demonstrate the convergence analysis. Moreover, we present some lemmas and theorems to prove the convergence of the reproducing kernel Hilbert space method. In this method, we give the approximate solution of the fuzzy partial Volterra integro-differential equation as a Fourier series in the Hilbert space. In order to clear the efficiency of the proposed method, some numerical examples have been solved.

1. Introduction

In recent years, a great interest in fuzzy mathematics has been observed, due to the presence of many applications in several fields, especially in physics, engineering and economics [1, 2]. On the other hand, and in particular, it was necessary to increase interest in studying fuzzy ordinary differential equations [3, 4], fuzzy partial differential equations [5, 6, 7, 8, 9] and fuzzy integro-differential equations [10] through the publication of many papers related to these fuzzy equations. However, as a result of the efforts of some researchers, the fuzzy derivative was defined in [11]. Based on this definition and due to the difficulty of finding the exact solution for many of the fuzzy equations, a lot of papers emerged that discuss the numerical solutions of fuzzy ordinary differential equations [12], fuzzy partial differential equations [13, 14, 15] and fuzzy fractional differential equations [16].

Through our random search, we found two papers discussing how to find the exact solution for the fuzzy partial Volterra integro-differential equation (FPVIDE) using the fuzzy Laplace transform and the fuzzy Fourier transform [17, 18]. In addition, the existence and uniqueness of the solution to the FPVIDE were discussed in [17], but we could not find any paper discussing the numerical solution of the FPVIDE. Our main motivation and the central question for studying this paper is to construct a reproducing kernel Hilbert space (RKHS) method for finding numerical solutions of FPVIDEs under Hukuhara differentiability. More specifically, we consider the following general form of the FPVIDE:

$$A(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}+B(x,t)\frac{\partial u(x,t)}{\partial x}+C(x,t)\frac{\partial u(x,t)}{\partial t}+D(x,t)u(x,t)+\int_0^t k(t,s)u(x,s)ds=r(x,t), t>0, 0\le x\le l, \quad (1)$$

2020 *Mathematics Subject Classification.* Primary 35R13; Secondary 45K05, 45K05
Keywords. Fuzzy partial integro-differential equation, Hilbert space, Reproducing kernel function
Received: 07 June 2022; Revised: 02 December 2022; Accepted: 18 May 2024
Communicated by Marko Petković
Email address: ghaleb_gumah@bau.edu.jo (Ghaleb Gumah)

subject to the following fuzzy initial and fuzzy boundary conditions:

$$u(x,0) = \sigma_1(\alpha)f(x),\ 0 \le x \le l, \tag{2}$$

$$\frac{\partial u(x,0)}{\partial t} = \sigma_2(\alpha)g(x),\ 0 \le x \le l, \tag{3}$$

$$u(0,t) = \sigma_3(\alpha)h(t),\ t > 0, \tag{4}$$

where A, B, C and D are real valued continuous functions, $r(x,t)$ is a fuzzy function, σ_1, σ_2, $\sigma_3 \in R_F$ such that R_F is the set of all fuzzy numbers for all $\alpha \in [0,1]$ and $u(x,t)$ is an unknown fuzzy function which is a fuzzy solution to the FPVIDE.

Recently, the RKHS method has important scientific applications in applied mathematics. Specially, this method with kernel functions and Gram-Schmidt orthogonalization process has been applied to solve several types of equations, including singular two-point boundary value problems [19], hybrid fuzzy differential equations [20], boundary value problems for hyperbolic and parabolic integro-differential equations [21], system of fuzzy Volterra integro-differential equations [22], fuzzy integral equations [23,24] and fuzzy fractional differential equations [25, 26, 27]. Therefore, it is an effective numerical method for nonlinear problems without discretization. Generally, to demonstrate the RKHS method to solve FPVIDEs, we need the following steps:

- Building some appropriate Hilbert spaces.
- Writing the FPVIDE in the operator form.
- Proving some lemmas and theorems which help us to show the convergence of the proposed method.
- Building the algorithm of the RKHS to find the approximate fuzzy solutions to the FPVIDE.
- Presenting two numerical examples to verify the proposed algorithm.

In recent papers, the RKHS method become a fundamental tool in computational mathematics for obtaining the approximate solution to many fuzzy problems. The main motivation of this method reveals a fast convergence rate when the approximate solution is compared with the exact solution for fuzzy problems.

This paper is divided into six sections. In Section 2, we present some basic concepts in fuzzy numbers and fuzzy calculus. In Section 3, we construct some suitable Hilbert spaces with a kernel function to build the RKHS method. The convergence analysis of the RKHS method is introduced in Section 4. Some numerical examples are presented in Section 5. At the end of the paper, we summarize some of the results that were reached.

2. Preliminaries

Definition 2.1 ([20]) A fuzzy set $\lambda : \mathbb{R} \longrightarrow [0,1]$ is called a fuzzy number with α-cut representation $[\underline{\lambda}(\alpha), \overline{\lambda}(\alpha)]$ for all $\alpha \in [0,1]$ if $\lambda(\alpha s + (1-\alpha)t) \geqslant \min(\lambda(s), \lambda(t))$ for each s, $t \in \mathbb{R}$, there is $s \in \mathbb{R}$ such that $\lambda(s) = 1$, the set $\{s \in \mathbb{R} \mid \lambda(s) > \alpha\}$ is closed and the α-cut representation of λ is a compact interval.

Definition 2.2 If $\lambda = [\underline{\lambda}(\alpha), \overline{\lambda}(\alpha)]$ and $\mu = [\underline{\mu}(\alpha), \overline{\mu}(\alpha)]$ are two fuzzy numbers. We define the basic operations in the α-cut representation of the fuzzy numbers as:

i) $\lambda + \mu = [\underline{\lambda}(\alpha) + \underline{\mu}(\alpha), \overline{\lambda}(\alpha) + \overline{\mu}(\alpha)]$.

ii) $\lambda - \mu = [\underline{\lambda}(\alpha) - \overline{\mu}(\alpha), \overline{\lambda}(\alpha) - \underline{\mu}(\alpha)]$.

iii) $k\lambda = \Big\{[k\underline{\lambda}(\alpha), k\overline{\lambda}(\alpha)],\ \text{if } k \ge 0, \quad [k\overline{\lambda}(\alpha), k\underline{\lambda}(\alpha)],\ \text{if } k < 0.$

Definition 2.3 ([6]) Let $u : [a,b] \times [a,b] \longrightarrow R_F$ be a fuzzy-valued function with α-cut representation $[\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$. If the partial derivatives $\frac{\partial \underline{u}(x,t;\alpha)}{\partial x}$ and $\frac{\partial \overline{u}(x,t;\alpha)}{\partial x}$ exists and $\frac{\partial \underline{u}(x,t;\alpha)}{\partial x} \leq \frac{\partial \overline{u}(x,t;\alpha)}{\partial x}$, then the first-partial fuzzy derivative of u is defined as $\left[\frac{\partial u(x,t)}{\partial x}\right]_\alpha = \left[\frac{\partial \underline{u}(x,t;\alpha)}{\partial x}, \frac{\partial \overline{u}(x,t;\alpha)}{\partial x}\right]$, where R_F be the set of all fuzzy numbers.

Definition 2.4 ([6]) Let $u : [a,b] \times [a,b] \longrightarrow R_F$ be a fuzzy-valued function with $\alpha-$cut representation $[\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$. If the partial derivatives $\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}$ and $\frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}$ exists and $\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2} \leq \frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}$, then the second-partial fuzzy derivative of u is defined as $\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]_\alpha = \left[\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x^2}, \frac{\partial^2 \overline{u}(x,t;\alpha)}{\partial x^2}\right]$.

To provide Def. (2.3) and Def. (2.4), we can see that from the function $u(x,t) = \left[(\alpha-1)x^2e^t, (1-\alpha)x^2e^t\right]$ such that $\frac{\partial u(x,t)}{\partial x} = [2(\alpha-1)xe^t, 2(1-\alpha)xe^t]$ and $\frac{\partial^2 u(x,t)}{\partial x^2} = [2(\alpha-1)e^t, 2(1-\alpha)e^t]$.

3. Reproducing kernel Hilbert spaces

In this section, we introduce some reproducing kernel Hilbert spaces which are used to build the RKHS method.

Definition 3.1 ([22]) Hilbert space $W_2^1[0,T]$ consists of functions $f(t)$ defined on $[0,T]$ such that $f(t)$ is absolutely continuous and $f'(t) \in L_2[0,T]$. The norm for the space is:

$$\|f\|_{W_2^1} = \sqrt{(f(0))^2 + \int_0^T (f'(t))^2\, dt}. \tag{5}$$

Definition 3.2 ([22]) Hilbert space $W_2^2[0,T]$ consists of functions $f(t)$ defined on $[0,T]$ such that $f'(t)$ is absolutely continuous, $f(0) = 0$ and $f''(t) \in L_2[0,T]$. The norm for the space is:

$$\|f\|_{W_2^2} = \sqrt{(f(0))^2 + (f'(0))^2 + \int_0^T (f''(t))^2\, dt}. \tag{6}$$

Definition 3.3 ([22]) Hilbert space $W_2^3[0,T]$ consists of functions $f(t)$ defined on $[0,T]$ such that $f''(t)$ is absolutely continuous, $f(0) = f'(0) = 0$ and $f'''(t) \in L_2[0,T]$. The norm for the space is:

$$\|f\|_{W_2^3} = \sqrt{(f(0))^2 + (f'(0))^2 + (f''(0))^2 + \int_0^T (f'''(t))^2\, dt}. \tag{7}$$

The space $W_2^m[0,T]$ is a RKHS for $m = 1,2,3$. This means there is a reproducing kernel function $K(t,s) \in W_2^m[0,T]$, for each $t,s \in [0,T]$ and any $f(t) \in W_2^m[0,T]$, such that $\langle f(t), K(t,s)\rangle_{W_2^m} = f(s)$.

Theorem 3.1 ([28]) Hilbert space $W_2^3[0,T]$ is a RKHS with reproducing kernel function

$$\text{K(t,s)}=\begin{cases} (\frac{3+t}{12})s^2t^2 + (\frac{24-t^4}{24})st + 1, & s \leq t, \\ (\frac{3+s}{12})s^2t^2 + (\frac{24-s^4}{24})st + 1, & s>t. \end{cases} \tag{8}$$

To build our method, we need put $R(x,t,y,s) = K(x,t)K(y,s)$. Since $R(x,t,y,s)$ is symmetric and positive semi-definite, then $R(x,t,y,s)$ is the reproducing kernel function on $[0,l] \times [0,T]$.

Definition 3.4 ([28]) Hilbert space $W(\Lambda)$ consists of functions $f(x,t)$ defined on $\Lambda = [0,l] \times [0,T]$ such that $f(x,t)$ is complete continuous function and $\dfrac{\partial^2 f(x,t)}{\partial x \partial t} \in L_2(\Lambda)$. For all $f, g \in W$, the inner product for the space $W(\Lambda)$ is:

$$\langle f, g \rangle_W = \langle f(x,0), g(x,0) \rangle_{W_2^1} + \int_0^T \frac{\partial f(0,t)}{\partial t} \frac{\partial g(0,t)}{\partial t} dt + \int_0^T \int_0^l \frac{\partial^2 f(x,t)}{\partial x \partial t} \frac{\partial^2 g(x,t)}{\partial x \partial t} dx dt. \tag{9}$$

Definition 3.5 ([28]) Hilbert space $H(\Lambda)$ consists of functions $f(x,t)$ defined on $\Lambda = [0,l] \times [0,T]$ such that $\dfrac{\partial^3 f(x,t)}{\partial x^2 \partial t}$ is complete continuous function, $f(x,0) = 0$, $\dfrac{\partial f(x,0)}{\partial t} = 0$ and $\frac{\partial^5 f(x,t)}{\partial x^3 \partial t^2} \in L_2(\Lambda)$. For all $f, g \in H$, the inner product for the space $H(\Lambda)$ is:

$$\langle f, g \rangle_H = \sum_{j=0}^{1} \left\langle \frac{\partial^j f(x,0)}{\partial t^j}, \frac{\partial^j g(x,0)}{\partial t^j} \right\rangle_{W_2^3} + \sum_{j=0}^{2} \int_0^T \frac{\partial^{2+j} f(0,t)}{\partial t^2 \partial x^j} \frac{\partial^{2+j} g(0,t)}{\partial t^2 \partial x^j} dt + \int_0^T \int_0^l \frac{\partial^5 f(x,t)}{\partial x^3 \partial t^2} \frac{\partial^5 g(x,t)}{\partial x^3 \partial t^2} dx dt. \tag{10}$$

Definition 3.6 If $Z(\Lambda) = W(\Lambda) \oplus W(\Lambda)$, then the RKHS $Z(\Lambda)$ consists of fuzzy functions $u(x,t) = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$ such that $\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha) \in W(\Lambda)$ with inner product $\langle u, w \rangle_Z = \sum_{j=1}^{2} \left\langle u_j, w_j \right\rangle_W$ for all $u, w \in Z(\Lambda)$. The norm for the space $Z(\Lambda)$ is defined as $\|u\|_Z = \sqrt{\|\underline{u}\|_W^2 + \|\overline{u}\|_W^2}$.

Definition 3.7 If $V(\Lambda) = H(\Lambda) \oplus H(\Lambda)$, then the RKHS $V(\Lambda)$ consists of fuzzy functions $u(x,t) = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$ such that $\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha) \in H(\Lambda)$ with inner product $\langle u, w \rangle_V = \sum_{j=1}^{2} \left\langle u_j, w_j \right\rangle_H$ for all $u, w \in V(\Lambda)$. The norm for the space $V(\Lambda)$ is defined as $\|u\|_V = \sqrt{\|\underline{u}\|_H^2 + \|\overline{u}\|_H^2}$.

To provide Def. (3.6) and Def. (3.7), we can see that from the function $u(x,t) = [(\alpha - 1)xe^t, (1-\alpha)xe^t]$, $0 \leq x \leq 1$ and $0 < t \leq 1$. Since $\dfrac{\partial \underline{u}(0,t;\alpha)}{\partial t} = 0$, $\frac{\partial^2 \underline{u}(x,t;\alpha)}{\partial x \partial t} = (\alpha - 1)e^t$ and $\left\langle \underline{u}(x,0;\alpha), \underline{u}(x,0;\alpha) \right\rangle_{W_2^1} = (\alpha - 1)^2$, then from Def. (3.4), we have $\|\underline{u}\|_W^2 = \dfrac{(\alpha-1)^2}{2}(e^2 + 1)$. Similarly, we have $\|\overline{u}\|_W^2 = \dfrac{(1-\alpha)^2}{2}(e^2+1)$. Thus, $\|u\|_Z = (1-\alpha)\sqrt{e^2+1}$. Now, from Def. (3.3), we have $\left\langle \underline{u}(x,0;\alpha), \underline{u}(x,0;\alpha) \right\rangle_{W_2^3} = 0$ and $\left\langle \dfrac{\partial \underline{u}(x,0;\alpha)}{\partial t}, \dfrac{\partial \underline{u}(x,0;\alpha)}{\partial t} \right\rangle_{W_2^3} = 0$. Since $\dfrac{\partial^3 \underline{u}(0,t;\alpha)}{\partial t^2 \partial x} = (\alpha - 1)e^t$, $\dfrac{\partial^4 \underline{u}(0,t;\alpha)}{\partial t^2 \partial x^2} = 0$ and $\dfrac{\partial^5 \underline{u}(x,t;\alpha)}{\partial x^3 \partial t^2} = 0$, then from Def. (3.5), we have $\|\underline{u}\|_H^2 = \dfrac{(\alpha-1)^2}{2}(e^2 - 1)$. Similarly, we have $\|\overline{u}\|_H^2 = \dfrac{(1-\alpha)^2}{2}(e^2 - 1)$. Thus, $\|u\|_V = (1-\alpha)\sqrt{e^2 - 1}$.

Lemma 3.1 Suppose that $V(\Lambda)$ is a fuzzy RKHS with reproducing kernel function $R(x,t,y,s) = K(x,y)K(t,s)$. For any $u(x,t) = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)] \in V(\Lambda)$, then $\langle u(x,t), R(x,t,y,s) \rangle_V = u(y,s)$, where $u(y,s) = [\underline{u}(y,s;\alpha), \overline{u}(y,s;\alpha)]$.

Proof. Note that

$$\begin{aligned}
\left\langle \underline{u}(x,t;\alpha), R(x,t,y,s)\right\rangle_H &= \left\langle \underline{u}(x,t;\alpha), K(x,y)K(t,s)\right\rangle_H \\
&= \sum_{j=0}^{1}\left\langle \frac{\partial^j \underline{u}(x,0;\alpha)}{\partial t^j}, K(x,y)\frac{\partial^j K(0,s)}{\partial t^j}\right\rangle_{W_2^3} \\
&+ \sum_{j=0}^{2}\int_0^T \left(\frac{\partial^{2+j}\underline{u}(0,t;\alpha)}{\partial t^2\partial x^j}\frac{\partial^2 K(t,s)}{\partial t^2}\frac{\partial^j K(0,y)}{\partial x^j}\right)dt \\
&+ \sum_{j=0}^{2}\int_0^T \left(\frac{\partial^{2+j}\underline{u}(0,t;\alpha)}{\partial t^2\partial x^j}\frac{\partial^2 K(t,s)}{\partial t^2}\frac{\partial^j K(0,y)}{\partial x^j}\right)dt \\
&= \sum_{j=0}^{1}\frac{\partial^j \underline{u}(y,0;\alpha)}{\partial t^j}\frac{\partial^j K(0,s)}{\partial t^j} \\
&+ \int_0^T \frac{\partial^2 K(t,s)}{\partial t^2}\left(\frac{\partial^2}{\partial t^2}\int_0^l \frac{\partial^3 \underline{u}(x,t;\alpha)}{\partial x^3}\frac{\partial^3 K(x,y)}{\partial x^3}dx\right)dt \\
&+ \int_0^T \frac{\partial^2 K(t,s)}{\partial t^2}\left(\frac{\partial^2}{\partial t^2}\sum_{j=0}^{2}\left(\frac{\partial^j \underline{u}(0,t;\alpha)}{\partial x^j}\frac{\partial^j K(0,y)}{\partial x^j}\right)\right)dt \\
&= \sum_{j=0}^{1}\frac{\partial^j \underline{u}(y,0;\alpha)}{\partial t^j}\frac{\partial^j K(0,s)}{\partial t^j} + \int_0^T \frac{\partial^2 K(t,s)}{\partial t^2}\frac{\partial^2}{\partial t^2}\left\langle \underline{u}(x,t;\alpha), K(x,y)\right\rangle_{W_2^3} dt \\
&= \sum_{j=0}^{1}\frac{\partial^j \underline{u}(y,0;\alpha)}{\partial t^j}\frac{\partial^j K(0,s)}{\partial t^j} + \int_0^T \frac{\partial^2 K(t,s)}{\partial t^2}\frac{\partial^2 \underline{u}(y,t;\alpha)}{\partial t^2}dt \\
&= \left\langle \underline{u}(y,t;\alpha), K(t,s)\right\rangle_{W_2^2} \\
&= \underline{u}(y,s;\alpha).
\end{aligned}$$

Similarly with the same procedure, we get $\langle \overline{u}(x,t), R(x,t,y,s)\rangle_H = \overline{u}(y,s)$. That is, $\langle u(x,t), R(x,t,y,s)\rangle_V = u(y,s)$. The proof of the lemma is completed. □

4. Numerical method with convergence analysis

To apply the RKHS method, the nonhomogeneous fuzzy initial conditions and the nonhomogeneous fuzzy boundary condition in Eqs. (2), (3) and (4) must be converted to homogeneous fuzzy initial and fuzzy boundary conditions. Therefore, we suppose the FPVIDE (1)−(4) using a specific transformation of the following form: $u(x,t) = v(x,t) + \rho(x,t)$, where $\rho(x,t) = \sigma_1(\alpha)f(x) + e^{\sigma_2(\alpha)tg(x)} - 1 + (x+1)\sigma_3(\alpha)h(t)$ such that $h(0) = h'(0) = f(0) = g(0)$. The Eqs. (1)−(4) can be converted into:

$$A(x,t)\frac{\partial^2 v(x,t)}{\partial x^2} + B(x,t)\frac{\partial v(x,t)}{\partial x} + C(x,t)\frac{\partial v(x,t)}{\partial t} + D(x,t)v(x,t) + \int_0^t k(t,s)v(x,s)ds = G(x,t), \tag{11}$$

$$v(x,0) = 0,\ 0 \le x \le l, \tag{12}$$

$$\frac{\partial v(x,0)}{\partial t} = 0,\ 0 \le x \le l, \tag{13}$$

$$v(0,t) = 0,\ t > 0, \tag{14}$$

where

$$G(x,t) = r(x,t) - A(x,t)\frac{\partial^2\rho(x,t)}{\partial x^2} - B(x,t)\frac{\partial\rho(x,t)}{\partial x} - C(x,t)\frac{\partial\rho(x,t)}{\partial t} - D(x,t)\rho(x,t) - \int_0^t k(t,s)\rho(x,s)ds.$$

To solve numerically the FPVIDE (11)−(14) in the RKHS $V(\Lambda)$, we define a bounded linear operator $L : V \to Z$ as $Lv(x,t) = A(x,t)\frac{\partial^2 v(x,t)}{\partial x^2} + B(x,t)\frac{\partial v(x,t)}{\partial x} + C(x,t)\frac{\partial v(x,t)}{\partial t} + D(x,t)v(x,t) + \int_0^t k(t,s)v(x,s)ds$, where $L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$.

To build an orthonormal basis functions $\{\bar{\psi}_i(x,t)\}_{i=1}^{\infty}$ in $V(\Lambda)$, we choose a countable dense subset $\{(x_i,t_i)\}_{i=1}^{\infty}$ in Λ, define $\phi_i(x,t) = \left(R(x,t,y,s)_{|(y,s)=(x_i,t_i)}, R(x,t,y,s)_{|(y,s)=(x_i,t_i)}\right)^T$ and $\psi_i(x,t) = L^*\phi_i(x,t)$, where $\phi_i(x,t) = \left(\phi_{i1}(x,t), \phi_{i2}(x,t)\right)^T$, $\psi_i(x,t) = (\psi_{i1}(x,t), \psi_{i2}(x,t))^T$ and $L^* = \begin{bmatrix} L_1^* & 0 \\ 0 & L_2^* \end{bmatrix}$ is the adjoint operator of L. For $\{\psi_i(x,t)\}_{i=1}^{\infty}$, the use of Gram-Schmidt orthogonalization process yields:

$$\bar{\psi}_i(x,t) = \left(\sum_{k=1}^{i} \xi_{ik}\psi_{k1}(x,t), \sum_{k=1}^{i} \xi_{ik}\psi_{k2}(x,t)\right)^T, \tag{15}$$

where

$$\xi_{11} = \frac{1}{\|\vartheta_1\|},\ \xi_{ij} = \frac{-\sum\limits_{k=j}^{i-1} c_{ik}\xi_{kj}}{\sqrt{\|\vartheta_i\|^2 - \sum\limits_{k=1}^{i-1}(c_{ik})^2}}\ (j < i),\ \xi_{ii} = \frac{1}{\sqrt{\|\vartheta_i\|^2 - \sum\limits_{k=1}^{i-1}(c_{ik})^2}}\ (i > 1).$$

Theorem 4.1 If the set $\{(x_i,t_i)\}_{i=1}^{\infty}$ is dense in Λ and the inverse linear operator exists, then $\{\psi_i(x,t)\}_{i=1}^{\infty}$ is a complete system in $V(\Lambda)$.

Proof. For every $v(x,t) = \left((\underline{v}(x,t;\alpha), \overline{v}(x,t;\alpha)\right)^T \in V(\Lambda)$, $\tilde{R}(x,t,y,s) = (R(x,t,y,s), R(x,t,y,s))^T$ at $(y,s) = (x_i,t_i)$ and $\langle v(x,t), \psi_i(x,t)\rangle_V = 0$, we have

$$\begin{aligned} \langle v(x,t), \psi_i(x,t)\rangle_V &= \left\langle v(x,t), L^*\tilde{R}(x,t,y,s)_{|(y,s)=(x_i,t_i)}\right\rangle_V \\ &= \left\langle Lv(x,t), \tilde{R}(x,t,y,s)_{|(y,s)=(x_i,t_i)}\right\rangle_G \\ &= Lv(x_i,t_i) \\ &= 0. \end{aligned}$$

But the set $\{(x_i,t_i)\}_{i=1}^{\infty}$ is dense in Λ. This means $Lv(x,t) = 0$. Since L^{-1} exists, then $v(x,t) = 0$. □

Theorem 4.2 Under the assumption of Theorem (4.1), the fuzzy solution of FPVIDE (11)−(14) is given by:

$$v(x,t) = \left[\sum_{i=1}^{\infty}\sum_{k=1}^{i} \xi_{ik}\underline{G}(x_k,t_k;\alpha)\bar{\psi}_{i1}(x,t), \sum_{i=1}^{\infty}\sum_{k=1}^{i} \xi_{ik}\overline{G}(x_k,t_k;\alpha)\bar{\psi}_{i2}(x,t)\right], \tag{16}$$

where $G(x,t) = [\underline{G}(x,t;\alpha), \overline{G}(x,t;\alpha)]$ in Eq. (11) be a fuzzy valued function.

Proof. Since $v \in V(\Lambda) = H(\Lambda) \oplus H(\Lambda)$ and $\{\bar{\psi}_i(x,t)\}_{i=1}^{\infty} = (\bar{\psi}_{i1}(x,t), \bar{\psi}_{i2}(x,t))^T$ is orthonormal functions in $V(\Lambda)$, then based on Fourier series expansion and the reproducing property of kernel function $R(x,t,y,s)$, we see

$$\begin{aligned}
v(x,t) &= \left[\underline{v}(x,t;\alpha), \overline{v}(x,t;\alpha)\right] \\
&= \left[\sum_{i=1}^{\infty}\left\langle \underline{v}(x,t;\alpha), \bar{\psi}_{i1}(x,t)\right\rangle_H \bar{\psi}_{i1}(x,t), \sum_{i=1}^{\infty}\left\langle \overline{v}(x,t;\alpha), \bar{\psi}_{i2}(x,t)\right\rangle_H \bar{\psi}_{i2}(x,t)\right] \\
&= \left[\sum_{i=1}^{\infty}\left\langle \underline{v}(x,t;\alpha), \sum_{k=1}^{i}\xi_{ik}\psi_{k1}(x,t)\right\rangle_H \overline{\psi}_{i1}(x,t), \sum_{i=1}^{\infty}\left\langle \overline{v}(x,t;\alpha), \sum_{k=1}^{i}\xi_{ik}\psi_{k2}(x,t)\right\rangle_H \overline{\psi}_{i2}(x,t)\right] \\
&= \left[\sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\left\langle \underline{v}(x,t;\alpha), \psi_{k1}(x,t)\right\rangle_H \bar{\psi}_{i1}(x,t), \sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\left\langle \overline{v}(x,t;\alpha), \psi_{k2}(x,t)\right\rangle_H \bar{\psi}_{i2}(x,t)\right] \\
&= \left[\sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\left\langle \underline{v}(x,t;\alpha), L_1^*\phi_{k1}(x,t)\right\rangle_H \bar{\psi}_{i1}(x,t), \sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\left\langle \overline{v}(x,t;\alpha), L_2^*\phi_{k2}(x,t)\right\rangle_H \bar{\psi}_{i2}(x,t)\right] \\
&= \left[\sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\left\langle L_1\underline{v}(x,t;\alpha), \phi_{k1}(x,t)\right\rangle_W \bar{\psi}_{i1}(x,t), \sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\left\langle L_2\overline{v}(x,t;\alpha), \phi_{k2}(x,t)\right\rangle_W \bar{\psi}_{i2}(x,t)\right] \\
&= \left[\begin{array}{l}\sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\left\langle L_1\underline{v}(x,t;\alpha), R(x,t,y,s)_{|(y,s)=(x_k,t_k)}\right\rangle_W \bar{\psi}_{i1}(x,t), \\ \sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\left\langle L_2\overline{v}(x,t;\alpha), R(x,t,y,s)_{|(y,s)=(x_k,t_k)}\right\rangle_W \bar{\psi}_{i2}(x,t)\end{array}\right] \\
&= \left[\sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}L_1\underline{v}(x_k,t_k;\alpha)\bar{\psi}_{i1}(x,t), \sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}L_2\overline{v}(x_k,t_k;\alpha)\bar{\psi}_{i2}(x,t)\right] \\
&= \left[\sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\underline{G}(x_k,t_k;\alpha)\bar{\psi}_{i1}(x,t), \sum_{i=1}^{\infty}\sum_{k=1}^{i}\xi_{ik}\overline{G}(x_k,t_k;\alpha)\bar{\psi}_{i2}(x,t)\right].
\end{aligned}$$

The proof is completed. □

Take the truncated series of $v(x,t)$ as:

$$v_n(x,t) = \left[\sum_{i=1}^{n}\sum_{k=1}^{i}\xi_{ik}\underline{G}(x_k,t_k;\alpha)\bar{\psi}_{i1}(x,t), \sum_{i=1}^{n}\sum_{k=1}^{i}\xi_{ik}\overline{G}(x_k,t_k;\alpha)\bar{\psi}_{i2}(x,t)\right], \tag{17}$$

which is an approximate fuzzy solution to the FPVIDE (11)−(14).

Theorem 4.3 Assume that $\left\|\underline{v}_n(x,t;\alpha)\right\|_H$ and $\|\bar{v}_n(x,t;\alpha)\|_H$ are bounded in Eq. (17). If $\{(x_i,t_i)\}_{i=1}^{\infty}$ is dense on Λ and $G(x,t) \in V(\Lambda)$ be a fuzzy continuous valued function, then the approximate fuzzy solution $v_n = [\underline{v}_n, \bar{v}_n]$ is convergent to the exact fuzzy solution $v = [\underline{v}, \bar{v}]$ in the RKHS $V(\Lambda)$.

Proof. Let $A_i = \sum_{k=1}^{i}\xi_{ik}\underline{G}(x_k,t_k;\alpha)$ and $B_i = \sum_{k=1}^{i}\xi_{ik}\overline{G}(x_k,t_k;\alpha)$. From Eq. (17), we get $\underline{v}_{n+1}(x,t;\alpha) = \underline{v}_n(x,t;\alpha) + A_{n+1}\bar{\psi}_{i1}(x,t)$ and $\bar{v}_{n+1}(x,t;\alpha) = \bar{v}_n(x,t;\alpha) + B_{n+1}\bar{\psi}_{i2}(x,t)$. From the orthogonality of $\{\bar{\psi}_{i1}(x,t)\}_{i=1}^{\infty}$ and $\{\bar{\psi}_{i2}(x,t)\}_{i=1}^{\infty}$, we have

$$\left\|\underline{v}_{n+1}(x,t;\alpha)\right\|_H^2 = \left\|\underline{v}_n(x,t;\alpha)\right\|_H^2 + A_{n+1}^2 = \left\|\underline{v}_{n-1}(x,t;\alpha)\right\|_H^2 + A_n^2 + A_{n+1}^2 = \left\|\underline{v}_0(x,t;\alpha)\right\|_H^2 + \sum_{i=1}^{n+1}A_i^2;$$

$$\|\bar{v}_{n+1}(x,t;\alpha)\|_H^2 = \|\bar{v}_n(x,t;\alpha)\|_H^2 + B_{n+1}^2 = \|\bar{v}_{n-1}(x,t;\alpha)\|_H^2 + B_n^2 + B_{n+1}^2 = \|\bar{v}_0(x,t;\alpha)\|_H^2 + \sum_{i=1}^{n+1}B_i^2.$$

Thus, $\left\|\underline{v}_n(x,t;\alpha)\right\|_H \leq \left\|\underline{v}_{n+1}(x,t;\alpha)\right\|_H$ and $\|\bar{v}_n(x,t;\alpha)\|_H \leq \|\bar{v}_{n+1}(x,t;\alpha)\|_H$. Since $\left\|\underline{v}_n(x,t;\alpha)\right\|_H$ and $\|\bar{v}_n(x,t;\alpha)\|_H$ are bounded, then $\underline{v}_n$ and $\bar{v}_n$ are convergent as $n \to \infty$ in the RKHS $H(\Lambda)$. This means $\{A_i\}_{i=1}^{\infty}$, $\{B_i\}_{i=1}^{\infty} \in l^2$. Moreover, $v_n = [\underline{v}_n, \bar{v}_n]$ is convergent as $n \to \infty$ in the RKHS $V(\Lambda)$. Since $(\underline{v}_m - \underline{v}_{m-1}) \perp (\underline{v}_{m-1} - \underline{v}_{m-2}) \perp ... \perp (\underline{v}_{n+1} - \underline{v}_n)$ and $(\bar{v}_m - \bar{v}_{m-1}) \perp (\bar{v}_{m-1} - \bar{v}_{m-2}) \perp ... \perp (\bar{v}_{n+1} - \bar{v}_n)$ for $m > n$, then

$$\begin{aligned}
\left\|\underline{v}_m(x,t;\alpha) - \underline{v}_n(x,t;\alpha)\right\|_H^2 &= \left\|\underline{v}_m(x,t;\alpha) - \underline{v}_{m-1}(x,t;\alpha) + \underline{v}_{m-1}(x,t;\alpha) - \ldots + \underline{v}_{n+1}(x,t;\alpha) - \underline{v}_n(x,t;\alpha)\right\|_H^2 \\
&\leq \left\|\underline{v}_m(x,t;\alpha) - \underline{v}_{m-1}(x,t;\alpha)\right\|_H^2 + \ldots + \left\|\underline{v}_{n+1}(x,t;\alpha) - \underline{v}_n(x,t;\alpha)\right\|_H^2 \\
&= \sum_{i=n+1}^{m} (A_i)^2 \to 0 \text{ as } n, m \to \infty.
\end{aligned}$$

$$\begin{aligned}
\left\|\bar{v}_m(x,t;\alpha) - \bar{v}_n(x,t;\alpha)\right\|_H^2 &= \left\|\bar{v}_m(x,t;\alpha) - \bar{v}_{m-1}(x,t;\alpha) + \bar{v}_{m-1}(x,t;\alpha) - \ldots + \bar{v}_{n+1}(x,t;\alpha) - \bar{v}_n(x,t;\alpha)\right\|_H^2 \\
&\leq \left\|\bar{v}_m(x,t;\alpha) - \bar{v}_{m-1}(x,t;\alpha)\right\|_H^2 + \ldots + \left\|\bar{v}_{n+1}(x,t;\alpha) - \bar{v}_n(x,t;\alpha)\right\|_H^2 \\
&= \sum_{i=n+1}^{m} (B_i)^2 \to 0 \text{ as } n, m \to \infty.
\end{aligned}$$

We know that $H(\Lambda)$ is a complete space. Thus, there exist $\underline{v}$, $\bar{v} \in H(\Lambda)$ such that $\underline{v}_n \to \underline{v}$ and $\bar{v}_n \to \bar{v}$ as $n \to \infty$ in the sense of $\|.\|_H$. This means $v_n \to v$ as $n \to \infty$ in the sense of $\|.\|_V$. Now, we prove $v(x,t) = [\underline{v}(x,t;\alpha), \bar{v}(x,t;\alpha)]$ is the fuzzy solution of the FPVIDE (11)–(14). Since $\underline{v}(x,t;\alpha) = \sum_{i=1}^{\infty} A_i \bar{\psi}_{i1}(x,t)$ and $\bar{v}(x,t;\alpha) = \sum_{i=1}^{\infty} B_i \bar{\psi}_{i2}(x,t)$, then

$$\begin{aligned}
(L_1\underline{v})(x_j,t_j;\alpha) &= \sum_{i=1}^{\infty} A_i \left\langle L_1\overline{\psi}_{i1}(x,t), \phi_{j1}(x,t)\right\rangle_W \\
&= \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_{i1}(x,t), L_1^*\phi_{j1}(x,t)\right\rangle_H \\
&= \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_{i1}(x,t), \psi_{j1}(x,t)\right\rangle_H.
\end{aligned}$$

$$\begin{aligned}
(L_2\bar{v})(x_j,t_j;\alpha) &= \sum_{i=1}^{\infty} B_i \left\langle L_2\overline{\psi}_{i2}(x,t), \phi_{j2}(x,t)\right\rangle_W \\
&= \sum_{i=1}^{\infty} B_i \left\langle \overline{\psi}_{i2}(x,t), L_2^*\phi_{j2}(x,t)\right\rangle_H \\
&= \sum_{i=1}^{\infty} B_i \left\langle \overline{\psi}_{i2}(x,t), \psi_{j2}(x,t)\right\rangle_H.
\end{aligned}$$

It follows that:

$$\sum_{j=1}^{n} \xi_{nj}(L_1\underline{v})(x_j,t_j;\alpha) = \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_{i1}(x,t), \sum_{j=1}^{n} \xi_{nj}\psi_{j1}(x,t)\right\rangle_H = \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_{i1}(x,t), \overline{\psi}_{n1}(x,t)\right\rangle_H = A_n,$$

$$\sum_{j=1}^{n} \xi_{nj}(L_2\bar{v})(x_j,t_j;\alpha) = \sum_{i=1}^{\infty} B_i \left\langle \overline{\psi}_{i2}(x,t), \sum_{j=1}^{n} \xi_{nj}\psi_{j2}(x,t)\right\rangle_H = \sum_{i=1}^{\infty} B_i \left\langle \overline{\psi}_{i2}(x,t), \overline{\psi}_{n2}(x,t)\right\rangle_H = B_n.$$

Now, if $n = 1$, then $(L_1\underline{v})(x_1,t_1;\alpha) = \underline{G}(x_1,t_1;\alpha)$ and $(L_2\bar{v})(x_1,t_1;\alpha) = \overline{G}(x_1,t_1;\alpha)$. Again, if $n = 2$, then

$(L_1\underline{v})(x_2, t_2; \alpha) = \underline{G}(x_2, t_2; \alpha)$ and $(L_2\overline{v})(x_2, t_2; \alpha) = \overline{G}(x_2, t_2; \alpha)$. By induction, it is easy to see that

$$(L_1\underline{v})\left(x_j, t_j; \alpha\right) = \underline{G}(x_j, t_j; \alpha), \tag{18}$$

$$(L_2\overline{v})\left(x_j, t_j; \alpha\right) = \overline{G}(x_j, t_j; \alpha). \tag{19}$$

where $j = 1,2,\ldots$. Since $\{(x_i, t_i)\}_{i=1}^{\infty}$ is dense on Λ, then for any $(y, z) \in \Lambda$, there is a subsequence $\{(x_{n_j}, t_{n_j})\}$ such that $(x_{n_j}, t_{n_j}) \longrightarrow (y, z)$ as $j \longrightarrow \infty$. Moreover, let $j \longrightarrow \infty$ in Eqs. (18) and (19), by the convergence of v_n and the continuity of G, we have

$$(L_1\underline{v})(y, z; \alpha) = \underline{G}(y, z; \alpha), \tag{20}$$

$$(L_2\overline{v})(y, z; \alpha) = \overline{G}(y, z; \alpha). \tag{21}$$

That is, $(Lv)(y, z) = G(y, z)$. This means $v(x, t) = [\underline{v}(x, t; \alpha), \bar{v}(x, t; \alpha)]$ is the solution of the FPVIDE (11)−(14). The proof is completed. □

5. Numerical examples

In this section, we test the accuracy and reliability of the RKHS method by presenting the approximate results and absolute error for some fuzzy partial Volterra integro-differential equations.

Example 1 Consider the following FPVIDE

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial t^2} + [2\alpha + 2, 6 - 2\alpha]e^x - 2\int_0^t (t-s)u(x,s)ds,\ 0 \le x \le 1,\ t > 0, \tag{22}$$

$$u(0,t) = [\alpha + 1, 3 - \alpha]\cos t,\ t > 0, \tag{23}$$

$$u(x,0) = [\alpha + 1, 3 - \alpha]e^x,\ 0 \le x \le 1, \tag{24}$$

$$\frac{\partial u(x,0)}{\partial t} = 0,\ 0 \le x \le 1, \tag{25}$$

for all $\alpha \in [0, 1]$. This problem has the exact solution of the Eqs. (22)−(25) is given by

$$u(x,t) = \left[(\alpha + 1)e^x \cos t, (3 - \alpha)e^x \cos t\right].$$

Table 1. The absolute errors of approximating $\underline{u}$ for Eqs. (22)−(25).

(x,t)	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
$(0.2, 0.2)$	1.206785×10^{-3}	1.407915×10^{-3}	1.609046×10^{-3}	1.810177×10^{-3}
$(0.4, 0.4)$	2.173558×10^{-3}	2.535818×10^{-3}	2.898077×10^{-3}	3.260337×10^{-3}
$(0.6, 0.6)$	9.968506×10^{-4}	2.760581×10^{-3}	3.154949×10^{-3}	3.549318×10^{-3}
$(0.8, 0.8)$	3.029001×10^{-4}	1.162992×10^{-4}	1.329134×10^{-4}	1.495275×10^{-4}

Table 2. The absolute errors of approximating $\overline{u}$ for Eqs. (22)−(25).

(x,t)	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
$(0.2, 0.2)$	2.815831×10^{-3}	2.614700×10^{-3}	2.413570×10^{-3}	3.984856×10^{-3}
$(0.4, 0.4)$	5.071636×10^{-3}	4.709376×10^{-3}	4.347116×10^{-3}	4.338056×10^{-3}
$(0.6, 0.6)$	5.521162×10^{-3}	5.126793×10^{-3}	4.732424×10^{-4}	1.827559×10^{-3}
$(0.8, 0.8)$	2.325984×10^{-4}	2.159843×10^{-4}	1.993701×10^{-4}	5.553168×10^{-4}

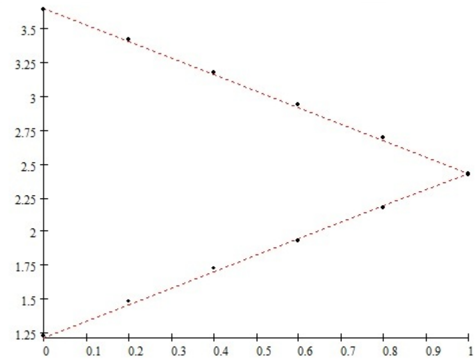

Figure 1: Exact solution (red) and numerical solution (black).

The approximate solutions of the RKHS method at different values of (x,t) in Λ has been shown in Tables (1) and (2), when $\alpha \in 0.2, 0.4, 0.6, 0.8$. From these tables, we can see that the absolute error show that the approximate solution to the FPVIDE (22)−(25) is close to the exact solution. Moreover, a more accurate approximate solution to the Eqs. (22)−(25) can be obtained with a larger value of n. In addition to that when $(x,t) = (0.2, 0.2)$, $\alpha \in 0, 0.2, 0.4, 0.6, 0.8, 1$ and $n = 50$, Figure (1) shows that this method is fastly convergent.

Example 2 Consider the following FPVIDE

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} + u(x,t) + [1+\alpha, 3-\alpha](2-(x^2+1)e^t) + \int_0^t e^{(t-s)} u(x,s)ds,\ 0 \le x \le 1,\ t > 0, \tag{26}$$

$$u(0,t) = [\alpha+1, 3-\alpha]t,\ t > 0, \tag{27}$$

$$\frac{\partial u(0,t)}{\partial x} = 0,\ t > 0, \tag{28}$$

$$u(x,0) = [1+\alpha, 3-\alpha]x^2,\ 0 \le x \le 1, \tag{29}$$

$$\frac{\partial u(x,0)}{\partial t} = [1+\alpha, 3-\alpha],\ 0 \le x \le 1, \tag{30}$$

for all $\alpha \in [0,1]$. The exact solution of the Eqs. (26)−(30) is given by

$$u(x,t) = [(1+\alpha)(x^2+t), (3-\alpha)(x^2+t)].$$

Table 3. The absolute errors of approximating $\underline{u}$ for Eqs. (26)−(30)at (0.25, 0.2).

n	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
10	2.37284×10^{-2}	2.84741×10^{-2}	3.32198×10^{-2}	3.79654×10^{-2}	4.27111×10^{-2}
100	2.34358×10^{-4}	2.81229×10^{-4}	3.28101×10^{-4}	3.74973×10^{-4}	4.21844×10^{-4}
200	2.34339×10^{-6}	2.81207×10^{-6}	3.28075×10^{-6}	3.74943×10^{-6}	4.21811×10^{-6}
300	2.34336×10^{-7}	2.81203×10^{-7}	3.28070×10^{-7}	3.74938×10^{-7}	4.21805×10^{-7}

Table 4. The absolute errors of approximating $\overline{u}$ for Eqs. (26)−(30)at (0.25, 0.2).

n	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
10	7.11852×10^{-2}	6.64395×10^{-2}	6.16938×10^{-2}	5.69481×10^{-2}	5.22025×10^{-2}
100	7.03074×10^{-4}	6.56202×10^{-4}	6.09331×10^{-4}	5.62459×10^{-4}	5.15588×10^{-4}
200	7.03018×10^{-6}	6.56150×10^{-6}	6.09283×10^{-6}	5.62415×10^{-6}	5.15547×10^{-6}
300	7.03008×10^{-7}	6.56141×10^{-7}	6.09274×10^{-7}	5.62406×10^{-7}	5.15539×10^{-7}

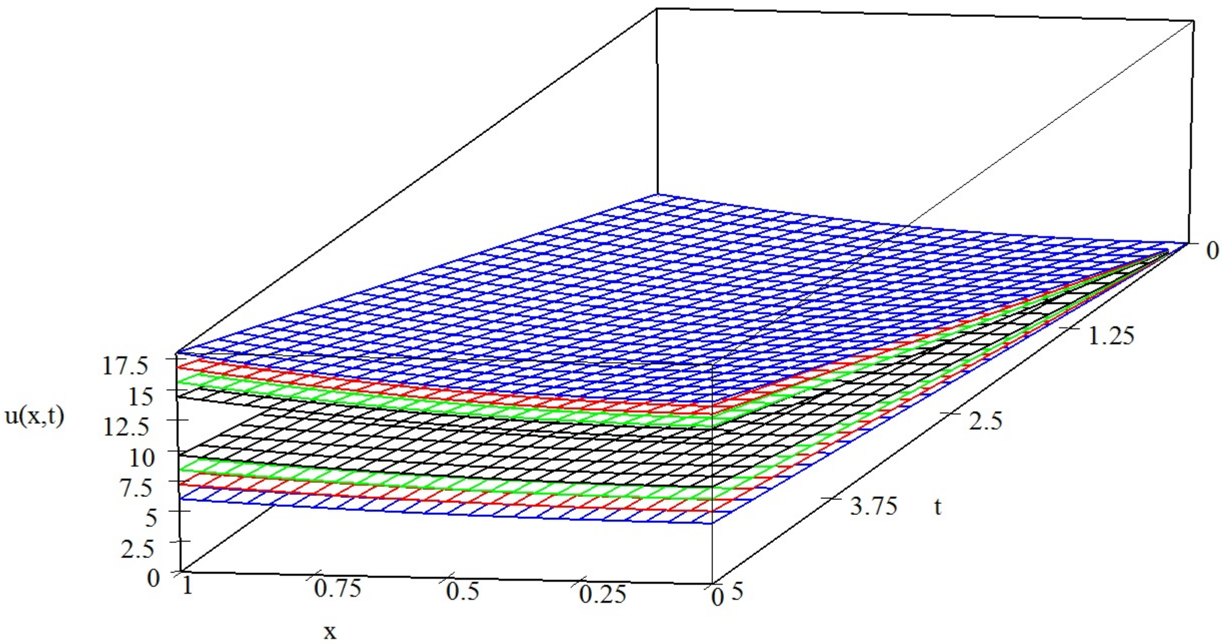

Figure 2: Fuzzy solution trajectories for Eqs. (26)-(30).

To examine the accuracy of the approximate fuzzy solution of the FPVIDE (26)−(30), we choose different values for n as shown in Tables (3) and (4). Based on this, we conclude that the RKHS method is more efficient to give better results because when the value of n increases, then the error of the RKHS method decreases very fast. On the other hand, Figure (2) presents the trajectories for the fuzzy solution graphically when $\alpha = 0$ (blue), $\alpha = 0.2$ (red), $\alpha = 0.4$ (green) and $\alpha = 0.6$ (black).

6. Conclusion

A novel numerical method is presented for fuzzy partial Volterra-integro differential equations in a reproducing kernel Hilbert space. The reproducing kernel functions are employed to generate the orthonormal functions. Also, the orthonormal functions satisfy the homogeneous fuzzy initial and fuzzy boundary conditions of the considered problem. Therefore, the RKHS method has been successfully applied to find the approximate fuzzy solution of the fuzzy partial Volterra-integro differential equations. From numerical results in Tables 1, 2, 3 and 4, we can see the present method is simple, impressive and wider applicability. Moreover, the RKHS method has never been applied to nonlinear fuzzy partial Volterra-integro differential equations. It can be also expanded to nonlinear fuzzy problems, fuzzy inverse problems and fuzzy perturbation problems as a future work.

References

[1] C. G. Almeida, J. F. Meyer and H. L. Vasconcelos, Fuzzy parameter in a partial differential equation model for population dispersal of Leaf-Cutting Ants, Nonlinear Analysis: Real World Applications, 12 (2011), 3397–3412.

[2] J. Leite, M. Cecconello, J. Leite, R. C. Bassanezi, On Fuzzy Solutions for Diffusion Equation, Journal of Applied Mathematics, 2015 (2015), Article ID 874931, 1–10.
[3] B. Bede, I. J. Rudas and A. L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, Information Sciences, 177 (2007), 1648–1662.
[4] J. J. Nieto, A. Khastan and K. Ivaz, Numerical solution of fuzzy differential equations under generalized differentiability, Nonlinear Analysis: Hybrid Systems, 3 (2009), 700–7007.
[5] J. J. Buckley and T. Feuring, Introduction to fuzzy partial differential equations, Fuzzy Sets and Systems, 105 (1999), 241–248.
[6] G. Gumah, Numerical solutions of special fuzzy partial differential equations in a reproducing kernel Hilbert space, Computational and Applied Mathematics, 41 (2) (2022), 1–17.
[7] A.M. Bertone, R.M. Jafelice, L.C. Barros and R.C. Bassanezi, On fuzzy solutions for partial
[8] Y.Y. Chen, Y.T. Chang and B.S. Chen, Fuzzy solutions to partial differential equations: adaptive approach, IEEE Transactions on Fuzzy Systems, 17 (2009), 116–127.
[9] Z. Gouyandeh, T. Allahviranloo, S. Abbasbandy and A. Armand, A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform, Fuzzy Sets and Systems, 309 (2017), 81–97.
[10] G. Gumah, M. Naser, M. Al-Smadi and S. Al-Omari, Application of reproducing kernel Hilbert space method for solving second-order fuzzy Volterra integro-differential equations, Advances in Difference Equations, (2018) 475, https://doi.org/10.1186/s13662-018-1937-8.
[11] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, 24 (1987), 301–317.
[12] M. Akram, G. Muhammad and T. Allahviranloo, Solution of initial-value problem for linear third-order fuzzy differential equations, Computational and Applied Mathematics, 41 (398) (2022), 1–31.
[13] E. Hussian and M. Suhhiem, Numerical solution of fuzzy partial differential equations by using modified fuzzy neural networks, British Journal of Mathematics and Computer Science, 12 (2) (2016), 1–20.
[14] U. Pirzada and D. Vakaskar, Solution of fuzzy heat equations using Adomian decomposition method, International Journal of Applied Mathematics and Mechanics, 3 (2015), 87–91.
[15] T. Allahviranloo, Difference methods for fuzzy partial differential equations, Computational Methods in Applied Mathematics, 2 (2002), 233–242.
[16] M. Al-Smadi, O. A. Arqub and D. Zeidan, Fuzzy fractional differential equations under the Mittag-Leffler kernel differential operator of the ABC approach: theorems and applications, Chaos Solitons Fractals, (2021) 146: 110891.
[17] E. Qahremani, T. Allahviranlooa, S. Abbasbandy and N. Ahmady, A study on the fuzzy parabolic Volterra partial integro-differential equations, Journal of Intelligent and Fuzzy Systems, 40 (1) (2021), 1639–1654.
[18] Z.Gouyandeh, T. Allahviranloo, S. Abbasbandy and A. Armand, A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform, Fuzzy Sets and Systems, 309 (2017), 81–97.
[19] M. Cui and F. Geng, Solving singular two-point boundary value problem in reproducing kernel space, Journal of Computational and Applied Mathematics, 205 (2007), 6–15.
[20] G. Gumah, M. F. M. Naser, M. Al-Smadi, S.K.Q. Al-Omari and D. Baleanu, Numerical solutions of hybrid fuzzy differential equations in a Hilbert space, Applied Numerical Mathematics, 151 (2020), 402–412.
[21] M. Fardi and M. Ghasemi, Solving nonlocal initial-boundary value problems for parabolic and hyperbolic integro-differential equations in reproducing kernel Hilbert space, Numerical Methods for Partial Differential Equations, 33 (2017), 174–198.
[22] G. Gumah, S. Al-Omari and D. Baleanu, Soft computing technique for a system of fuzzy Volterra integro-differential equations in a Hilbert space, Applied Numerical Mathematics, 152 (2020), 310–322.
[23] T. Rasham, M.S. Shabbir, P. Agarwal and S. Momani, On a pair of fuzzy dominated mappings on closed ball in the multiplicative metric space with applications, Fuzzy Sets and Systems, 437 (2202), 81–96.
[24] T. Rasham, A. Asif, H. Aydi and M.D. Sen, On pairs of fuzzy dominated mappings and applications, Advances in Difference Equations, 2021 (1), 1–22.
[25] T. Rasham, A. Shoaib, C. Park, R.P. Agarwal and H. Aydi, On a pair of fuzzy mappings in modular-like metric spaces with applications, Advances in difference equations, 2021 (1), 1–17.
[26] T. Rasham, G. Marino, A. Shahzad, C. Park and A. Shoaib, Fixed point results for a pair of fuzzy mappings and related applications in b-metric like spaces, Advances in difference equations, 2021 (1), 1–18.
[27] M. Gamal, T. Rasham, W. Cholamjiak, F.G. Shi and C. Park, New iterative scheme for fixed point results of weakly compatible maps in multiplicative Gm metric space via various contractions with application, AIMS Mathematics, 7 (8), 13681–13703.
[28] O. A. Arqub and M. Al-Smadi, Numerical algorithm for solving time-fractional partial integro-differential equations subject to initial and Dirichlet boundary conditions, Numerical Methods for Partial Differential Equations, 34 (2018), 1577–1597.