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Existence analysis on a (k,φ)-Hilfer FBVPs with a linear combination of (k,φ)-Hilfer derivatives

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Abstract. In this paper, we apply classical Banach and Krasnosel'skiĭ's fixed point theorems along with Leray-Schauder alternative to study the existence and uniqueness of solutions for FDEs involving (k,φ)-Hilfer fractional derivative with a linear combination of two (k,φ)-Hilfer derivatives in boundary value conditions. Finally, we present some examples to validate our theoretical outcomes.

1. Introduction

Fractional-order integral and derivative operators appear in fractional calculus to study many scientific phenomena associated with physics, chemistry and engineering problems, see [1]-[9]. Fractional derivative operators are constructed by applying fractional integral operators of many kinds such as Riemann-Liouville, Caputo, Hadamard, Katugampola, Hilfer and etc. Some fractional derivative operators are special cases of the other types of fractional derivatives. For example, Riemann-Liouville and Hadamard fractional derivatives are obtained as special cases of the generalized fractional derivative which have been introduced by Katugampola [10, 11]. On the other hand, fractional derivative operators Caputo, Caputo-Hadamard and Caputo-Erdelyi are introduced by the ψ-fractional derivative operator [12]. Authors in [19] introduced the (k,ψ)-Hilfer fractional derivative operator which generalize some known fractional derivative operators. In [14], authors studied the multi-order boundary value problem (BVP) consisting of two fractional derivatives supplemented with a linear composition of fractional integral in the boundary conditions:

$$\begin{cases}(\hat{r}\, D_{0^+}^{\hat{\sigma}_1} + (1-\hat{r})\, D_{0^+}^{\hat{\sigma}_2})z(r) = f(r,z(r)), \quad r\in[0,\tau^*],\\ z(0)=0, \quad \hat{r}\, I_{0^+}^{v_1} z(\tau^*) + (1-\hat{r})\, I_{0^+}^{v_2} z(\tau^*) = a_0,\end{cases} \tag{1}$$

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in which $D^{\eta}_{0^+}$ is the Riemann-Liouville fractional derivative of order $\eta \in \{\hat{\sigma}_1, \hat{\sigma}_2\}$ such that $1 < \hat{\sigma}_1, \hat{\sigma}_2 < 2$, $I^{\eta'}_{0^+}$ is the Riemann-Liouville integral with $\eta' \in \{v_1, v_2\}$, $0 < v_1, v_2 \leq 1$ and $a_0 \in \mathbb{R}$. The investigation of the BVP:

$$\begin{cases}(\hat{r}\, D^{\hat{\sigma}}_{0^+} + (1-\hat{r})\, D^{v}_{0^+})z(r) = f(r, z(r)), \quad r \in [0, \tau^*],\\ z(0) = 0, \quad \hat{r}\, D^{\hat{\sigma}_1} z(\tau^*) + I^{\hat{\sigma}_2}_{0} z(\tau_0) = a_0,\end{cases} \tag{2}$$

was initiated in [15], in which $D^{\hat{\sigma}}_{0^+}$ and $D^{v}_{0^+}$ indicate the Riemann-Liouville fractional derivatives with $1 < \hat{\sigma} \leq 2$, $1 \leq v < \sigma$, $0 < \hat{r} \leq 1$, $0 \leq \hat{\sigma}_1 \leq \hat{\sigma} - v$, $\hat{\sigma}_2 \geq 0$, $a_0 \in \mathbb{R}$ and $0 < \tau_0 < \tau^*$.

For more information about boundary value problems with fractional derivatives see, [19]-[23]. Inspired by the above works, Rezapour et al. [16] have studied the Liouville-Caputo integro-differential BVP of the form:

$$\begin{cases}(\hat{r}\, {}^cD^{\hat{\sigma}_1}_{0^+} + (1-\hat{r})\, I^{\hat{\sigma}_2}_{0^+})z(r) = f(r, z(r)) + {}^cD^{\hat{\sigma}_3}_{0^+}\tilde{f}(r, z(r)), \quad r \in [0, \tau^*],\\ z(0) = 0, \quad \hat{r}\, {}^cD^{v_1}_{0^+} z(\tau^*) + (1-\hat{r})\, {}^cD^{v_2}_{0^+} z(\tau^*) = a_0,\end{cases} \tag{3}$$

in which ${}^cD^{\gamma}_{0^+}$ is the Caputo derivative with $\gamma \in \{\hat{\sigma}_1, \hat{\sigma}_2, v_2, v_1\}$, $a_0 \in \mathbb{R}$ and $I^{\hat{\sigma}_2}_{0^+}$ indicates the Riemann-Liouville fractional integral such that $1 < \hat{\sigma}_1, \hat{\sigma}_3 \leq 2$, $\hat{\sigma}_1 > \hat{\sigma}_3$, $0 < \hat{\sigma}_2 \leq 1$, $0 < r \leq 1$, $0 < v_1, v_2 < \hat{\sigma}_1 - \hat{\sigma}_3$ and $f, \tilde{f} \in C([0, \tau^*] \times \mathbb{R}, \mathbb{R})$.

Recently, Ntouyas et al. [17] have considered the BVP involving (k, ψ)-Hilfer type fractional derivatives of order in $(1, 2]$, supplemented with a linear combination of (k, ψ)-Hilfer type derivative and integral operators of the form

$$\begin{cases}{}^{k,H}D^{\alpha,\beta,\psi} u(r) = f(r, u(r)), \quad r \in (a, b],\\ u(a) = 0, \quad u(b) = \lambda\, {}^{k,H}D^{p,q,\psi} u(\eta) + \mu\, {}^kI^{v,\psi} u(\sigma).\end{cases} \tag{4}$$

Here, ${}^{k,H}D^{\alpha,\beta,\psi}$ is the (k, ψ)-Hilfer-type fractional derivative of order α, $1 < \alpha < 2$ and parameter β, $0 \leq \beta \leq 1$, $k > 0$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, ${}^{k,H}D^{p,q,\psi}$ indicates the (k, ψ)-Hilfer-type fractional derivative of order p, $1 < p < 2$ and parameter q, $0 \leq q \leq 1$, $p < \alpha$, ${}^kI^{v,\psi}$ denotes the (k, ψ)-Riemann-Liouville fractional integral of order $v > 0$, $\lambda, \mu \in \mathbb{R}$, and $a < \xi, v < b$.

Motivated by the above works, our goal of this study is to consider multi-order BVP with linear combination of fractional derivatives in boundary conditions:

$$\begin{cases}(\hat{r}\, {}^{k,H}D^{\hat{\sigma}_1,\beta,\varphi} + (1-\hat{r})\, {}^{k,H}I^{\hat{\sigma}_2,\varphi}_{0^+})\breve{z}(r) = f(r, \breve{z}(r)) + {}^{k,H}D^{\hat{\sigma}_3,\beta,\varphi}\tilde{f}(r, \breve{z}(r)), \quad r \in [0, \tau^*],\\ \breve{z}(0) = 0, \quad , \hat{r}\, {}^{k,H}D^{v_1,\beta,\varphi}\breve{z}(\tau^*) + (1-\hat{r})\, {}^{k,H}D^{v_2,\beta,\varphi}\breve{z}(\tau^*) = a_0,\end{cases} \tag{5}$$

where ${}^{k,H}D^{\eta,\beta,\varphi}$ indicates the (k, φ)-Hilfer-type fractional derivative of order η, with $\eta \in \{\hat{\sigma}_1, \hat{\sigma}_3, v_1, v_2\}$, $a_0 \in \mathbb{R}$ and ${}^{k,H}I^{\hat{\sigma}_2,\varphi}_{0^+}$ stands for the (k, φ)-Riemann-Liouville fractional integral of order $\hat{\sigma}_2 > 0$ such that $k > 0$, $1 < \hat{\sigma}_1, \hat{\sigma}_3 \leq 2$, $\hat{\sigma}_1 > \hat{\sigma}_3$, $0 < \hat{\sigma}_2 \leq 1$, $0 < \hat{r} \leq 1$, $0 < v_1, v_2 < \hat{\sigma}_1 - \hat{\sigma}_3$. The presented results will be considered via classical Banach principle along with Leray-Schauder nonlinear alternative and Krasnoselskiǐ fixed point theorem.

The remainder of this paper will be arranged as follows: First the main concepts are recalled in Section 2. In Section 3, a basic lemma is proved to convert the problem (5) into a fixed point problem. The existence and uniqueness result concerning the problem (5) are presented in Section 4. Section 5 contains illustrative numerical examples.

2. Preliminaries

First, some definitions and lemmas related to this work are recalled.

Definition 2.1. *[18] Let $h \in L^1([a,b],\mathbb{R})$, $k > 0$ and φ is an increasing function with $\varphi'(t) \neq 0$ for all $t \in [a,b]$. Then the (k,φ)-Riemann-Liouville fractional integral of order $\alpha > 0$ ($\alpha \in \mathbb{R}$) of the function f is given by*

$$ {}^{k}\mathcal{I}^{\alpha;\varphi}_{a+} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_{a+}^{\theta} \varphi'(s)(\varphi(t) - \varphi(s))^{\frac{\alpha}{k}-1} f(s) ds. $$

Definition 2.2. *[19] Let $\alpha, k \in \mathbb{R}^+ = (0,\infty)$, $\beta \in [0,1]$, φ is an increasing function such that $\varphi \in C^n([a,b],\mathbb{R})$, $\varphi'(t) \neq 0, t \in [a,b]$ and $f \in C^n([a,b],\mathbb{R})$. Then the (k,φ)-Hilfer fractional derivative of the function h of order α and type β, is defined by*

$$ {}^{k,H}\mathcal{D}^{\alpha,\beta;\varphi} f(t) = \mathcal{I}^{\beta(nk-\alpha);\varphi}_{a+} \left(\frac{k}{\varphi'(t)} \frac{d}{dt} \right)^{n} {}^{k}I^{(1-\beta)(nk-\alpha);\varphi}_{a+} f(t), \quad n = \left\lceil \frac{\alpha}{k} \right\rceil. $$

Lemma 2.3. *[19] Let $\mu, k \in \mathbb{R}^+$ and $n = \left\lceil \frac{\mu}{k} \right\rceil$. Assume that $\mathfrak{h} \in C^n([a,b],\mathbb{R})$ and ${}^{k}\mathfrak{I}^{nk-\mu;\psi}_{a+}\mathfrak{h} \in C^n([a,b],\mathbb{R})$. Then*

$$ {}^{k}\mathcal{I}^{\mu;\psi} \left({}^{k,RL}\mathcal{D}^{\mu;\psi} \mathfrak{h}(w) \right) = \mathfrak{h}(w) - \sum_{j=1}^{n} \frac{(\psi(w) - \psi(a))^{\frac{\mu}{k}-j}}{\Gamma_k(\mu - jk + k)} \left[\left(\frac{k}{\psi'(w)} \frac{d}{dw} \right)^{n-j} {}^{k}\mathcal{I}^{nk-\mu;\psi}_{a+} \mathfrak{h}(w) \right]_{z=a}. $$

Lemma 2.4. *[19] Let $\alpha, k \in \mathbb{R}^+$ with $\alpha < k$, $\beta \in [0,1]$ and $\theta_k = \alpha + \beta(k-\alpha)$. Then*

$$ {}^{k}\mathcal{I}^{\theta_k;\psi} \left({}^{k,RL}\mathcal{D}^{\theta_k;\psi} \mathfrak{h} \right)(w) = {}^{k}\mathcal{I}^{\alpha;\psi} \left({}^{k,H}\mathcal{D}^{\alpha,\beta;\psi} \mathfrak{h} \right)(w), \quad \mathfrak{h} \in C^n([a,b],\mathbb{R}). $$

Lemma 2.5. *[19] Let $\zeta, k \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$ such that $\frac{\eta}{k} > -1$. Then*

$$ (i).\ {}^{k}\mathcal{I}^{\zeta,\psi}(\psi(t) - \psi(a))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta + k)}{\Gamma_k(\eta + k + \zeta)} (\psi(t) - \psi(a))^{\frac{\eta+\zeta}{k}}. $$

$$ (ii).\ {}^{k}\mathcal{D}^{\zeta,\psi}(\psi(t) - \psi(a))^{\frac{\eta}{k}} = \frac{\Gamma_k(\eta + k)}{\Gamma_k(\eta + k - \zeta)} (\psi(t) - \psi(a))^{\frac{\eta-\zeta}{k}}. $$

Lemma 2.6. *[2] Let $\alpha_1, \alpha_2, \beta, k \in (0,\infty)$ with $\alpha_2 > \alpha_1$, $k > 0$ and $\beta \in [0,1]$. Then,*

$$ {}^{k,H}\mathcal{D}^{\alpha_1,\beta;\varphi} \left({}^{k}\mathcal{I}^{\alpha_2;\varphi}_{0^+} \right) h(r) = {}^{k}\mathcal{I}^{\alpha_2-\alpha_1;\varphi}_{0^+} h(r), \quad h \in C([a,b],\mathbb{R}). $$

3. An auxiliary Result

Lemma 3.1. *Let $k > 0$, $1 < \hat{\sigma}_1, \hat{\sigma}_3 \leq 2$, $\hat{\sigma}_1 > \hat{\sigma}_3$, $0 < \hat{\sigma}_2 \leq 1$, $0 < \hat{r} \leq 1$, $0 < \hat{v}_1, \hat{v}_2 < \hat{\sigma}_1 - \hat{\sigma}_3$, $p = \hat{\sigma}_1 + \beta(2k - \hat{\sigma}_1)$ and $h, \tilde{h} \in C^2([0,\tau^*],\mathbb{R})$. Then, if*

$$ \Delta := (\hat{r} - 1) \frac{1}{\Gamma_k(p - v_2)} (\varphi(\tau^*) - \varphi(0))^{\frac{p-v_2}{k}-1} - \hat{r} \frac{1}{\Gamma_k(p - v_1)} (\varphi(\tau^*) - \varphi(0))^{\frac{p-v_1}{k}-1} \neq 0, $$

the unique solution of the linear fractional BVP:

$$ \begin{cases} (\hat{r}\, {}^{k,H}\mathcal{D}^{\hat{\sigma}_1,\beta;\varphi} + (1 - \hat{r})\, {}^{k}\mathcal{I}^{\hat{\sigma}_2;\varphi}_{0^+}) \breve{z}(r) = h(r) + {}^{k,H}\mathcal{D}^{\hat{\sigma}_3,\beta;\varphi} \tilde{h}(r), \quad r \in [0,\tau^*], \\ \breve{z}(0) = 0, \quad \hat{r}\, {}^{k,H}\mathcal{D}^{v_1,\beta;\varphi} \breve{z}(\tau^*) + (1 - \hat{r})\, {}^{k,H}\mathcal{D}^{v_2,\beta;\varphi} \breve{z}(\tau^*) = a_0, \end{cases} \tag{6} $$

is given by

$$ \begin{aligned} \breve{z}(r) \quad = \quad & \frac{\hat{r} - 1}{\hat{r}}\, {}^{k}\mathcal{I}^{\hat{\sigma}_1+\hat{\sigma}_2;\varphi}_{0^+} \breve{z}(r) + \frac{1}{\hat{r}}\, {}^{k}\mathcal{I}^{\hat{\sigma}_1;\varphi}_{0^+} h(r) + \frac{1}{\hat{r}}\, {}^{k}\mathcal{I}^{\hat{\sigma}_1-\hat{\sigma}_3;\varphi}_{0^+} \tilde{h}(r) \\ & + \frac{1}{\Delta} \Big[(\hat{r} - 1)\, {}^{k}\mathcal{I}^{\hat{\sigma}_1+\hat{\sigma}_2-v_1;\varphi}_{0^+} \breve{z}(\tau^*) + {}^{k}\mathcal{I}^{\hat{\sigma}_1-v_1;\varphi}_{0^+} h(\tau^*) + {}^{k}\mathcal{I}^{\hat{\sigma}_1-\hat{\sigma}_3-v_1;\varphi}_{0^+} \tilde{h}(\tau^*) \\ & + \frac{(1 - \hat{r})^2}{\hat{r}}\, {}^{k}\mathcal{I}^{\hat{\sigma}_1+\hat{\sigma}_2-v_2;\varphi}_{0^+} \breve{z}(\tau^*) + \frac{(1 - \hat{r})}{\hat{r}}\, {}^{k}\mathcal{I}^{\hat{\sigma}_1-v_2;\varphi}_{0^+} h(\tau^*) \\ & + \frac{(1 - \hat{r})}{\hat{r}}\, {}^{k}\mathcal{I}^{\hat{\sigma}_1-\hat{\sigma}_3-v_2;\varphi}_{0^+} \tilde{h}(\tau^*) - a_0 \Big] \frac{(\varphi(\tau^*) - \varphi(0))^{\frac{p}{k}-1}}{\Gamma_k(p)}. \end{aligned} \tag{7} $$

Proof. Assume that $\breve{z}$ is a solution of the boundary value problem (3.1). Taking fractional integral operator ${}^{k}I^{\hat{\sigma}_1;\varphi}$ on both sides of the first equation in (6) and using Lemmas 2.3 and 2.4, we conclude that

$$\begin{aligned}\breve{z}(r) &= \frac{\hat{r}-1}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2;\varphi}\breve{z}(r)+\frac{1}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1;\varphi}h(r)+\frac{1}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3;\varphi}\tilde{h}(r)\\ &+c_0\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{\Gamma_k(p)}+c_1\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-2}}{\Gamma_k(p-k)}. \end{aligned} \tag{8}$$

Using $\breve{z}(0)=0$ with (8) we conclude that $c_1=0$ since $\frac{p}{k}-2<0$. By using Lemma 2.5, we have

$$ {}^{k}\mathcal{D}^{v_1,\beta,\varphi}(\varphi(t)-\varphi(0))^{\frac{p}{k}-1}=\frac{\Gamma_k(p)}{\Gamma_k(p-v_1)}(\varphi(t)-\varphi(0))^{\frac{p-v_1}{k}-1} \tag{9}$$

and

$$ {}^{k}\mathcal{D}^{v_2,\beta,\varphi}(\varphi(t)-\varphi(0))^{\frac{p}{k}-1}=\frac{\Gamma_k(p)}{\Gamma_k(p-v_2)}(\varphi(t)-\varphi(0))^{\frac{p-v_2}{k}-1} \tag{10}$$

Now applying (9), (10) and the condition $\hat{r}\,{}^{k,H}\mathcal{D}^{v_1,\beta;\varphi}\breve{z}(\tau^*)+(1-\hat{r})\,{}^{k,H}\mathcal{D}^{v_2,\beta;\varphi}\breve{z}(\tau^*)=a_0$ in (8), after inserting $c_1=0$, we get

$$\begin{aligned}&(\hat{r}-1)\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_1;\varphi}\breve{z}(\tau^*)+{}^{k}I_{0^+}^{\hat{\sigma}_1-v_1;\varphi}h(\tau^*)+{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_1;\varphi}\tilde{h}(\tau^*)\\ &+\frac{(1-\hat{r})^2}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_2;\varphi}\breve{z}(\tau^*)+\frac{(1-\hat{r})}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_2;\varphi}h(\tau^*)+\\ &+\frac{(1-\hat{r})}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_2;\varphi}\tilde{h}(\tau^*)-a_0\\ &=c_0\Big((\hat{r}-1)\frac{1}{\Gamma_k(p-v_2)}(\varphi(\tau^*)-\varphi(0))^{\frac{p-v_2}{k}-1}-\hat{r}\frac{1}{\Gamma_k(p-v_1)}(\varphi(\tau^*)-\varphi(0))^{\frac{p-v_1}{k}-1}\Big).\end{aligned}$$

Consequently, we conclude that

$$\begin{aligned}c_0 &= \frac{1}{\Delta}\Big[(\hat{r}-1)\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_1;\varphi}\breve{z}(\tau^*)+{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_1;\varphi}h(\tau^*)+{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_1;\varphi}\tilde{h}(\tau^*)\\ &+\frac{(1-\hat{r})^2}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_2;\varphi}\breve{z}(\tau^*)+\frac{(1-\hat{r})}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_2;\varphi}h(\tau^*)\\ &+\frac{(1-\hat{r})}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_2;\varphi}\tilde{h}(\tau^*)-a_0\Big].\end{aligned}$$

Replacing c_0 in (8) we get (7). We can prove the converse by direct computation. The proof is finished. □

4. Existence and uniqueness results

Let $\mathfrak{X}=C([0,\tau^*],\mathbb{R})$ be the Banach space endowed with the norm $\|\breve{x}\|=\max\{|\breve{x}(r)|: r\in[0,\tau^*]\}$. It is obvious that the space $\mathfrak{X}$ is a Banach space.

Now, using Lemma 3.1, assume that the operator $\mathcal{Q}:\mathfrak{X}\longrightarrow\mathfrak{X}$ has been defined as follows:

$$\begin{aligned}\mathcal{Q}(\breve{x})(r) &= \frac{\hat{r}-1}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2;\varphi}\breve{x}(r)+\frac{1}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1;\varphi}f(r,\breve{x}(r))+\frac{1}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3;\varphi}\tilde{f}(r,\breve{x}(r))\\ &+\frac{1}{\Delta}\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{\Gamma_k(p)}\Big[(\hat{r}-1)\,{}^{k}I_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_1;\varphi}\breve{x}(\tau^*)\\ &+{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_1;\varphi}f(\tau^*,\breve{x}(\tau^*))+{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_1;\varphi}\tilde{f}(\tau^*,\breve{x}(\tau^*))\\ &+\frac{(1-\hat{r})^2}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_2;\varphi}\breve{x}(\tau^*)+\frac{(1-\hat{r})}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_2;\varphi}f(\tau^*,\breve{x}(\tau^*))\\ &+\frac{(1-\hat{r})}{\hat{r}}\,{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_2;\varphi}\tilde{f}(\tau^*,\breve{x}(\tau^*))-a_0\Big].\end{aligned} \tag{11}$$

For convince, we set:

$$
\begin{aligned}
\mathcal{G}_1 &= \Big\{\frac{|\hat{r}-1|}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2+k)} \\
&+\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)}\Big[\frac{(|\hat{r}-1|)(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_1}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_1+k)} \\
&+\frac{(1-\hat{r})^2}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_2+k)}\Big]\Big\}, \\
\mathcal{G}_2 &= \Big\{\frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1}}{\Gamma_k(\hat{\sigma}_1+k)}+\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)}\Big[\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_1}}{\Gamma_k(\hat{\sigma}_1-v_1+k)} \\
&\frac{(1-\hat{r})}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_2}}{\Gamma_k(\hat{\sigma}_1-v_2+k)}\Big]\Big\}, \\
\mathcal{G}_3 &= \Big\{\frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3+k)}+\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)}\Big[\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v1}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_1+k)} \\
&+\frac{(1-\hat{r})}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v2}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_2+k)}\Big]\Big\}, \\
\mathcal{G}_4 &= |a_0|\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)}. \qquad (12)
\end{aligned}
$$

In the following theorem, Banach's fixed-point theorem is applied to present an existence and uniqueness result to the problem (5).

Theorem 4.1. *Suppose that the following condition holds:*

$(\mathcal{D}_1)$

$$
\begin{aligned}
&|f(r,\breve{x})-f(r,\breve{y})| \le \mathcal{G}|\breve{x}-\breve{y}|, \\
&|\breve{f}(r,\breve{x})-\breve{f}(r,\breve{y})| \le \overline{\mathcal{G}}|\breve{x}-\breve{y}|,
\end{aligned}
$$

where $\mathcal{G},\overline{\mathcal{G}} > 0$ *for each* $r \in [0,\tau^*]$ *and* $\breve{x},\breve{y} \in \mathbb{R}$.

Then a unique solution of the problem (5) is obtained on $[0,\tau^*]$, *provided that*

$$\mathcal{S} := \mathcal{G}_1 + \mathcal{G}_2\mathcal{G} + \mathcal{G}_3\overline{\mathcal{G}} < 1. \qquad (13)$$

Proof. We show that the hypotheses of Banach's fixed point theorem are satisfied by the operator $\mathcal{Q}$. Assume that $\mathcal{M} = \sup_{r\in[0,\tau^*]}|f(r,0)|$ and $\mathcal{N} = \sup_{r\in[0,\tau^*]}|\breve{f}(r,0)|$. Now assume that $B_\delta = \{\breve{x} \in C([0,\tau^*],\mathbb{R}) : \|\breve{x}\| \le \delta\}$ with

$$\delta \ge \frac{\mathcal{G}_2\mathcal{M}+\mathcal{G}_3\mathcal{N}+\mathcal{G}_4}{1-\mathcal{G}_1-\mathcal{G}_2\mathcal{G}-\mathcal{G}_3\overline{\mathcal{G}}}. \qquad (14)$$

First, we indicate that $\mathcal{Q}(B_\delta) \subseteq B_\delta$. Applying condition $(\mathcal{D}_1)$, we get

$$
\begin{aligned}
|f(r,\breve{x}(r))| &\le |f(r,\breve{x}(r))-f(r,0)|+|f(r,0)|, \\
&\le \mathcal{G}\|\breve{x}\|+\mathcal{M} \le \mathcal{G}\delta+\mathcal{M}.
\end{aligned}
$$

Similarly, we conclude that

$$|\tilde{f}(r,\breve{x}(r))| \le \overline{\mathcal{G}}\delta+\mathcal{N}.$$

Accordingly, for all $\check{x} \in B_\delta$, we get

$$
\begin{aligned}
|(\mathcal{Q})(\check{x})(r)| \leq\ & \frac{|\hat{r}-1|}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2;\varphi}|\check{x}(r)| + \frac{1}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1;\varphi}(|f(r,\check{x}(r)) - f(r,0)| + |f(r,0)|) \\
&+\frac{1}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3;\varphi}(|\tilde{f}(r,\check{x}(r)) - \check{f}(r,0)| + |\check{f}(r,0)|) \\
&+\frac{1}{|\Delta|}\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{\Gamma_k(p)}\Big[\,|(\hat{r}-1)|\ {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_1;\varphi}|\check{x}|(\tau^*) \\
&+{}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_1;\varphi}(|f(\tau^*,\check{x}(\tau^*)) - \check{f}(\tau^*,0)| + |\check{f}(\tau^*,0)|) \\
&+{}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_1;\varphi}(|\tilde{f}(\tau^*,\check{x}(\tau^*)) - \tilde{f}(\tau^*,0)| + |\tilde{f}(\tau^*,0)|) \\
&+\frac{(1-\hat{r})^2}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_2;\varphi}|\check{x}(\tau^*)| \\
&+\frac{(1-\hat{r})}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_2;\varphi}(|f(\tau^*,\check{x}(\tau^*)) - f(\tau^*,0)| + |f(\tau^*,0)|) \\
&+\frac{(1-\hat{r})}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_2;\varphi}(|\check{f}(\tau^*,\check{x}(\tau^*)) - \check{f}(\tau^*,0)| + |\check{f}(\tau^*,0)|) + |a_0|\Big] \\
\leq\ & \frac{|\hat{r}-1|}{\hat{r}}\|\check{x}\|\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2+k)} + \frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1}}{\Gamma_k(\hat{\sigma}_1+k)}(\mathcal{G}\delta+\mathcal{M}) \\
&+\frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3+k)}(\overline{\mathcal{G}}\delta+\mathcal{N}) \\
&+\frac{1}{|\Delta|}\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{\Gamma_k(p)}\Big[\,|(\hat{r}-1)|\,\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_1}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_1+k)}\|\check{x}\| \\
&+\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_1}}{\Gamma_k(\hat{\sigma}_1-v_1+k)}(\mathcal{G}\delta+\mathcal{M}) + \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v_1}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_1+k)}(\overline{\mathcal{G}}\delta+\mathcal{N}) \\
&+\frac{(1-\hat{r})^2}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_2+k)}\|\hat{x}\| \\
&+\frac{1-\hat{r}}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_2}}{\Gamma_k(\hat{\sigma}_1-v_2+k)}(\mathcal{G}\delta+\mathcal{M}) + \\
&+\frac{1-\hat{r}}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v_2}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_2+k)}(\overline{\mathcal{G}}\delta+\mathcal{N}) + |a_0|\Big] \\
\leq\ & \Big\{\frac{|\hat{r}-1|}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2+k)} \\
&+\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)}\Big[\frac{(\hat{r}-1)(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_1}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_1+k)} \\
&+\frac{(1-\hat{r})^2}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_2+k)}\Big]\Big\}\delta \\
&+\Big\{\frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1}}{\Gamma_k(\hat{\sigma}_1+k)} + \frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)}\Big[\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_1}}{\Gamma_k(\hat{\sigma}_1-v_1+k)} \\
&\frac{(1-\hat{r})}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_2}}{\Gamma_k(\hat{\sigma}_1-v_2+k)}\Big]\Big\}(\mathcal{G}\delta+\mathcal{M}) + \Big\{\frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3}}{\Gamma_k(\hat{\sigma}_1-(\hat{\sigma}_3+k)} \\
&+\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)}\Big[\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v1}}{\Gamma_k(\hat{\sigma}_1-(\hat{\sigma}_3-v_1+k)} \\
&+\frac{(1-\hat{r})}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v2}}{\Gamma_k(\hat{\sigma}_1-(\hat{\sigma}_3-v_2+k)}\Big]\Big\}(\overline{\mathcal{G}}r+N) \\
&+|a_0|\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)} \\
=\ & \mathcal{G}_1\delta + \mathcal{G}_2(\mathcal{G}\delta+\mathcal{M}) + \mathcal{G}_3(\overline{\mathcal{G}}\delta+\mathcal{N}) + \mathcal{G}_4 \leq \delta.
\end{aligned}
$$

Inconsequence, $\|\mathcal{Q}(\check{x})\| \le \delta$ and we have $Q(B_\delta) \subseteq B_\delta$. Now, we indicate that the operator $\mathcal{Q}$ is a contraction. For all $r \in [0, \tau^*]$ and $\check{x}, \check{y} \in B_\delta$ we have

$$\begin{aligned}
\mathcal{Q}(\check{x})(r) - \mathcal{Q}(\check{y})(r) \mid \ \le\ & \frac{\mid \hat{r}-1 \mid}{\hat{r}}\, {}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2;\varphi}|\check{x}(r)-\check{y}(r)| \\
&+\frac{1}{\hat{r}}\, {}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1;\varphi}(|f(r,\check{x}(r)) - f(r,\check{y}(r))|) + \frac{1}{\hat{r}}\, {}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3;\varphi}(|\tilde{f}(r,\check{x}(r)) - \check{f}(r,\check{y}(r))| \\
&+\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{\mid\Delta\mid\Gamma_k(p)}\Big[(\mid \hat{r}-1\mid)\, {}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_1;\varphi}|\check{x}-\check{y}|(\tau^*) \\
&+{}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_1;\varphi}|f(\tau^*,\check{x}(\tau^*)) - f(\tau^*,\check{y}(\tau^*))| + {}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_1;\varphi}|\tilde{f}(\tau^*,\check{x}(\tau^*)) - \tilde{f}(\tau^*,\check{y}(\tau^*))| \\
&+\frac{(1-\hat{r})^2}{\hat{r}}\, {}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_2;\varphi}|\check{x}(\tau^*)-\check{y}(\tau^*)| \\
&+\frac{(1-\hat{r})}{\hat{r}}\, {}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_2;\varphi}|f(\tau^*,\check{x}(\tau^*)) - f(\tau^*,\check{y}(\tau^*))| \\
&+\frac{(1-\hat{r})}{\hat{r}}\, {}^{k}\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_2;\varphi}|\check{f}(\tau^*,\check{x}(\tau^*)) - \check{f}(\tau^*,\check{y}(\tau^*))|\Big] \\
\le\ & \Big\{\frac{\mid \hat{r}-1\mid}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2+k)}\|\check{x}-\check{y}\| + \frac{1}{\hat{r}}\mathcal{G}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1}}{\Gamma_k(\hat{\sigma}_1+k)}\|\check{x}-\check{y}\| \\
&+\frac{1}{\hat{r}}\overline{\mathcal{G}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3+k)}\|\check{x}-\check{y}\| \\
&+\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{\mid\Delta\mid\Gamma_k(p)}\Big[(\mid \hat{r}-1\mid)\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_2-v_1}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_2-v_1+k)}\|\check{x}-\check{y}\| \\
&+\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_1}}{\Gamma_k(\hat{\sigma}_1-v_1+k)}\mathcal{G}\|\check{x}-\check{y}\| + \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v_1}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_1+k)}\overline{\mathcal{G}}\|\check{x}-\check{y}\| \\
&+\frac{(1-\hat{r})^2}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_2+k)}\|\check{x}-\check{y}\| + \frac{(1-\hat{r})}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_2}}{\Gamma_k(\hat{\sigma}_1-v_2+k)}\mathcal{G}\|\check{x}-\check{y}\| \\
&+\frac{(1-\hat{r})}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v_2}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_2+k)}\overline{\mathcal{G}}\|\check{x}-\check{y}\|\Big]\Big\} \\
=\ & \mathcal{S}\|\check{x}-\check{y}\|.
\end{aligned}$$

Hence, $\|\mathcal{Q}(\check{x}) - \mathcal{Q}(\check{y})\| \le \mathcal{S}\|\check{x}-\check{y}\|$ and using $\mathcal{S} < 1$ we conclude that $\mathcal{Q}$ is contraction. Now, applying Banach contraction principle we conclude that $\mathcal{Q}$ has a unique fixed point which is a unique solution of the problem (5). □

In the following theorem, Krasnosel'skiĭ fixed point theorem [24] is applied to present an existence result.

Theorem 4.2. *Assume that the functions $f, \tilde{f} : [0,\tau^*] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions satisfying condition $(\mathcal{D}_2)$. Moreover we assume that:*

$(\mathcal{D}_2)$

$$\mid f(r,\check{x}(r)) \mid \le \mathcal{P}(r), \quad \mid \tilde{f}(r,\check{x}(r)) \mid \le \overline{\mathcal{P}}(r),$$

for all $(r,\check{x}) \in [0,\tau^*] \times \mathbb{R}$, and $\mathcal{P}, \overline{\mathcal{P}} \in C([0,\tau^*],[0,\infty))$.
Then the problem (5) has at least one solution on $[0,\tau^*]$, if $\mathcal{S}_1 < 1$, in which

$$\begin{aligned}
\mathcal{S}_1 :=& \frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{\mid\Delta\mid\Gamma_k(p)}\Big[\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_1}}{\Gamma_k(\hat{\sigma}_1-v_1+k)}\frac{(1-\hat{r})}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_2}}{\Gamma_k(\hat{\sigma}_1-v_2+k)}\Big]\Big\}\parallel\mathcal{P}\parallel + \Big\{\frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3+k)} \\
&+\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{\mid\Delta\mid\Gamma_k(p)}\Big[\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v1}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_1+k)} + \frac{(1-\hat{r})}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v2}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_2+k)}\Big]\Big\}\parallel\overline{\mathcal{P}}\parallel. \qquad (15)
\end{aligned}$$

Proof. Let $\sup_{r\in[0,\tau^*]} \mid \mathcal{P}(r) \mid = \parallel \mathcal{P} \parallel$, $\sup_{r\in[0,\tau^*]} \mid \overline{\mathcal{P}} \mid = \parallel \overline{\mathcal{P}} \parallel$ and $\mathcal{B}_\delta = \{\check{x} \in C([0,\tau^*],\mathbb{R}) : \parallel \check{x} \parallel \le \delta\}$, with $\delta \ge \mathcal{G}_1\delta + \mathcal{G}_2 \parallel \mathcal{P} \parallel + \mathcal{G}_3 \parallel \overline{\mathcal{P}} \parallel + \mathcal{G}_4$. Assume that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have been defined on B_δ as follows:

$$\mathcal{Q}_1\check{x}(r) = \frac{1}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1;\varphi} f(r,\check{x}(r)) + \frac{1}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3;\varphi} \tilde{f}(r,\check{x}(r)),$$

and

$$\begin{aligned}\mathcal{Q}_2\check{x}(r) =& \frac{\hat{r}-1}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2;\varphi}\check{x}(r) + \frac{1}{\mid\Delta\mid}\frac{(\varphi(r)-\varphi(0))^{\frac{p}{k}-1}}{\Gamma_k(p)}\Big[(\mid \hat{r}-1 \mid)\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_1;\varphi}\check{x}(\tau^*) \\ &+ {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_1;\varphi} f(\tau^*,\check{x}(\tau^*)) + {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_1;\varphi}\tilde{f}(\tau^*,\check{x}(\tau^*)) \\ &+ \frac{(1-\hat{r})^2}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_2;\varphi}\check{x}(\tau^*) + \frac{(1-\hat{r})}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_2;\varphi} f(\tau^*,\check{x}(\tau^*)) \\ &+ \frac{(1-\hat{r})}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_2;\varphi}\tilde{f}(\tau^*,\check{x}(\tau^*)) - a_0\Big].\end{aligned}$$

For all $\check{x},\check{y} \in B_\delta$, we have

$$\begin{aligned}\mid \mathcal{Q}_1\check{x}(r) + \mathcal{Q}_1\check{y}(r) \mid \le& \sup_{r\in[0,\tau^*]}\Big\{\frac{\mid \hat{r}-1\mid}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2;\varphi} \mid \check{x}(r) \mid + \frac{1}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1;\varphi} \mid f(r,\check{x}(r)) \mid + \frac{1}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3;\varphi} \mid \tilde{f}(r,\check{x}(r)) \mid \\ &+ \frac{1}{\mid\Delta\mid}\frac{(\varphi(r)-\varphi(0))^{\frac{p}{k}-1}}{\Gamma_k(p)}\Big[(\mid \hat{r}-1\mid)\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_1;\varphi} \mid \check{y}(\tau^*) \mid \\ &+ {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_1;\varphi} \mid f(\tau^*,\check{y}(\tau^*)) \mid + {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_1;\varphi} \mid \tilde{f}(\tau^*,\check{y}(\tau^*)) \mid \\ &+ \frac{(1-\hat{r})^2}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1+\hat{\sigma}_2-v_2;\varphi} \mid \check{y}(\tau^*) \mid + \frac{(1-\hat{r})}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-v_2;\varphi} \mid f(\tau^*,\check{y}(\tau^*)) \mid \\ &+ \frac{(1-\hat{r})}{\hat{r}}\, {}^k\mathcal{I}_{0^+}^{\hat{\sigma}_1-\hat{\sigma}_3-v_2;\varphi} \mid \tilde{f}(\tau^*,\check{y}(\tau^*)) \mid + \mid a_0 \mid \Big].\Big\} \\ \le& \frac{\mid \hat{r}-1 \mid}{\hat{r}}\delta\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2+k)} + \frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1}}{\Gamma_k(\hat{\sigma}_1+k)} \parallel \mathcal{P} \parallel \\ &+ \frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3+k)} \parallel \overline{\mathcal{P}} \parallel \\ &+ \frac{1}{\mid\Delta\mid}\frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{\Gamma_k(p)}\Big[(\mid \hat{r}-1 \mid)\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_1}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_1+k)}\delta \\ &+ \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_1}}{\Gamma_k(\hat{\sigma}_1-v_1+k)} \parallel \mathcal{P} \parallel + \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v_1}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_1+k)} \parallel \overline{\mathcal{P}} \parallel \\ &+ \frac{(1-\hat{r})^2}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_2+k)}\delta \\ &+ \frac{1-\hat{r}}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_2}}{\Gamma_k(\hat{\sigma}_1-v_2+k)} \parallel \mathcal{P} \parallel + \\ &+ \frac{1-\hat{r}}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v_2}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_2+k)} \parallel \overline{\mathcal{P}} \parallel + |a_0|\Big] \\ =& \mathcal{G}_1\delta + \mathcal{G}_2 \parallel \mathcal{P} \parallel + \mathcal{G}_3 \parallel \overline{\mathcal{P}} \parallel + \mathcal{G}_4 \le \delta.\end{aligned}$$

Hence $\parallel \mathcal{Q}_1\check{x}(r) + \mathcal{Q}_1\check{y}(r) \parallel \le \delta$ which implies that $\mathcal{Q}_1\check{x} + \mathcal{Q}_2\check{y} \in \mathcal{B}_\delta$. On the other hand, by applying (15) the operator $\mathcal{Q}_2$ is a contraction mapping and the details are omitted. By applying the continuity property of f

and $\tilde{f}$, we conclude that the operator Q_1 is continuous. Besides, Q_1 is uniformly bounded on $\mathcal{B}_\delta$ as

$$\begin{aligned} & \| Q_1\check{x} \| \\ \leq & \frac{\delta(|\hat{r}-1|)}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2+k)}+\frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1}}{\Gamma_k(\hat{\sigma}_1+k)}\|\mathcal{P}\| \\ & +\frac{1}{\hat{r}}\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3+k)}\|\overline{\mathcal{P}}\| \end{aligned}$$

Now let $r_1, r_2 \in [0, \tau^*]$ with $r_1 < r_2$. Thus we have

$$\begin{aligned} & |(Q_1)(\check{x})(r_2)-Q_1)(\check{x})(r_2)| \\ \leq & \Big|\frac{1}{\Gamma_k(\hat{\sigma}_1)}\int_0^{r_1}\varphi'(s)[(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1}{k}-1}-(\varphi(r_1)-\varphi(s))^{\frac{\hat{\sigma}_1}{k}-1}]f(s,\check{x}(s))ds \\ & +\int_{r_1}^{r_2}\varphi'(s)(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1}{k}-1}f(s,\check{x}(s))ds\Big| \\ & +\Big|\frac{1}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3)}\int_0^{r_1}\varphi'(s)[(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1-\hat{\sigma}_3}{k}-1}-(\varphi(r_1)-\varphi(s))^{\frac{\hat{\sigma}_1-\hat{\sigma}_3}{k}-1}]f(s,\check{x}(s))ds \\ & +\int_{r_1}^{r_2}\varphi'(s)(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1-\hat{\sigma}_3}{k}-1}\tilde{f}(s,\check{x}(s))ds\Big| \\ \leq & \frac{\|\mathcal{P}\|}{\Gamma_k(\hat{\sigma}_1+k)}[2(\varphi(r_2)-\varphi(r_1))^{\frac{\hat{\sigma}_1}{k}}+|(\varphi(r_2)-\varphi(0))^{\frac{\hat{\sigma}_1}{k}}+(\varphi(r_1)-\varphi(0))^{\frac{\hat{\sigma}_1}{k}}|] \\ & +\frac{\|\overline{\mathcal{P}}\|}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3+k)}[2(\varphi(r_2)-\varphi(r_1))^{\frac{\hat{\sigma}_1-\hat{\sigma}_3}{k}}+|(\varphi(r_2)-\varphi(0))^{\frac{\hat{\sigma}_1-\hat{\sigma}_3}{k}} \\ & +(\varphi(r_1)-\varphi(0))^{\frac{\hat{\sigma}_1-\hat{\sigma}_3}{k}}|]. \end{aligned}$$

As $r_2 - r_1 \longrightarrow 0$, the right hand of the above inequality tends to zero indecently of $\check{x}$. Thus Q_1 is equicontinuous and by Arzelá–Ascoli theorem, we conclude that Q_1 is completely continuous. Now applying Krasnoselskiǐ fixed point theorem we conclude that the problem (5) has at least one solution on $[0, \tau^*]$ □

Now we apply Leray-Schauder alternative type [25] to present another existence result.

Theorem 4.3. *Assume that the functions $f, \tilde{f} : [0, \tau^*]\times\mathbb{R} \longrightarrow \mathbb{R}$ are two continuous functions. Moreover we assume that there exist continuous, nondecreasing function $\mathcal{U} : [0, \infty) \longrightarrow (0, \infty)$ and continuous function $\overline{\mathcal{U}} : [0, \infty) \longrightarrow (0, \infty)$ such that*

$$|f(r,\check{x})| \leq \overline{\mathcal{U}}(r)\mathcal{U}(|\check{x}|), \quad |f(r,\check{x})| \leq \overline{\mathcal{U}}(r)\mathcal{U}(|\check{x}|),$$

for all $(r, \check{x}) \in [0, \tau^*] \times \mathbb{R}$.

($\mathcal{D}_3$) *There exist a constant $\nu > 0$ such that*

$$\frac{\nu}{\mathcal{G}_1\nu+\mathcal{G}_2\|\overline{\mathcal{U}}\|\mathcal{U}(\|\nu\|)+\mathcal{G}_3\|\overline{\mathcal{U}}\|\mathcal{U}(\|\nu\|)+\mathcal{G}_4} > 1$$

Then the problem (5) has at least one solution on $[0, \tau^*]$

Proof. First we indicate that the bounded sets are mapped into bounded sets in $C([0, \tau^*], \mathbb{R})$. For $\delta > 0$, let

$\mathcal{B}_\delta = \{\check{x} \in C([0, \tau^*], \mathbb{R}) :\| \check{x} \|\leq \delta\}$. Thus for $r \in [0, \tau^*]$ we get

$$\begin{aligned}
|(Qx)(r)| \leq & \Big\{ \frac{|\hat{r}-1|}{\hat{r}} \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2+k)} + \frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)} \Big[\frac{(|\hat{r}-1|)(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_1}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_1+k)} \\
& + \frac{(1-\hat{r})^2}{\hat{r}} \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_2+k)} \Big] \Big\} \delta \\
& + \Big\{ \frac{1}{\hat{r}} \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1}}{\Gamma_k(\hat{\sigma}_1+k)} + \frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)} \Big[\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_1}}{\Gamma_k(\hat{\sigma}_1-v_1+k)} \\
& \frac{(1-\hat{r})}{\hat{r}} \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_2}}{\Gamma_k(\hat{\sigma}_1-v_2+k)} \Big] \Big\} \overline{\mathcal{U}}(r)\mathcal{U}(|\check{x}|) + \Big\{ \frac{1}{\hat{r}} \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3}}{\Gamma_k(\hat{\sigma}_1-(\hat{\sigma}_3+k)} \\
& + \frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)} \Big[\frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v1}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_1+k)} \\
& + \frac{(1-\hat{r})}{\hat{r}} \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v2}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_2+k)} \Big] \Big\} \overline{\mathcal{U}}(r)\mathcal{U}(|\check{x}|) \\
& + |a_0| \frac{(\varphi(\tau^*)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)} = \mathcal{G}_1\delta + \mathcal{G}_2\overline{\mathcal{U}}(r)\mathcal{U}(|\check{x}|) + \mathcal{G}_3\overline{\mathcal{U}}(r)\mathcal{U}(|\check{x}|) + \mathcal{G}_4
\end{aligned}$$

Now we indicate that bounded subsets of $C([0, \tau^*], \mathbb{R})$ are mapped into equicontinuous sets by the operator $\mathcal{Q}$.
Let $r_2, r_1 \in [0, \tau^*]$ with $r_1 < r_2$. Thus we get

$$\begin{aligned}
& |(\mathcal{Q}_1)(\check{x})(r_2) - \mathcal{Q}_1)(\check{x})(r_2)| \\
\leq \; & \Big| \frac{1}{\Gamma_k(\hat{\sigma}_1)} \int_0^{r_1} \varphi'(s)[(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1}{k}-1} - (\varphi(r_1)-\varphi(s))^{\frac{\hat{\sigma}_1}{k}-1}] f(s,\check{x}(s))ds \\
& + \int_{r_1}^{r_2} \varphi'(s)[(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1}{k}-1} f(s,\check{x}(s))ds \Big| \\
& + \Big| \frac{1}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3)} \int_0^{r_1} \varphi'(s)[(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1-\hat{\sigma}_3}{k}-1} - (\varphi(r_1)-\varphi(s))^{\frac{\hat{\sigma}_1-\hat{\sigma}_3}{k}-1}] \tilde{f}(s,\check{x}(s))ds \\
& + \int_{r_1}^{r_2} \varphi'(s)[(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1-\hat{\sigma}_3}{k}-1} \tilde{f}(s,\check{x}(s))ds \Big| \\
& + \Big| \frac{1}{\Gamma_k(\hat{\sigma}_1)} \int_0^{r_1} \varphi'(s)[(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1+\hat{\sigma}_2}{k}-1} - (\varphi(r_1)-\varphi(s))^{\frac{\hat{\sigma}_1}{k}-1}] \check{x}(s)ds \\
& + \int_{r_1}^{r_2} \varphi'(s)[(\varphi(r_2)-\varphi(s))^{\frac{\hat{\sigma}_1+\hat{\sigma}_2}{k}-1} \check{x}(s)ds \Big| \\
& + \frac{(\varphi(r_2)-\varphi(0))^{\frac{p}{k}-1} - (\varphi(r_1)-\varphi(0))^{\frac{p}{k}-1}}{|\Delta|\Gamma_k(p)} \Big[(|\hat{r}-1)|\delta \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_1}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_1+k)} \\
& + \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_1}}{\Gamma_k(\hat{\sigma}_1-v_1+k)} \|\overline{\mathcal{U}}\| \mathcal{U}(\delta) + \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_1}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_1+k)} \|\overline{\mathcal{U}}\| \mathcal{U}(\delta) \\
& + \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v_1}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_1+k)} \|\overline{\mathcal{U}}\| \mathcal{U}(\delta) + \frac{(1-\hat{r})^2}{\hat{r}} \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1+\hat{\sigma}_2-v_2}}{\Gamma_k(\hat{\sigma}_1+\hat{\sigma}_2-v_2+k)} \delta \\
& + \frac{(1-\hat{r})}{\hat{r}} \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-v_2}}{\Gamma_k(\hat{\sigma}_1-v_2+k)} \|\overline{\mathcal{U}}\| \mathcal{U}(\delta) \\
& + \frac{(1-\hat{r})}{\hat{r}} \frac{(\varphi(\tau^*)-\varphi(0))^{\hat{\sigma}_1-\hat{\sigma}_3-v_2}}{\Gamma_k(\hat{\sigma}_1-\hat{\sigma}_3-v_2+k)} \|\overline{\mathcal{U}}\| \mathcal{U}(\delta) + |a_0| \Big]
\end{aligned}$$

As $r_2 - r_1 \longrightarrow 0$, the right hand of the above inequality tends to zero. Hence Arzelá–Ascoli theorem implies that the operator $Q : C([0,\tau^*],\mathbb{R}) \longrightarrow C([0,\tau^*],\mathbb{R})$ is completely continuous.

Finally, For $\lambda \in (0,1)$, the set of all solutions to equations $\check{x} = \lambda Q(\check{x})$ is proved.

Assume that $\check{x}$ is a solution. Then, for $r \in [0,\tau^*]$ we have

$$|\check{x}(r)| \leq \mathcal{G}_1\delta + \mathcal{G}_2 \parallel \overline{\mathcal{U}} \parallel \mathcal{U}(\parallel \check{x} \parallel) + \mathcal{G}_3 \parallel \overline{\mathcal{U}} \parallel \mathcal{U}(\parallel \check{x} \parallel) + \mathcal{G}_4.$$

or

$$\frac{\parallel \check{x} \parallel}{\mathcal{G}_1\delta + \mathcal{G}_2 \parallel \overline{\mathcal{U}} \parallel \mathcal{U}(\parallel \check{x} \parallel) + \mathcal{G}_3 \parallel \overline{\mathcal{U}} \parallel \mathcal{U}(\parallel \check{x} \parallel) + \mathcal{G}_4} \leq 1.$$

Due to $(\mathcal{D}_3)$, there exists $\nu > 0$ such that $\parallel \check{x} \parallel \neq \nu$. Let us put

$$\mathcal{W} = \{\check{x} \in C([0,\tau^*],\mathbb{R}); \parallel x \parallel < \nu\}.$$

The operator $Q : \overline{\mathcal{W}} \longrightarrow C([0,\tau^*],\mathbb{R})$ is continuous and completely continuous. Besides, there is no $\check{x} \in \partial\mathcal{W}$ such that $\check{x} = \lambda Q\check{x}$ for some $\lambda \in (0,1)$. Consequently, applying nonlinear alternative of Leray–Schauder type, a fixed point of Q is obtained on $\overline{\mathcal{W}}$ which is a solution of the problem (5). □

Example 4.4. *Consider the BVP of the form:*

$$\begin{cases} \left(\frac{1}{3}\ {}^{\frac{1}{2},H}\mathcal{D}^{2,\frac{1}{2};1+\frac{r}{1000}} + \dfrac{2}{3}\ {}^{\frac{1}{2},H}\mathcal{D}^{\frac{1}{2},\frac{1}{2};1+\frac{r}{1000}}\right)\check{x}(r) = f(r,x(r)) + \dfrac{1}{3}\ {}^{\frac{1}{2},H}\mathcal{D}^{\frac{3}{2},\frac{1}{2};1+\frac{r}{1000}}\tilde{f}(r,\check{x}(r)) \\ \check{x}(0) = 0, \qquad \frac{1}{3}\ {}^{\frac{1}{2},H}\mathcal{D}^{\frac{1}{2},\frac{1}{2};1+\frac{r}{1000}}\check{x}(1) + \dfrac{2}{3}\ {}^{\frac{1}{2},H}\mathcal{D}^{\frac{1}{2},\frac{1}{2};1+\frac{r}{1000}}\check{x}(1) = \dfrac{1}{2} \end{cases} \tag{16}$$

Here, $\hat{r} = \frac{1}{3}$, $k = \frac{1}{2}$, $\hat{\sigma}_1 = 2$, $\hat{\sigma}_2 = \frac{1}{2}$, $\hat{\sigma}_3 = \frac{3}{2}$, $\beta = \frac{1}{2}$, $\varphi(r) = 1 + \frac{r}{1000}$, $\tau^* = 1$. *By computation, we have* $\mathcal{G}_1 \approx .003$, $\mathcal{G}_3 \approx= .006, .009$, $\mathcal{G}_4 \approx \frac{3}{2}$, $\Delta = -2$.

(i) Assume that the functions $f, \tilde{f} : [0,\tau^*] \times \mathbb{R} \longrightarrow \mathbb{R}$ *have been defined by*

$$f(r,\check{x}) = \frac{e^{-r}}{10^3(r+10)}\cos\left(\frac{|\check{x}|}{1+|\check{x}|}\right), \quad \tilde{f}(r,\check{x}) = \arctan\left(\frac{|\check{x}|}{10^3(1+|\check{x}|)}\right). \tag{17}$$

Thus, we get

$$|f(r,\check{x}) - f(r,\check{y})| \leq \frac{1}{10^4}|\check{x} - \check{y}|, \quad |\tilde{f}(r,\check{x}) - \tilde{f}(r,\check{y})| \leq \frac{1}{10^3}|\check{x} - \check{y}|$$

for $r \in [0,1]$ *and* $\check{x}, \check{y} \in \mathbb{R}$. *Therefore, we have* $\mathcal{G} = \frac{1}{10^4}$ *and* $\overline{\mathcal{G}} = \frac{1}{10^3}$, *which implies that* $\mathcal{S} := \mathcal{G}_1 + \mathcal{G}_2\mathcal{G} + \mathcal{G}_3\overline{\mathcal{G}} \approx .003 + \frac{.006}{10^4} + \frac{.009}{10^3} \approx .0045 < 1$. *Consequently, by Theorem 4.1 the boundary value problem (16) with the functions* f *and* $\tilde{f}$ *given by (17) has a unique solution on* $[0,1]$.

(ii) Assume that the functions $f, \tilde{f} : [0,\tau^*] \times \mathbb{R} \longrightarrow \mathbb{R}$ *have been defined by*

$$f(r,\check{x}) = \frac{1}{10^5}\cos\Big(|\check{x}|\Big) + \frac{r}{4} + \frac{1}{10^5}, \quad \tilde{f}(r,\check{x}) = \frac{1}{10^6}\sin\Big(|\check{x}|\Big) + \frac{\check{r}}{8} + \frac{1}{10^6}. \tag{18}$$

Now, we see that

$$|f(r,\check{x})| \leq \frac{2}{10^5} + \frac{r}{4} = \mathcal{P}(r), \quad |\tilde{f}(r,\check{x})| \leq \frac{2}{10^6} + \frac{r}{8} = \overline{\mathcal{P}}(r).$$

Moreover, the functions f *and* $\tilde{f}$ *satisfy condition* $(\mathcal{D}_1)$ *with* $\mathcal{G} = \frac{2}{10^5}$ *and* $\overline{\mathcal{G}} = \frac{2}{10^6}$. *On the other hand,* $\mathcal{S}_1 \approx .83317 < 1$. *Consequently, by applying Theorem 4.2, we conclude that the problem (16) with* f *and* $\tilde{f}$ *given by (18) has at least one solution on* $[0,1]$.

(iii) Let now the functions $f, \tilde{f} : [0,\tau^*] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be presented by*

$$f(r,\check{x}) = \frac{1}{10r+5}\left(\frac{e^{-\check{x}}}{8+\mid \check{x} \mid^2} + \frac{1}{8}\right), \quad \tilde{f}(r,\check{x}) = \frac{1}{5}(e^{-\check{x}} + \frac{1}{8}). \tag{19}$$

By putting $\mathcal{U}(\mid \check{x} \mid) = e^{-|\check{x}|} + \frac{1}{8}$ *and* $\overline{\mathcal{U}}(r) = \frac{1}{5}$, *we have* $\| \overline{\mathcal{U}} \| = \frac{1}{5}$. *Consequently, there exists* $\nu > 0$ *such that*

$$\frac{\nu}{.003\nu + \frac{.006}{5}(e^{-\nu} + \frac{1}{8}) + \frac{.009}{5}(e^{-\nu} + \frac{1}{8}) + \frac{3}{2}} > 1.$$

Hence, by applying Theorem 4.3 we conclude that BVP (16) with the function given by (19) has at least one solution on $[0,1]$.

5. Conclusions

In this work, the existence and uniqueness results have been considered for a system of (k,π)- Hilfer fractional differential equations with linear combinations of fractional derivatives in boundary conditions. The standard tools of fixed point theory have been applied to construct the desired results. our results are new and extend the results of [16]. In the future directions, the coupled version of the problem (5) will be considered.

Declarations

Acknowledgement
Not applicable.
Availability of data and material
Not applicable.
Competing interests
The authors declare that they have no competing interests.
Authors contributions
All the authors contributed equally and significantly in writing this paper.

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