Filomat 38:24 (2024), 8565–8581
https://doi.org/10.2298/FIL2424565J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Markov chain approach to the coupon collector problem with universal coupon

**Jelena Jocković[a,*], Bojana Todić[a]**

[a]*Faculty of Mathematics, University of Belgrade, Serbia*

**Abstract.** The classical coupon collector problem has various modifications and generalizations. One group of generalizations is based on the idea of introducing additional coupons, with special purposes, to the set of available coupons. We consider the case when this set consists of standard coupons (that can belong to the collection), a null coupon (which can be drawn, but does not belong to any collection), and an additional universal coupon, that can replace any of the standard coupons. By employing a Markov chain approach, we derive the exact forms of the $k$-step transition matrix and the fundamental matrix, which we use to obtain the properties of the waiting time until a subcollection, or a full collection is sampled, and some additional characteristics of the collecting process (probability that the coupon collecting procedure ends in a particular way). We also provide numerical examples and explain possible applications of the variant of the coupon collector problem we considered.

## 1. Introduction

The coupon collector problem belongs to the family of urn problems, and can be formulated as follows: a company issues coupons of different types (say, elements of the set $\mathbb{N}_n = \{1, 2, \dots, n\}$), each with a particular probability of being drawn. The object of interest is the number of coupons that must be drawn to obtain a full collection, or a part of the collection.

The classical coupon collector problem (CCCP) refers to the case when all coupons from the set $\mathbb{N}_n = \{1, 2, \dots, n\}$ have equal probability of being drawn. Although very simple, this problem led to various limit results and generalizations (see, for example, [3], [8], [11]).

We consider the following generalization of CCCP (which, to the best of our knowledge, has not been considered yet): We assume that, apart from elements of $\mathbb{N}_n$, the set of coupons consists of a null coupon (which does not belong to the collection) and, in addition, a universal coupon (so called, joker), an element that can replace any of the elements from the set $\mathbb{N}_n$ (one at a time). Therefore, we work with an augmented set of available coupons:

$$\mathbb{N}_n^{\star,\circ} = \{1, 2, \dots, n, \star, \circ\}, \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 60C05, Secondary 60J10.

*Keywords*. coupon collector problem, waiting time, universal coupon, Markov chain, transition probability matrix, fundamental matrix.

Received: 18 August 2023; Revised: 25 April 2024; Accepted: 09 June 2024

Communicated by Marko Petković

This work was supported by Ministry of Science, Technological Development and Innovation of the Republic of Serbia: Grant No. 451 - 03 - 47/2023 - 01/200104.

* Corresponding author: Jelena Jocković

*Email addresses:* jelena.jockovic@matf.bg.ac.rs (Jelena Jocković), bojana.todic@matf.bg.ac.rs (Bojana Todić)

where $\star$ denotes the joker, and $\circ$ denotes the null coupon. We assume that coupons are drawn one by one, that sampling is done with replacement, that all coupons from $\mathbb{N}_n$ are drawn with equal probability $p$, joker is drawn with probability $p_J$ and null coupon is drawn with probability $p_N$, where $p_N + p_J \le 1$. Therefore, $p = (1 - p_N - p_J)/n$. We will call this version of the problem coupon collector problem with universal coupon, and denote as CCPUC in the rest of the paper.

There are a number of recent generalizations of CCCP based on adding additional coupons (coupons with special purposes) to the coupon set ([1], [2], [9], [13], [14]).

In [1] and [2], the authors consider the case where there is only one additional coupon (null coupon), and deal with the case of arbitrary distribution of available coupons (unequal probability case).

The CCPUC we consider is somewhat similar to the situation where coupons have more than one purpose, considered in [14]. Using the terminology introduced in [14], we can say that joker coupon has $n$ possible goals. The difference is that, in our case, the experiment does not end when all possible goals are listed (which could be when the first joker is drawn), but when the sum of the number of different coupons and the number of jokers reaches a certain limit.

Another generalization of the classical coupon collector problem which is, at first, similar to CCPUC, is proposed in [9], where the appearance of the additional coupon (so called, bonus coupon) leads to obtaining one more coupon. However, the principal difference between the two versions of the problem is that, in case of CCPUC no additional coupons are drawn.

In [13], the author considers the case where the additional coupon (so called, penalty coupon) interferes with collecting standard coupons, in the sense that the collection process ends when the absolute difference between the number of collected standard coupons and the number of collected penalty coupons is equal to the total number of standard coupons.

Let $W_{n,c}$ be the waiting time until a portion of the size $c$, $1 \le c \le n$, of the collection $\mathbb{N}_n$ is sampled, although some or all of the standard coupons can be replaced by jokers. In this paper, we obtain the distribution function, the first and the second moment of $W_{n,c}$, as well as some additional properties of the collecting process, using Markov chain techniques and mathematical induction.

The Markov chain approach has already been successfully applied to several variants of the coupon collector problem (see [1], [5], [13]), as well as to other types of waiting time problems (see, for example, [7]).

This paper is organized as follows: in Section 2 we explain how the CCPUC translates into a Markov chain model, and provide some auxiliary results. In Section 3 we obtain the explicit expressions for the $k$-step transition probability matrix and the fundamental matrix, which is our main result. In Section 4 we derive the distributional properties of the waiting times $W_{n,n}$ and $W_{n,c}$. Numerical examples are given in Section 5. Conclusions and possible applications of this work are presented in Section 6.

## 2. Markov chain approach for CCPUC

First, we introduce some notation. In the rest of the text, we will denote with $\mathbf{I}$ the identity matrix, and with $\mathbf{0}$ matrix with all entries equal to 0. With $(\mathbf{M})_{i,j}$ we will denote the element in the $i$-th row and the $j$-th column of the matrix $\mathbf{M}$.

Let $X_t = (Y_t, Z_t)$ be the number of different types of standard coupons, and number of jokers, respectively, sampled after $t$ units of time. Therefore,

$$W_{n,c} = \inf\{t \in \mathbb{N} \mid Y_t + Z_t = c\}.$$

We can notice that $\{X_t, t \in \mathbb{N}\}$ is a Markov chain on the state space:

$$S = \bigcup_{i=0}^{n-1} T_{n-1-i} \cup \Lambda, \tag{2}$$

where

$$T_{n-1-i} = \{(0, i), (1, i), \dots, (n-1-i, i)\}, \quad i \in \{0, 1, \dots, n-1\} \tag{3}$$

are sets of transient states and

$$\Lambda = \{(n-a,a), \quad a \in \{0,1,\dots,n\}\} \tag{4}$$

is the set of absorbing states. Therefore, $\mid S \mid = n(n+1)/2 + n + 1$.

In this case, the transition probability matrix for one step is

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q}_n & \mathbf{R}_{\frac{n(n+1)}{2}\times(n+1)} \\ \mathbf{0}_{(n+1)\times\frac{n(n+1)}{2}} & \mathbf{I}_{(n+1)\times(n+1)} \end{pmatrix}. \tag{5}$$

Here, matrix $\mathbf{Q}_n$ is describing transitions between transient states, and can be partitioned as follows:

$$\mathbf{Q}_n = \begin{array}{c} \\ T_{n-1} \\ T_{n-2} \\ T_{n-3} \\ \vdots \\ T_1 \\ T_0 \end{array} \begin{array}{c} \begin{array}{ccccccc} T_{n-1} & T_{n-2} & T_{n-3} & T_{n-4} & \dots & T_1 & T_0 \end{array} \\ \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{B}_{n-1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{n-2} & \mathbf{B}_{n-2} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{n-3} & \mathbf{B}_{n-3} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_0 \end{pmatrix}_{\frac{n(n+1)}{2}\times\frac{n(n+1)}{2}} \end{array}. \tag{6}$$

Matrix $\mathbf{A}_k$, $k \in \{0,1,\dots,n-1\}$ is related to transitions between states $T_k$.
Probabilities of these transitions are:

$$p_{(i,n-1-k),(j,n-1-k)} = \begin{cases} p_N + ip, & i = j; \\ 1 - p_N - p_J - ip, & j = i+1; \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

Observing that $1 - p_N - p_J - ip = (n-i)p$, we can write the matrix $\mathbf{A}_k$ (of the type $(k+1)\times(k+1)$) in the following form:

$$\mathbf{A}_k = \begin{pmatrix} p_N & np & 0 & 0 & \dots & 0 & 0 \\ 0 & p_N + p & (n-1)p & 0 & \dots & 0 & 0 \\ 0 & 0 & p_N + 2p & (n-2)p & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (n-k+2)p & 0 \\ 0 & 0 & 0 & 0 & \dots & p_N + (k-1)p & (n-k+1)p \\ 0 & 0 & 0 & 0 & \dots & 0 & p_N + kp \end{pmatrix}. \tag{8}$$

Matrix $\mathbf{B}_k$, $k \in \{1,\dots,n-1\}$, is describing transitions from states $T_k$ to states $T_{k-1}$. Probabilities of these transitions are:

$$p_{(i,n-1-k),(j,n-k)} = \begin{cases} p_J, & i = j; \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

and the matrix has the form:

$$\mathbf{B}_k = \begin{pmatrix} p_J \mathbf{I}_{k\times k} \\ \mathbf{0}_{1\times k} \end{pmatrix}_{(k+1)\times k}. \tag{10}$$

Matrix $\mathbf{R}$ is describing transitions from transient to absorbing states, and can be partitioned as follows:

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_{n-1} \\ \mathbf{R}_{n-2} \\ \vdots \\ \mathbf{R}_0 \end{pmatrix}_{\frac{n(n+1)}{2}\times(n+1)}, \mathbf{R}_k = \begin{pmatrix} \mathbf{0}_{k\times(n-k-1)} & \mathbf{0}_{k\times 1} & \mathbf{0}_{k\times 1} & \mathbf{0}_{k\times k} \\ \mathbf{0}_{1\times(n-k-1)} & (n-k)p & p_J & \mathbf{0}_{1\times k} \end{pmatrix}. \tag{11}$$

The form of the matrix $\mathbf{R}_k$ follows from:

$$p_{(i,n-1-i),(j,m)} = \begin{cases} p_J, & i = j = k, m = n-k; \\ (n-k)p & i = k, j = k+1, m = n-k-1; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

In the next two lemmas, we derive some properties of the matrices $\mathbf{A}_k$ and $\mathbf{B}_k$, that will be needed later.

**Lemma 2.1.** *1. For matrices $\mathbf{B}_k$, $k \in \{2, \ldots, n-1\}$, the following equalities hold:*

$$\mathbf{B}_k\mathbf{B}_{k-1} = \begin{pmatrix} p_J^2 \mathbf{I}_{(k-1)\times(k-1)} \\ \mathbf{0}_{2\times(k-1)} \end{pmatrix}. \tag{13}$$

*2. For matrices $\mathbf{A}_k$ and $\mathbf{B}_k$, $k \in \{1, \ldots, n-1\}$, the following equalities hold:*

$$\mathbf{A}_k^s\mathbf{B}_k = \mathbf{B}_k\mathbf{A}_{k-1}^s = \begin{pmatrix} p_J \mathbf{A}_{k-1}^s \\ \mathbf{0}_{1\times k} \end{pmatrix}, s \in \mathbb{N}. \tag{14}$$

*Proof.* The equality (13) is obtained by block matrix multiplication. The equality (14) is obtained by block matrix multiplication, using the following representation of the matrix $\mathbf{A}_k$:

$$\mathbf{A}_k = \begin{pmatrix} \mathbf{A}_{k-1} & \mathbf{A}^* \\ \mathbf{0}_{1\times k} & p_N + kp \end{pmatrix}, \quad \mathbf{A}^* = \begin{pmatrix} \mathbf{0}_{(k-1)\times 1} \\ (n-k+1)p \end{pmatrix}. \tag{15}$$

□

**Lemma 2.2.** *1. Element in the m-th row and j-th column of the matrix $\mathbf{A}_k^s$, where $k \in \{0, 1, \ldots, n-1\}$, $m, j \in \{1, 2, \ldots, k+1\}$, $s \in \mathbb{N}$ is:*

$$\left(\mathbf{A}_k^s\right)_{m,j} = \sum_{i=0}^{k} (-1)^{j-1+i} \binom{n-m+1}{i-m+1} \binom{n-i}{j-1-i} (p_N + ip)^s. \tag{16}$$

*2. Element in the m-th row and j-th column of the matrix $(\mathbf{I} - \mathbf{A}_k)^{-s}$, where $k \in \{0, 1, \ldots, n-1\}$, $m, j \in \{1, 2, \ldots, k+1\}$, $s \in \mathbb{N}$ is:*

$$((\mathbf{I} - \mathbf{A}_k)^{-s})_{m,j} = \sum_{i=0}^{k} (-1)^{j-1+i} \binom{n-m+1}{i-m+1} \binom{n-i}{j-1-i} \frac{1}{(1 - p_N - ip)^s}. \tag{17}$$

**Remark 2.3.** *For convenience, instead of changing limits of the sums, we define:*

$$\binom{n}{k} = 0 \text{ if } k < 0 \text{ or } n < k. \tag{18}$$

*Proof.* 1. The matrix $\mathbf{A}_k$ is upper-triangular, therefore, eigenvalues coincide with the diagonal elements:

$$\mu_i = p_N + ip, \quad i \in \{0, 1, \ldots, k\}. \tag{19}$$

Straightforward calculation of the corresponding eigenvectors leads to representation:

$$\mathbf{A}_k = \mathbf{M}_k\mathbf{G}\mathbf{M}_k^{-1}, \tag{20}$$

where

$$\mathbf{G}_k = \begin{pmatrix} \mu_0 & 0 & 0 & \ldots & 0 \\ 0 & \mu_1 & 0 & \ldots & 0 \\ 0 & 0 & \mu_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \mu_k \end{pmatrix},$$

and elements in the $m$-th row and $j$-th column of matrices $\mathbf{M}_k$ and $\mathbf{M}_k^{-1}$ are given by:

$$(\mathbf{M}_k)_{m,j} = \begin{cases} \binom{n-m+1}{j-m,} & m \le j; \\ 0, & m > j, \end{cases} \text{ and } \left(\mathbf{M}_k^{-1}\right)_{m,j} = \begin{cases} (-1)^{m+j}\binom{n-m+1}{j-m}, & m \le j; \\ 0, & m > j, \end{cases} \tag{21}$$

respectively. We directly obtain:

$$\mathbf{A}_k^s = \mathbf{M}_k \mathbf{G}_k^s \mathbf{M}_k^{-1}. \tag{22}$$

Multiplying matrices in (22) leads to the expression (16).
2. Similarly to (22), using basic properties of matrix multiplication, we obtain:

$$\mathbf{I} - \mathbf{A}_k = \mathbf{M}_k(\mathbf{I} - \mathbf{G}_k)\mathbf{M}_k^{-1}, \tag{23}$$

$$(\mathbf{I} - \mathbf{A}_k)^{-1} = \mathbf{M}_k(\mathbf{I} - \mathbf{G}_k)^{-1}\mathbf{M}_k^{-1}, \tag{24}$$

and so

$$(\mathbf{I} - \mathbf{A}_k)^{-s} = \mathbf{M}_k(\mathbf{I} - \mathbf{G}_k)^{-s}\mathbf{M}_k^{-1}, \tag{25}$$

which yields the required statement. □

## 3. Transition probabilities in $k \geq 1$ steps and fundamental matrix

Our main result is contained in the next two theorems, where we provide the exact expressions for the $k$-step transition probability matrix $\mathbf{Q}_n^k$, and the fundamental matrix $\mathbf{F}_n = (\mathbf{I} - \mathbf{Q}_n)^{-1}$. We use the classical reference on Markov chain techniques [6], mathematical induction, and basic matrix calculus.

**Theorem 3.1.** *For $k \geq 1$ the following equality holds:*

$$\mathbf{Q}_n^k = \begin{pmatrix} \mathbf{D}_{n-1}^{(k,0)} & \mathbf{D}_{n-2}^{(k,1)} & \mathbf{D}_{n-3}^{(k,2)} & \dots & \mathbf{D}_1^{(k,n-2)} & \mathbf{D}_0^{(k,n-1)} \\ \mathbf{0} & \mathbf{D}_{n-2}^{(k,0)} & \mathbf{D}_{n-3}^{(k,1)} & \dots & \mathbf{D}_1^{(k,n-3)} & \mathbf{D}_0^{(k,n-2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{D}_1^{(k,0)} & \mathbf{D}_0^{(k,1)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{D}_0^{(k,0)} \end{pmatrix}, \tag{26}$$

*where*

$$\mathbf{D}_i^{(k,j)} = \begin{pmatrix} \binom{k}{j} p_J^j \mathbf{A}_i^{k-j} \\ \mathbf{0}_{j\times(i+1)} \end{pmatrix}_{(i+j+1)\times(i+1)} \tag{27}$$

*is the block describing the k-step transitions from states $T_{i+j}$ to states $T_i$.*

*Proof.* We use mathematical induction on the power $k$. For $k = 1$, we easily check that the representation (26) is valid. Next, assuming that (26) holds for the power $k - 1$, we prove that it holds for the power $k$. We have $\mathbf{Q}_n^k = \mathbf{Q}_n^{k-1} \cdot \mathbf{Q}_n$. On the other hand, we can write the matrix $\mathbf{Q}_n^k$ in the block-matrix form as follows:

$$\mathbf{Q}_n^k = \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} & \mathbf{V}_{1,3} & \dots & \mathbf{V}_{1,n-1} & \mathbf{V}_{1,n} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} & \mathbf{V}_{2,3} & \dots & \mathbf{V}_{2,n-1} & \mathbf{V}_{2,n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{V}_{n-1,1} & \mathbf{V}_{n-1,2} & \mathbf{V}_{n-1,3} & \dots & \mathbf{V}_{n-1,n-1} & \mathbf{V}_{n-1,n} \\ \mathbf{V}_{n,1} & \mathbf{V}_{n,2} & \mathbf{V}_{n,3} & \dots & \mathbf{V}_{n,n-1} & \mathbf{V}_{n,n} \end{pmatrix}, \tag{28}$$

where $\mathbf{V}_{i,j}$ is the block of the type $(n-i+1)\times(n-j+1)$. Multiplying matrices and using induction hypothesis, we obtain:

$$\mathbf{V}_{i,j} = \begin{cases} \mathbf{0}, & j < i; \\ \mathbf{D}_{n-i}^{(k-1,0)}\mathbf{A}_{n-i}, & j = i; \\ \mathbf{D}_{n-j+1}^{(k-1,j-i-1)}\mathbf{B}_{n-j+1} + \mathbf{D}_{n-j}^{(k-1,j-i)}\mathbf{A}_{n-j}, & j > i \end{cases} \tag{29}$$

$$= \begin{cases} \mathbf{0}, & j < i; \\ \mathbf{A}_{n-i}^{k-1}\mathbf{A}_{n-i}, & j = i; \\ \begin{pmatrix} \binom{k-1}{j-i-1}p_J^{j-i-1}\mathbf{A}_{n-j+1}^{k-j+i} \\ \mathbf{0}_{(j-i-1)\times(n-j+2)} \end{pmatrix}\mathbf{B}_{n-j+1} + \begin{pmatrix} \binom{k-1}{j-i}p_J^{j-i}\mathbf{A}_{n-j}^{k-j+i-1} \\ \mathbf{0}_{(j-i)\times(n-j+1)} \end{pmatrix}\mathbf{A}_{n-j}, & j > i. \end{cases}$$

For $j > i$, from Lemma 2.1 follows:

$$\mathbf{A}_{n-j+1}^{k-j+i}\mathbf{B}_{n-j+1} = \begin{pmatrix} p_J\mathbf{A}_{n-j}^{k-j+i} \\ \mathbf{0}_{1\times(n-j+1)} \end{pmatrix}, \tag{30}$$

and we have

$$\mathbf{0}_{(j-i-1)\times(n-j+2)}\mathbf{B}_{n-j+1} = \mathbf{0}_{(j-i-1)\times(n-j+1)}. \tag{31}$$

Finally, combining (29), (30) and (31), we obtain

$$\mathbf{V}_{i,j} = \begin{cases} \mathbf{0}, & j < i; \\ \mathbf{A}_{n-i}^{k}, & j = i; \\ \begin{pmatrix} \binom{k-1}{j-i-1}p_J^{j-i}\mathbf{A}_{n-j}^{k-j+i} \\ \mathbf{0}_{(j-i)\times(n-j+1)} \end{pmatrix} + \begin{pmatrix} \binom{k-1}{j-i}p_J^{j-i}\mathbf{A}_{n-j}^{k-j+i} \\ \mathbf{0}_{(j-i)\times(n-j+1)} \end{pmatrix}, & j > i \end{cases}$$

$$= \begin{cases} \mathbf{0}, & j < i; \\ \mathbf{A}_{n-i}^{k}, & j = i; \\ \begin{pmatrix} \binom{k}{j-i}p_J^{j-i}\mathbf{A}_{n-j}^{k-j+i} \\ \mathbf{0}_{(j-i)\times(n-j+1)} \end{pmatrix}, & j > i \end{cases}$$

$$= \begin{cases} \mathbf{0}, & j < i; \\ \mathbf{D}_{n-j}^{(k,j-i)}, & j \geq i, \end{cases} \tag{32}$$

which completes the proof of the theorem. □

**Theorem 3.2.** *Fundamental matrix* $\mathbf{F} = \mathbf{F}_n = (\mathbf{I} - \mathbf{Q}_n)^{-1}$ *can be partitioned as follows:*

$$\mathbf{F}_n = \begin{pmatrix} \mathbf{C}_{n-1}^{(0)} & \mathbf{C}_{n-2}^{(1)} & \mathbf{C}_{n-3}^{(2)} & \dots & \mathbf{C}_{1}^{(n-2)} & \mathbf{C}_{0}^{(n-1)} \\ \mathbf{0} & \mathbf{C}_{n-2}^{(0)} & \mathbf{C}_{n-3}^{(1)} & \dots & \mathbf{C}_{1}^{(n-3)} & \mathbf{C}_{0}^{(n-2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{C}_{1}^{(0)} & \mathbf{C}_{0}^{(1)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{C}_{0}^{(0)} \end{pmatrix}, \tag{33}$$

*where*

$$\mathbf{C}_i^{(j)} = \begin{pmatrix} p_J^j(\mathbf{I} - \mathbf{A}_i)^{-(j+1)} \\ \mathbf{0}_{j\times(i+1)} \end{pmatrix}_{(i+j+1)\times(i+1)} \tag{34}$$

*is the block describing the paths from states* $T_{i+j}$ *to states* $T_i$.

*Proof.* We use Theorem 3.1 and the relation:

$$\mathbf{F}_n = \sum_{k=0}^{\infty} \mathbf{Q}_n^k \tag{35}$$

(see, for example, [6], Theorem 3.2.1, p. 46).

We can write the matrix $\mathbf{F}_n$ in the following block-matrix form:

$$\mathbf{F}_n = \begin{pmatrix} \mathbf{U}_{1,1} & \mathbf{U}_{1,2} & \mathbf{U}_{1,3} & \dots & \mathbf{U}_{1,n-1} & \mathbf{U}_{1,n} \\ \mathbf{U}_{2,1} & \mathbf{U}_{2,2} & \mathbf{U}_{2,3} & \dots & \mathbf{U}_{2,n-1} & \mathbf{U}_{2,n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{U}_{n-1,1} & \mathbf{U}_{n-1,2} & \mathbf{U}_{n-1,3} & \dots & \mathbf{U}_{n-1,n-1} & \mathbf{U}_{n-1,n} \\ \mathbf{U}_{n,1} & \mathbf{U}_{n,2} & \mathbf{U}_{n,3} & \dots & \mathbf{U}_{n,n-1} & \mathbf{U}_{n,n} \end{pmatrix}, \tag{36}$$

where $\mathbf{U}_{i,j}$ is the block of the type $(n-i+1)\times(n-j+1)$. We have:

$$\mathbf{U}_{i,j} = \begin{cases} \mathbf{0}, & j < i; \\ \sum_{k=0}^{\infty} \mathbf{D}_{n-j}^{(k,j-i)}, & j \geq i, \end{cases} \tag{37}$$

and the result follows because for $j \geq i$

$$\sum_{k=j-i}^{\infty} \binom{k}{j-i} \mathbf{A}_{n-j}^{k-j+i} = (\mathbf{I} - \mathbf{A}_{n-j})^{-(j-i+1)}. \tag{38}$$

(see, for example, [10], p. 17). □

## 4. Properties of the waiting times $W_{n,n}$ and $W_{n,c}$

Here we obtain expressions for the distribution function, and the first and the second moment of the waiting times $W_{n,c}$ and $W_{n,n}$ using the results obtained in Section 3. We also provide additional results that are characteristic for CCPUC and do not appear in relation to the other variants of the problem.

We will need some more notation. With $S_1(\mathbf{M})$ we denote the sum of entries of the first row of the matrix $\mathbf{M}$, and $S_1^{(m)}(\mathbf{M})$ is the sum of the first $m$ entries of the first row of $\mathbf{M}$.

**Theorem 4.1.** *1. For the waiting time $W_{n,n}$, for any $t \in \mathbb{N}_0$, the following relation holds:*

$$P\{W_{n,n} > t\} = \sum_{i=0}^{n-1} \binom{t}{i} p_J{}^i \sum_{k=0}^{n-i-1} (-1)^{n-i-k-1} \binom{n-k-1}{i} \binom{n}{k} (kp+p_N)^{t-i}. \tag{39}$$

*2. For the waiting time $W_{n,c}$, $1 \leq c \leq n$, for any $t \in \mathbb{N}_0$, the following relation holds:*

$$P\{W_{n,c} > t\} = \sum_{i=0}^{c-1} \binom{t}{i} p_J{}^i \sum_{k=0}^{c-i-1} (-1)^{c-i-k-1} \binom{n-k-1}{n-c+i} \binom{n}{k} (kp + p_N)^{t-i}. \tag{40}$$

*Proof.* 1. The expression $P\{W_{n,n} > t\}$ is equal to probability that, if the initial state of the chain is $(0,0)$, the chain is, after $t$ steps, still in one of the transient states. Therefore,

$$P\{W_{n,n} > t\} = S_1\left(\mathbf{Q}^t\right) = \sum_{i=0}^{n-1} S_1\left(\mathbf{D}_{n-1-i}^{(t,i)}\right) = \sum_{i=0}^{n-1} \binom{t}{i} p_J^i S_1\left(\mathbf{A}_{n-1-i}{}^{t-i}\right). \tag{41}$$

Using Lemma 2.2, we can see that the element in the $m$-th row and $j$-th column of the matrix $\mathbf{A}_{n-1-i}{}^{t-i}$ is:

$$\left(\mathbf{A}_{n-1-i}{}^{t-i}\right)_{m,j} = \sum_{s=0}^{n-i-1}(-1)^{j-1+s}\binom{n-m+1}{s-m+1}\binom{n-s}{j-1-s}\mu_s^{t-i}, \tag{42}$$

where $\mu_s = p_N + sp$. Using (42), (18), and the relation

$$\sum_{m=0}^{a}(-1)^m\binom{b}{m} = (-1)^a\binom{b-1}{a}, \quad 0 \le a < b, \tag{43}$$

we obtain:

$$S_1\left(\mathbf{A}_{n-1-i}{}^{t-i}\right) = \sum_{j=1}^{n-i}\sum_{s=0}^{n-i-1}\binom{n}{s}\mu_s^{t-i}(-1)^{j-1+s}\binom{n-s}{j-1-s} = \sum_{s=0}^{n-i-1}(-1)^{n-i-1+s}\binom{n}{s}\binom{n-s-1}{i}\mu_s^{t-i}. \tag{44}$$

Therefore, from (41)

$$P\{W_{n,n} > t\} = \sum_{i=0}^{n-1}\binom{t}{i}p_J^i\sum_{s=0}^{n-i-1}(-1)^{n-i-1+s}\binom{n}{s}\binom{n-s-1}{i}(p_N+sp)^{t-i}. \tag{45}$$

2. Proof is a simple modification of the proof of the first part of Theorem. We define the column matrix $\mathbf{H}_c$ as follows:

$$\mathbf{H}_c = \begin{pmatrix} \mathbf{1}_{c\times 1} \\ \mathbf{0}_{(n-c)\times 1} \\ \mathbf{1}_{(c-1)\times 1} \\ \mathbf{0}_{(n-c)\times 1} \\ \vdots \\ \mathbf{1}_{1\times 1} \\ \mathbf{0}_{(n-c)\times 1} \\ \mathbf{0}_{\left(\frac{n(n+1)}{2}-\frac{c(2n-c+1)}{2}\right)\times 1} \end{pmatrix}. \tag{46}$$

We notice that if $i + j > k + l$, the transition from state $(i, j)$ to state $(k, l)$ is not possible. Therefore, we can rearrange the transition probability matrix (5) in the following form:

$$\mathbf{P}^{(c)} = \begin{pmatrix} \mathbf{Q}^{(c)} & \mathbf{R}^{(c)} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{47}$$

where the matrix $\mathbf{Q}^{(c)}$ is describing transitions between states

$$T^{(c)} = \{(i, j), i + j \le c - 1\}. \tag{48}$$

In this case, sum of any row of the matrix $\left(\mathbf{Q}^{(c)}\right)^t$ is equal to the element in the same row of the matrix $\mathbf{Q}^t\mathbf{H}_c$, for any $t \in \mathbb{N}$. This is because the matrix $\mathbf{H}_c$ is canceling (setting to zero) all the transition probabilities to states from the set $S \setminus T^{(c)}$.
The expression $P\{W_{n,c} > t\}$ is equal to probability that the chain, starting from the state $(0, 0)$, is, after $t$ steps, still in one of the states from $T^{(c)}$. Therefore,

$$P\{W_{n,c} > t\} = S_1\left(\left(\mathbf{Q}^{(c)}\right)^t\right) = S_1\left(\mathbf{Q}^t\mathbf{H}_c\right) = \sum_{i=0}^{c-1} S_1^{(c-i)}\left(\mathbf{D}_{n-1-i}{}^{(t,i)}\right) = \sum_{i=0}^{c-1}\binom{t}{i}p_J^iS_1^{(c-i)}\left(\mathbf{A}_{n-1-i}{}^{t-i}\right). \tag{49}$$

Using (42),(18), and the relation (43), we obtain:

$$S_1^{(c-i)}\left(\mathbf{A}_{n-1-i}{}^{t-i}\right) = \sum_{j=1}^{c-i}\sum_{s=0}^{n-i-1}\binom{n}{s}\mu_s^{t-i}(-1)^{j-1+s}\binom{n-s}{j-1-s} = \sum_{s=0}^{c-i-1}(-1)^{c-i-1+s}\binom{n}{s}\binom{n-s-1}{n-c+i}\mu_s^{t-i}, \quad (50)$$

where $\mu_s = p_N + sp$. Therefore,

$$P\{W_{n,c} > t\} = \sum_{i=0}^{c-1}\binom{t}{i}p_J^i\sum_{s=0}^{c-i-1}(-1)^{c-i-1+s}\binom{n}{s}\binom{n-s-1}{n-c+i}(p_N+sp)^{t-i}. \quad (51)$$

□

Next, we provide expressions for the first and the second moment of the waiting time $W_{n,c}$.

**Theorem 4.2.** *For the waiting time $W_{n,c}$, $1 \le c \le n$, the following relations hold:*
*1.*

$$E(W_{n,c}) = \sum_{i=0}^{c-1}p_J{}^i\sum_{k=0}^{c-i-1}(-1)^{c-i-1-k}\binom{n-k-1}{n-c+i}\binom{n}{k}\frac{1}{(1-p_N-kp)^{i+1}}, \quad (52)$$

*2.*

$$E(W_{n,c}^2) = \sum_{i=0}^{c-1}p_J{}^i\sum_{k=0}^{c-i-1}(-1)^{c-i-1-k}\binom{n-k-1}{n-c+i}\binom{n}{k}\frac{2i+1+kp+p_N}{(1-p_N-kp)^{i+2}}. \quad (53)$$

*Proof.* 1. We have:

$$E(W_{n,c}) = \sum_{t=0}^{+\infty}P\{W_{n,c} > t\} = \sum_{i=0}^{c-1}p_J{}^i\sum_{k=0}^{c-i-1}(-1)^{c-i-1-k}\binom{n-k-1}{n-c+i}\binom{n}{k}\sum_{t=i}^{+\infty}\binom{t}{i}(kp+p_N)^{t-i}. \quad (54)$$

The required statement follows from the fact:

$$\sum_{t=i}^{+\infty}\binom{t}{i}a^{t-i} = \frac{1}{(1-a)^{i+1}}, \quad |\, a \,| < 1 \quad (55)$$

(see [10], p. 17).
2. We have:

$$E(W_{n,c}^2) = \sum_{t=0}^{+\infty}P\{W_{n,c} > t\} + 2\sum_{t=0}^{+\infty}tP\{W_{n,c} > t\}, \quad (56)$$

and:

$$\sum_{t=0}^{+\infty}tP\{W_{n,c} > t\} = \sum_{i=0}^{c-1}p_J{}^i\sum_{k=0}^{c-i-1}(-1)^{c-i-1-k}\binom{n-k-1}{n-c+i}\binom{n}{k}\sum_{t=i}^{+\infty}t\binom{t}{i}(kp+p_N)^{t-i}. \quad (57)$$

On the other hand, for $|\, a \,| < 1$, using (55) we obtain:

$$\begin{aligned}\sum_{t=i}^{+\infty}t\binom{t}{i}a^{t-i} &= \sum_{t=i}^{+\infty}i\binom{t}{i}a^{t-i} + \sum_{t=i}^{+\infty}(t-i)\binom{t}{i}a^{t-i}\\ &= i\sum_{t=i}^{+\infty}\binom{t}{i}a^{t-i} + a(i+1)\sum_{t=i+1}^{+\infty}\binom{t}{i+1}a^{t-i-1}\\ &= \frac{i}{(1-a)^{i+1}} + \frac{a(i+1)}{(1-a)^{i+2}}\\ &= \frac{i+a}{(1-a)^{i+2}}. \quad (58)\end{aligned}$$

Using (52), (56), (57) and (58) we obtain the required statement. □

For $c = n$ the expected waiting times obtained in Theorem 4.2 can be written in simpler form.

**Proposition 4.3.** *For the waiting time $W_{n,n}$ the following relations hold:*
*1.*

$$E(W_{n,n}) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{(kp)^{k-1}}{(kp+p_J)^k}. \tag{59}$$

*2.*

$$E(W_{n,n}^2) = 2p_J\sum_{k=2}^{n}(-1)^{k-2}\binom{n}{k}(k-1)\frac{(kp)^{k-2}}{(kp+p_J)^{k+1}} + 2\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{(kp)^{k-1}}{(kp+p_J)^{k+1}} - E(W_{n,n}). \tag{60}$$

*Proof.* 1. Using the binomial theorem and basic properties of binomial identities, we obtain:

$$\begin{aligned} E(W_{n,n}) &= \sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{n}{k}\sum_{i=0}^{n-k-1}(-1)^i\binom{n-k-1}{i}\frac{{p_J}^i}{(1-p_N-kp)^{i+1}} \\ &= \sum_{k=0}^{n-1}(-1)^{n-k-1}\binom{n}{k}\frac{1}{(n-k)p+p_J}\left(\frac{(n-k)p}{(n-k)p+p_J}\right)^{n-k-1} \\ &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{kp+p_J}\left(\frac{kp}{kp+p_J}\right)^{k-1} \\ &= \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{(kp)^{k-1}}{(kp+p_J)^k}. \end{aligned} \tag{61}$$

2. From (53) we have:

$$E(W_{n,n}^2) = 2\sum_{i=0}^{n-1}{p_J}^i\sum_{k=0}^{n-i-1}(-1)^{n-i-1-k}\binom{n-k-1}{i}\binom{n}{k}\frac{i+1}{(1-p_N-kp)^{i+2}} - E(W_{n,n}), \tag{62}$$

and using the binomial theorem and basic properties of binomial identities, we obtain:

$$\begin{aligned}
&\sum_{i=0}^{n-1} p_J{}^i \sum_{k=0}^{n-i-1} (-1)^{n-i-1-k} \binom{n-k-1}{i}\binom{n}{k} \frac{i+1}{(1-p_N-kp)^{i+2}} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \sum_{i=1}^{n-k-1} (-1)^{n-i-1-k} \binom{n-k-1}{i} p_J{}^i \frac{i}{(1-p_N-kp)^{i+2}} \\
&\quad + \sum_{k=0}^{n-1} \binom{n}{k} \sum_{i=0}^{n-k-1} (-1)^{n-i-1-k} \binom{n-k-1}{i} p_J{}^i \frac{1}{(1-p_N-kp)^{i+2}} \\
&= \sum_{k=0}^{n-2} \binom{n}{k} \sum_{i=0}^{n-k-2} (-1)^{n-i-2-k} \binom{n-k-2}{i} p_J{}^{i+1} \frac{n-k-1}{(1-p_N-kp)^{i+3}} \\
&\quad + \sum_{k=0}^{n-1} \binom{n}{k} \sum_{i=0}^{n-k-1} (-1)^{n-i-1-k} \binom{n-k-1}{i} p_J{}^i \frac{1}{(1-p_N-kp)^{i+2}} \\
&= p_J \sum_{k=0}^{n-2} \binom{n}{k} (-1)^{n-k-2} \frac{n-k-1}{(1-p_N-kp)^3} \left(1 - \frac{p_J}{1-p_N-kp}\right)^{n-k-2} \\
&\quad + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k-1} \frac{1}{(1-p_N-kp)^2} \left(1 - \frac{p_J}{1-p_N-kp}\right)^{n-k-1} \\
&= p_J \sum_{k=0}^{n-2} \binom{n}{k} (-1)^{n-k-2} \frac{n-k-1}{((n-k)p+p_J)^3} \left(\frac{(n-k)p}{(n-k)p+p_J}\right)^{n-k-2} \\
&\quad + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k-1} \frac{1}{((n-k)p+p_J)^2} \left(\frac{(n-k)p}{(n-k)p+p_J}\right)^{n-k-1} \\
&= p_J \sum_{k=2}^{n} \binom{n}{k} (-1)^{k-2} \frac{k-1}{(kp+p_J)^3} \left(\frac{kp}{kp+p_J}\right)^{k-2} + \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{1}{(kp+p_J)^2} \left(\frac{kp}{kp+p_J}\right)^{k-1} \\
&= p_J \sum_{k=2}^{n} (-1)^{k-2} \binom{n}{k} (k-1) \frac{(kp)^{k-2}}{(kp+p_J)^{k+1}} + \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{(kp)^{k-1}}{(kp+p_J)^{k+1}},
\end{aligned} \tag{63}$$

which completes the proof of the proposition. □

Using Proposition 4.3, we obtain a simple upper bound for the expected waiting time $E(W_{n,n})$.

**Proposition 4.4.** *For the expected waiting time $E(W_{n,n})$ the following inequality holds:*

$$E(W_{n,n}) \leq \frac{1}{2p}\left(2^n + \frac{1}{n} - \frac{p}{p+p_J}\right). \tag{64}$$

*Proof.* From the inequalities:

$$\frac{1}{np} \leq \frac{1}{kp} \leq \frac{1}{p}, \quad \frac{p}{p+p_J} \leq \frac{kp}{kp+p_J} \leq \frac{np}{np+p_J}, \tag{65}$$

that hold for $1 \leq k \leq n$, we obtain:

$$\frac{1}{np}\left(\frac{p}{p+p_J}\right)^k \leq \frac{(kp)^{k-1}}{(kp+p_J)^k} \leq \frac{1}{p}\left(\frac{np}{np+p_J}\right)^k, \quad 1 \leq k \leq n, \tag{66}$$

Using the relation (66) we obtain:

$$\begin{aligned}
E(W_{n,n}) &= \sum_{1\le 2k-1\le n}\binom{n}{2k-1}\frac{((2k-1)p)^{2k-2}}{((2k-1)p+p_J)^{2k-1}} - \sum_{2\le 2k\le n}\binom{n}{2k}\frac{(2kp)^{2k-1}}{(2kp+p_J)^{2k}} \\
&\le \frac{1}{p}\sum_{1\le 2k-1\le n}\binom{n}{2k-1}\left(\frac{np}{np+p_J}\right)^{2k-1} - \frac{1}{np}\sum_{2\le 2k\le n}\binom{n}{2k}\left(\frac{p}{p+p_J}\right)^{2k} \\
&= \frac{1}{2p}\left(\left(1+\frac{np}{np+p_J}\right)^n - \left(1-\frac{np}{np+p_J}\right)^n\right) - \frac{1}{np}\left(\frac{1}{2}\left(\left(1+\frac{p}{p+p_J}\right)^n + \left(1-\frac{p}{p+p_J}\right)^n\right) - 1\right). \qquad (67)
\end{aligned}$$

Since the following inequalities hold:

$$\left(1+\frac{np}{np+p_J}\right)^n \le 2^n, \quad \left(1-\frac{np}{np+p_J}\right)^n \ge 0, \quad \left(1+\frac{p}{p+p_J}\right)^n \ge 1+\frac{np}{p+p_J}, \quad \left(1-\frac{p}{p+p_J}\right)^n \ge 0, \qquad (68)$$

we have:

$$E(W_{n,n}) \le \frac{1}{2p}\left(2^n - \frac{1}{n}\left(1+\frac{np}{p+p_J}-2\right)\right) = \frac{1}{2p}\left(2^n + \frac{1}{n} - \frac{p}{p+p_J}\right), \qquad (69)$$

which completes the proof of the statement. □

**Remark 4.5.** *An expression for the variance of the waiting time $W_{n,c}$ follows directly from Theorem 4.2. Corresponding results related to the waiting time $W_{n,n}$ follow from Theorem 4.2 when $c = n$.*

**Remark 4.6.** *Another way to obtain the results stated in Theorem 4.2 would be to use directly the fundamental matrix obtained in Theorem 3.2 and the fact that the sum of the first row of the fundamental matrix is equal to $E(W_{n,n})$. Similar procedure can be applied to obtain the variance of $W_{n,n}$ (see [6]), as well as the corresponding results related to the waiting time $W_{n,c}$.*

**Remark 4.7.** *Using the fact that $W_{n,c} \ge c$ with probability 1, and Theorem 4.1, we were able to obtain new combinatorial identities:*

$$\sum_{i=0}^{c-1}\binom{t}{i}{p_J}^i\sum_{k=0}^{c-i-1}(-1)^{c-i-k-1}\binom{n-k-1}{n-c+i}\binom{n}{k}(kp+p_N)^{t-i} = 1, \qquad (70)$$

*for any $n \in \mathbb{N}$, $1 \le c \le n$, $0 \le t \le c-1$, $0 \le p_N, p_J < 1$, and $p = (1-p_N-p_J)/n$.*

Combining Theorem 3.1 and Theorem 3.2 with the representation (11), it is possible to obtain results related to the type of coupons collected, and the probability mass function for the waiting time $W_{n,c}$. We formulate one such result, related to the waiting time $W_{n,n}$.

**Theorem 4.8.** *If the full collection of coupons is sampled, probability that exactly $m$ coupons from $\mathbb{N}_n$ and $n-m$ jokers are sampled is equal to*

$$p_{(m,n-m)} = \begin{cases} pS_0, & m = n; \\ (n-m+1)pS_{n-m} + p_JS_{n-m-1}, & m = 1,\dots n-1; \\ p_JS_{n-1}, & m = 0, \end{cases} \qquad (71)$$

*where:*

$$S_r = \binom{n}{r+1}p_J^r\sum_{i=0}^{n-r-1}(-1)^i\binom{n-r-1}{i}\frac{1}{((r+1+i)p+p_J)^{r+1}}, \quad r = 0,1,\dots,n-1. \qquad (72)$$

*Proof.* The first row of the matrix $\mathbf{F}_n\mathbf{R}$ consists of probabilities that the chain starting in the transient state $(0, 0)$ ends up in an absorbing state (see Theorem 3.3.7 (page 52) in [6]). Therefore, the probability that exactly $m$ standard coupons and $n-m$ jokers are sampled, if the full collection is sampled, is the element of the first row and $(n-m+1)$-th column of the matrix $\mathbf{F}_n\mathbf{R}$. Using the representation of the fundamental matrix $\mathbf{F}_n$ obtained in Theorem 3.2, and the form of the matrix $\mathbf{R}$ in (11), we can write:

$$p_{(m,n-m)} = \begin{cases} p\left(\mathbf{C}_{n-1}^{(0)}\right)_{1,n}, & m = n; \\ (n-m+1)p\left(\mathbf{C}_{m-1}^{(n-m)}\right)_{1,m} + p_J\left(\mathbf{C}_{m}^{(n-1-m)}\right)_{1,m+1}, & m = 1, \dots n-1; \\ p_J\left(\mathbf{C}_{0}^{(n-1)}\right)_{1,1}, & m = 0. \end{cases} \tag{73}$$

Using Lemma 2.2, we obtain:

$$\left(\mathbf{C}_{n-r-1}^{(r)}\right)_{1,n-r} = \left(p_J^r\left(\mathbf{I}-\mathbf{A}_{n-r-1}\right)^{-(r+1)}\right)_{1,n-r} = \sum_{i=0}^{n-r-1}(-1)^{n-r+i-1}\binom{n}{i}\binom{n-i}{r+1}\frac{p_J^r}{(1-p_N-ip)^{r+1}}. \tag{74}$$

Finally, using the identity:

$$\binom{n}{i}\binom{n-i}{r+1} = \binom{n}{r+1}\binom{n-r-1}{i} \tag{75}$$

and simple sum manipulations, we complete the proof of the theorem. □

**Example 4.9.** *Let $p_J = p_N = p = 1/(n+2)$. Using Theorem 4.8 we obtain that the probability that all the coupons from the set $\mathbb{N}_n$ are sampled before a joker is equal to*

$$pS_0 = \frac{1}{n+2}\binom{n}{1}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{1}{\frac{i+2}{n+2}} = n\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{1}{i+2} = \frac{1}{n+1}.$$

*The last equality follows from the identity (see [10], eq. (7.7), p. 82), that holds for any $r \in \mathbb{N}$:*

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{1}{k+r} = \frac{m!(r-1)!}{(m+r)!}. \tag{76}$$

*Probability that $n$ joker coupons are sampled before the first coupon from the set $\mathbb{N}_n$ is sampled is equal to*

$$p_J S_{n-1} = \frac{1}{(n+1)^n}.$$

## 5. Numerical examples

In this section, we present numerical results for the problem we have considered.

First we consider a very simple case $n = 3$. In this case, the set of available coupons is $\mathbb{N}_3^{\star,\circ} = \{1, 2, 3, \star, \circ\}$. If a joker is drawn with probability $p_J = 1/8$, and null coupon is drawn with probability $p_N = 1/8$, then any coupon $k \in \mathbb{N}_3$ is drawn with probability $p = 1/4$. The transition probability matrix for one step is

$$\mathbf{P} = \begin{pmatrix} 0.125 & 0.75 & 0 & 0.125 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.375 & 0.5 & 0 & 0.125 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.625 & 0 & 0 & 0 & 0.25 & 0.125 & 0 & 0 \\ 0 & 0 & 0 & 0.125 & 0.75 & 0.125 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.375 & 0 & 0 & 0.5 & 0.125 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.125 & 0 & 0 & 0.75 & 0.125 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Next, we consider different combinations of sampling probabilities $p_N$ and $p_J$, and, by direct matrix manipulation, repeat some of the results presented in Section 4.

Numerical results corresponding to Theorems 4.1 - 4.8 in Section 4 are presented in Tables 1 - 6. Statistical software R was used for all the calculations.

| $t$ | $p_J = 0$, $p_N = 0$ | $p_J = \frac{1}{8}$, $p_N = 0$ | $p_J = 0$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{2}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{2}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0.875 | 0.875 | 0.875 | 0.5 |
| 2 | 1 | 1 | 0.9844 | 0.9844 | 0.9844 | 0.75 |
| 3 | 1 | 1 | 0.9989 | 0.9989 | 0.9989 | 0.875 |
| 4 | 1 | 1 | 0.9997 | 0.9997 | 0.9997 | 0.93750 |
| 5 | 1 | 1 | 0.9999 | 0.9999 | 0.9999 | 0.96875 |

Table 1: $P\{W_{3,1} \le t\}$ for different values of $t$

| $t$ | $p_J = 0$, $p_N = 0$ | $p_J = \frac{1}{8}$, $p_N = 0$ | $p_J = 0$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{2}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{2}$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0.6667 | 0.7448 | 0.5104 | 0.5781 | 0.7187 | 0.2031 |
| 3 | 0.8889 | 0.9256 | 0.7869 | 0.8398 | 0.9336 | 0.4238 |
| 4 | 0.963 | 0.9783 | 0.9101 | 0.9402 | 0.9849 | 0.6047 |
| 5 | 0.9876 | 0.9937 | 0.9624 | 0.9776 | 0.9965 | 0.7373 |

Table 2: $P\{W_{3,2} \le t\}$ for different values of $t$

| $t$ | $p_J = 0$, $p_N = 0$ | $p_J = \frac{1}{8}$, $p_N = 0$ | $p_J = 0$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{2}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{2}$ |
|---|---|---|---|---|---|---|
| 3 | 0.2222 | 0.3832 | 0.1489 | 0.2715 | 0.5586 | 0.0664 |
| 4 | 0.4444 | 0.6371 | 0.335 | 0.5227 | 0.8428 | 0.1814 |
| 5 | 0.6173 | 0.7901 | 0.5027 | 0.699 | 0.9477 | 0.3154 |
| 6 | 0.7407 | 0.8789 | 0.6368 | 0.8128 | 0.9829 | 0.4469 |
| 7 | 0.8258 | 0.93 | 0.7381 | 0.8841 | 0.9944 | 0.5641 |
| 8 | 0.8834 | 0.9594 | 0.8126 | 0.9282 | 0.9981 | 0.6626 |
| 9 | 0.9221 | 0.9764 | 0.8665 | 0.9555 | 0.9994 | 0.7423 |

Table 3: $P\{W_{3,3} \le t\}$ for different values of $t$

| $t$ | $p_J = 0$, $p_N = 0$ | $p_J = \frac{1}{8}$, $p_N = 0$ | $p_J = 0$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{2}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{2}$ |
|---|---|---|---|---|---|---|
| 3 | 0.2222 | 0.3832 | 0.1489 | 0.2715 | 0.5586 | 0.0664 |
| 4 | 0.2222 | 0.2539 | 0.1861 | 0.2512 | 0.2842 | 0.115 |
| 5 | 0.1728 | 0.153 | 0.1677 | 0.1763 | 0.1049 | 0.134 |
| 6 | 0.1234 | 0.0888 | 0.1341 | 0.1138 | 0.0352 | 0.1315 |
| 7 | 0.085 | 0.0511 | 0.1014 | 0.0712 | 0.0115 | 0.1172 |
| 8 | 0.0576 | 0.0294 | 0.0745 | 0.0441 | 0.0038 | 0.0985 |
| 9 | 0.0381 | 0.017 | 0.0539 | 0.0273 | 0.0012 | 0.0796 |

Table 4: $P\{W_{3,3} = t\}$ for different values of $t$

| | $p_J = 0$, $p_N = 0$ | $p_J = \frac{1}{8}$, $p_N = 0$ | $p_J = 0$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{2}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{2}$ |
|---|---|---|---|---|---|---|
| $E(W_{3,3})$ | 5.5 | 4.48 | 6.29 | 5 | 3.68 | 7.79 |

Table 5: Expectation of the waiting time $W_{3,3}$

| $m$ | $p_J = 0$, $p_N = 0$ | $p_J = \frac{1}{8}$, $p_N = 0$ | $p_J = 0$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{2}$, $p_N = \frac{1}{8}$ | $p_J = \frac{1}{8}$, $p_N = \frac{1}{2}$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0.002 | 0 | 0.0029 | 0.1866 | 0.0156 |
| 1 | 0 | 0.0602 | 0 | 0.0763 | 0.4937 | 0.1927 |
| 2 | 0 | 0.4334 | 0 | 0.4637 | 0.2912 | 0.5417 |
| 3 | 1 | 0.5044 | 1 | 0.4571 | 0.0286 | 0.25 |

Table 6: Probability of collecting $m$ standard coupons and $3 - m$ jokers, if the full collection is sampled

Next, we provide graphical representation of the expected waiting time $E(W_{n,n})$ for different values of parameters $n$, and the following combinations of the parameters $p_N$ and $p_J$:

1. $p_J \in [0, 1]$ and $p_N = p = \frac{1-p_J}{n+1}$;
2. $p_N \in [0, 1]$ and $p_J = p = \frac{1-p_N}{n+1}$;
3. $p_N = p_J \in [0, 0.5]$.

These results are displayed in Figures 1-3.

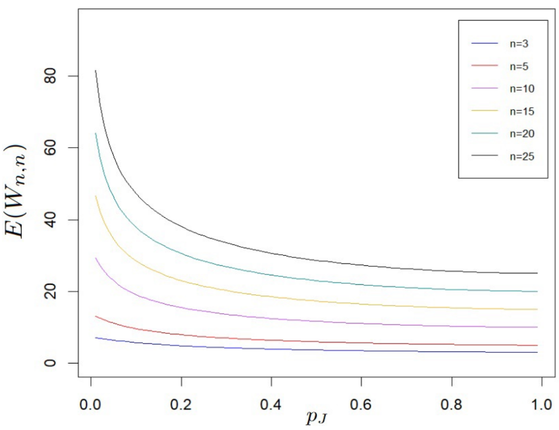


Figure 1: Expected waiting time $E(W_{n,n})$ in terms of $p_J$, for different values of $n$, if $p_N = p = \frac{1-p_J}{n+1}$

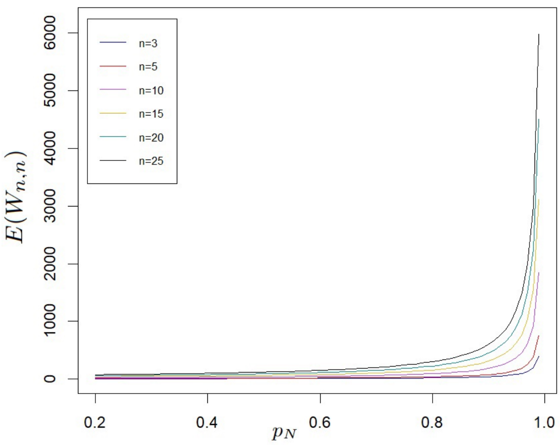


Figure 2: Expected waiting time $E(W_{n,n})$ in terms of $p_N$ for different values of $n$, if $p_J = p = \frac{1-p_N}{n+1}$

Note that the behavior of the expected waiting time $E(W_{n,n})$, depicted in Figure 1 and Figure 2, is consistent with the intuition we have about the problem: $E(W_{n,n})$ increase when the probability $p_N$ increase (as the null coupons slows down the collecting process, and therefore extends the expected waiting time) and decrease when the probability $p_J$ increase (as the joker coupon speeds up the collecting process).

We can also notice that $E(W_{n,n})$ increase when $n$ increase, as expected.

Behavior of $E(W_{n,n})$ as the function of $p_N = p_J$, depicted in Figure 3 is much more intriguing, as it suggests that the minimal expected waiting time is obtained when $p_N = p_J \approx 1/3$ in most cases considered.

These facts may be addressed in future research.

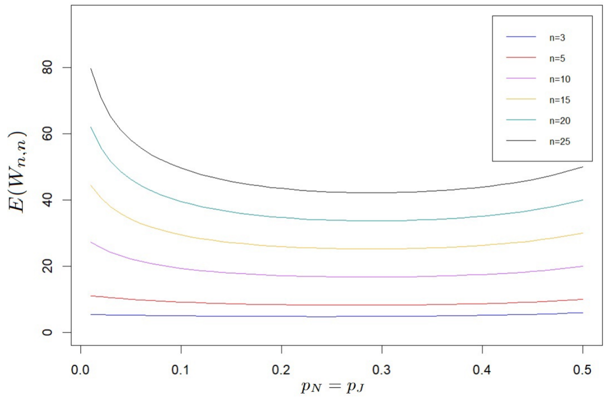


Figure 3: Expected waiting time $E(W_{n,n})$ in terms of $p_N = p_J$, for different values of $n$

## 6. Discussion

In this work, we considered a generalization of the CCCP, such that there are additional coupons that can slow down, or speed up the collection process.

Using more or less elementary mathematics and the well known Markov chain techniques, we obtained the exact forms of the transition probability matrix for more than one step, as well as of the fundamental matrix, which enabled us to obtain exact expressions for probabilities related to the waiting time until a full collection, or a part of the collection is sampled, generalizing the corresponding results for several existing versions of CCCP.

It is well known that CCCP and its generalizations may have applications in engineering (see, for example, [1], [4], [12], and the references therein). There are also recent applications in the field of biology ([15]).

Possible applications of this work relate to the detection of distributed deny of service (DDoS) attacks, which are explained, for example, in [1]. The authors conclude that, since the solution to these attacks is to continuously monitor Internet traffic, the occurrence of standard coupons in the coupon collector problem corresponds to tracking $c$ (out of $n$) recent high traffic flows, where the portion size $c$ is determined by server capacity, and that the probability of null coupon, $p_N$, corresponds to the case of flows with small probabilities that sum up to $p_N$. In this context, joker coupon can be considered as flows (not necessarily high flows), that need to be monitored regardless of their size because they are related to, for example, a highly suspicious source or a malfunctioning router.

## Acknowledgments

The authors would like to express their gratitude to the anonymous referee for his/hers comments towards an improved final version of the paper.

## References

[1] E. Anceaume, Y. Busnel, B. Sericola, *New results on a generalized coupon collector problem using Markov chains*, J. Appl. Probab. **52(2)** (2015), 405–418.
[2] E. Anceaume, Y. Busnel, E. Schulte-Geers, B. Sericola, *Optimization Results for a Generalized Coupon Collector Problem*, J. Appl. Probab. **53** (2016), 622–629.
[3] L. E. Baum, P. Billingsley, *Asymptotic distributions for the coupon collector's problem*, Ann. Math. Statist. **36** (1965), 1835–1839.
[4] A. Boneh, M. Hofri, *The Coupon-Collector Problem Revisited*, Department of Computer Science Purdue University Technical Reports. Paper 807. https://docs.lib.purdue.edu/cstech/807(1989).
[5] M. Ferrante, A. Tagliavini, *On the coupon-collector's problem with several parallel collections*, arXiv:1609.04174 (2016).
[6] J. G. Kemeny, L. J. Snell, *Finite Markov chains*, Springer - Verlag, New York - Heidelberg, 1976.

[7] B. Kim, J. Kim, *Sooner waiting time problems in a sequence of multi-state trials with random rewards*, Statist. Probab. Lett. **153** (2019), 171–179.
[8] P. Mladenović, *Limit distributions for the problem of collecting pairs*, Bernoulli **14(2)** (2008), 419–439.
[9] T. Nakata, I. Kubo, *A coupon collector's problem with bonuses*, In: Proceedings of the 4th Colloquium on Mathematics and Computer Science, DMTCS Proceedings, (2006), 215–224.
[10] J. Quaintance, H. W. Gould, *Combinatorial identities for Stirling numbers, The unpublished notes of H. W. Gould*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2016.
[11] E. Samuel-Cahn, *Asymptotic distributions for occupancy and waiting time problems with positive probability of falling through the cells*, Ann. Probab. **2(3)** (1974), 515–521.
[12] S. Shioda, *Some upper and lower bounds on the coupon collector problem*, J. Comput. Appl. Math. **200** (2007), 154–167.
[13] B. Todić, *Coupon collector problem with penalty coupon*, Matematički Vesnik **76 (1-2)** (2024), 15–28.
[14] M. Wild, S. Janson, S. Wagner, D. Laurie, *Coupon collecting and transversals of hypergraphs*, Discrete Math. Theor. Comput. Sci.**15(2)** (2013), 259–270.
[15] N. Zoroa, E. Lesigne, M. J. Fernández-Sáez, P. Zoroa, J. Casas, *The coupon collector urn model with unequal probabilities in ecology and evolution*, J. R. Soc. Interface **14** (2016), 20160643.