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# The stable solvability in weak topology setting and application to fixed point theory

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**Abstract.** The central objective of this research is to investigate the stable solvability notion within the context of the weak topology. Furthermore, we endeavor to deduce novel Leary-Schauder fixed point results, thereby contributing to the advancement of knowledge in this field.

## 1. Introduction

Spectral theory and fixed point theory for nonlinear operators have indeed been active areas of research in recent years. These fields are closely related to functional analysis and have applications in various mathematical and physical problems, see [1, 4, 6, 7, 9, 15, 16, 18, 21–25, 30] and the references therein contained. The interest in the subject owes to its extensive applications in many phenomenons in various fields of applied sciences.

Researchers have used different notions in the strong topology setting such as the measure of noncompactness, Stable solvability of nonlinear operators to analyze various aspects of nonlinear operators, such as the existence of fixed points, stability of solutions to nonlinear equations, and convergence of iterative methods. It plays a crucial role in proving results related to the spectral theory of nonlinear operators and the study of attractors and stability regions, see [4, 15–19].

In infinite-dimensional Banach spaces, the lack of local compactness poses challenges when dealing with various mathematical problems. The weak topology is a key concept that comes to the rescue in such cases, as it provides a useful alternative to the strong topology. Here's why the weak topology is crucial in dealing with the existence of solutions in infinite-dimensional Banach spaces.

By using the weak topology, researchers can often establish the existence of solutions of several equations, prove compactness results, and analyze the behavior of operators in infinite-dimensional Banach spaces. It has become an indispensable tool in various branches of mathematics and has found applications in functional analysis, partial differential equations, optimal control, and more. The weak topology is a powerful tool in the study of infinite-dimensional Banach spaces, overcoming some of the difficulties caused by the lack of local compactness and enabling the analysis of equations and operators in a more tractable manner.

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This work is focused on introducing stable-solvability notion of nonlinear operators within the context of the weak topology. It also offers new insights into fixed point results by employing Leray-Schauder type theorems.

This paper is arranged as follows. Firstly, some definitions, notations and some auxiliary results which will be used in the sequel are presented in Section 2. In Section 3, we introduce the notion of measure of weak noncompactness of nonlinear operators and its important properties. Section 4 is devoted to introduce the weak stable solvability of nonlinear operators and investigate its important properties. In section 5, we establish some fixed point theorems of Leary-Schauder type under appropriate assumptions. In Section 6, to illustrate the applicability of the theories, we prove the existence of a continuous solution for the following Volterra integral equation on a Banach space $X$ :

$$x(t) = f(x)(t) + \int_0^t k(t,s)g(s,x(s))\,ds,\ t \in J, \tag{1}$$

where $J = [0,1]$, $k : J \times J \to \mathbb{R}$ is continuous, $f : C(J,X) \to C(J,X)$ and $g : J \times X \to X$ are given.

## 2. Preliminaries results

Let $X$ and $Y$ be two Banach spaces and $T : X \longrightarrow Y$ a continuous operator, which in general be nonlinear. By $\mathcal{C}(X,Y)$ we denote the set of all continuous operators from $X$ into $Y$. Of course, this set forms a linear space. $\mathcal{C}(X) := \mathcal{C}(X,X)$ is an algebra with respect to the composition. However, in the case of linear operators the space $\mathcal{L}(X,Y)$ is normed by the operator norm.

**Definition 2.1.** *For $T \in \mathcal{C}(X,Y)$, we define*

$$[T]_Q = \limsup_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|}$$

*and*

$$[T]_q = \liminf_{\|x\|\to\infty} \frac{\|T(x)\|}{\|x\|},$$

*as elements of $[0,\infty]$. If $[T]_Q < \infty$, we call $T$ quasibounded. By $\mathcal{Q}(X,Y)$ we denote the set of all quasibounded continuous maps from $X$ into $Y$.*

**Remark 2.2.** *In particular, the fact that $[T]_Q = \lambda$ or $[T]_q = \lambda$ implies that there exists an unbounded sequence $(x_n)$ in $X$ such that*

$$\lim_{n\to\infty} \frac{\|T(x_n)\|}{\|x_n\|} = \lambda.$$

*Furthermore, the inequality*

$$[T]_q \le [T]_Q$$

*is obviously true. Consequently, $[T]_Q = 0$ actually implies that*

$$\lim_{n\to\infty} \frac{\|T(x_n)\|}{\|x_n\|} = 0$$

*for every sequence $(x_n)_n$ with $\|x_n\| \to \infty$.*

For a comprehensive list of properties of $[T]_q$ and $[T]_Q$ we refer the reader to [3]. The following lemma, proved in [31], will be used throughout the following section.

**Lemma 2.3.** *Let $T, S \in \mathcal{C}(X,Y)$ and $R \in \mathcal{C}(Y,Z)$. Then,*

*(i)* $[T]_q > 0$ *implies that $T$ is coercive i.e.,* $\lim_{|x\|\to\infty} \|T(x)\| = \infty$.

*(ii)* *One of the quantities on the left being finite,*

$$[T]_q - [S]_Q \leq [T+S]_q \leq [T]_q + [S]_Q.$$

*(iii)* *One of the quantities on the left being finite*

$$|[T]_q - [S]_q| \leq [T-S]_Q.$$

*In particular,* $[T-S]_Q = 0$ *implies* $[T]_q = [S]_q$.

*(iv)* $[T^{-1}]_Q = [T]_q^{-1}$ *if T is a homeomorphism and either T is linear or X and Y are finite dimensional.*

*(v)* $[R \circ T]_q \geq [R]_q [T]_q$.

For any $r > 0$, $B_r$ denotes the closed ball in $X$ centered at $0_X$ with radius $r$, and $B_X$ denotes the closed ball in $X$ centered at $0_X$ with radius 1. We write $\Omega_X$ to denotes the collection of all nonempty bounded subsets of $X$, and $\mathcal{K}^w$ to denotes the subset of $\Omega_X$ consisting of all weakly compact subsets of $X$. Recall that the notion of the measure of weak noncompactness was introduced by De Blasi [14]; it is the map $\omega : \Omega_X \longrightarrow [0, +\infty)$ defined in the following way:

$$\omega(\mathcal{M}) = \inf\Big\{r > 0 : \text{ there exists } \mathrm{K} \in \mathcal{K}^w \text{ such that } \mathcal{M} \subset \mathrm{K} + B_r\Big\},$$

for all $\mathcal{M} \in \Omega_X$. For more convenience, let us recall some basic properties of $\omega(\cdot)$ needed below (see, for example, [2, 14]) (see also [5], where an axiomatic approach to the notion of a measure of weak noncompactness is presented).

**Lemma 2.4.** *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be two elements of* $\Omega_X$. *Then, the following conditions are satisfied:*

*(1)* $\mathcal{M}_1 \subset \mathcal{M}_2$ *implies* $\omega(\mathcal{M}_1) \leq \omega(\mathcal{M}_2)$.

*(2)* $\omega(\mathcal{M}_1) = 0$ *if, and only if,* $\overline{\mathcal{M}_1}^w \in \mathcal{K}^w$, *where* $\overline{\mathcal{M}_1}^w$ *is the weak closure of the subset* $\mathcal{M}_1$.

*(3)* $\omega(\overline{\mathcal{M}_1}^w) = \omega(\mathcal{M}_1)$.

*(4)* $\omega(\mathcal{M}_1 \cup \mathcal{M}_2) = \max\{\omega(\mathcal{M}_1), \omega(\mathcal{M}_2)\}$.

*(5)* $\omega(\lambda\mathcal{M}_1) = |\lambda|\omega(\mathcal{M}_1)$ *for all* $\lambda \in \mathbb{R}$.

*(6)* $\omega(co(\mathcal{M}_1)) = \omega(\mathcal{M}_1)$, *where* $co(\mathcal{M}_1)$ *is the convex hull of* $\mathcal{M}_1$.

*(7)* $\omega(\mathcal{M}_1 + \mathcal{M}_2) \leq \omega(\mathcal{M}_1) + \omega(\mathcal{M}_2)$.

*(8)* *If* $(\mathcal{M}_n)_{n\geq 1}$ *is a decreasing sequence of nonempty, bounded, and weakly closed subsets of X with* $\lim_{n\to\infty} \omega(\mathcal{M}_n) = 0$, *then* $\mathcal{M}_\infty := \cap_{n=1}^{\infty} \mathcal{M}_n$ *is nonempty and* $\omega(\mathcal{M}_\infty) = 0$, *i.e.* $\mathcal{M}_\infty$ *is relatively weakly compact.*

**Remark 2.5.** *Notice that* $\omega(B_X) \in \{0, 1\}$. *Indeed, it is obvious that* $\omega(B_X) \leq 1$. *Let* $r > 0$ *be given such that there is a weakly compact K of X satisfying* $B_X \subset K + rB_X$. *Hence,* $\omega(B_X) \leq r\omega(B_X)$. *If* $\omega(B_X) \neq 0$, *then* $r \geq 1$. *Thus,* $\omega(B_X) \geq 1$.

**Definition 2.6.** *[29] Let X be a Banach space. Suppose* $T : Q \subset X \to X$ *maps bounded sets into bounded sets. We call T an* $\alpha$-$\omega$-*contractive if* $0 \leq \alpha < 1$ *and* $\omega(T(M)) \leq \alpha\omega(M)$ *for all bounded sets* $M \subset Q$.

**Remark 2.7.** *Notice that every weakly compact operator is* $\alpha$-$\omega$-*contractive with* $\alpha = 0$.

**Theorem 2.8.** *[29] Let Q be a nonempty, bounded, convex, closed set in a Banach space X. Assume* $T : Q \to Q$ *is weakly sequentially continuous and* $\alpha$-$\omega$-*contractive. Then T has a fixed point.*

The following results are crucial in our consideration.

**Theorem 2.9.** *[28] Let X be a Banach space and M be a bounded and equi-continuous subset of $C(J, X)$. Then the mapping $t \to \beta(M(t))$ is continuous on J and*

$$\beta(M) = \sup_{t\in J} \beta(M(t)) = \beta(M(J)).$$

*Here for a fixed number $t \in J$, $M(t) = \{x(t) : x \in M\}$ and $M(J) = \bigcup_{t\in J} \{x(t) : x \in M\}$.*

**Lemma 2.10.** *[13] Let X be a Banach space, and let $\xi : [a, b] \to X$ be a Pettis integrable function. Then*

$$\int_a^b \xi(s)ds \in (b - a)\overline{co}\{\xi([a, b])\}.$$

**Definition 2.11.** *[12] A Banach space X has the Dunford-Pettis property (DPP, in short) if, for any Banach space Y every weakly compact operator $T : X \to Y$ transforms weakly convergent sequences into convergent sequences.*

## 3. Measure of Weak Noncompactness

Throughout this section, $X$ denotes a Banach space. In the following definition we introduced the notion of measure of weak noncompactness of a nonlinear operator.

**Definition 3.1.** *Let $Z \subset X$. For $T \in C(Z, Y)$, we define*

$$[T]_A = \inf\{k : k > 0,\ \omega(T(M)) \leq k\omega(M) \text{ for all bounded } M \in Z\}$$

*and*

$$[T]_a = \sup\{k : k > 0,\ \omega(T(M)) \geq k\omega(M) \text{ for all bounded } M \in Z\}$$

*as elements of* $[0, \infty]$.

We call $[T]_A$ the measure of weak noncompactness of $T$ and denote by $\mathcal{U}(Z, Y)$ the set of all continuous maps $T$ from $Z$ into $Y$ with $[T]_A < \infty$. Note that in finite dimensional spaces we always have $[T]_A = 0$ and $[T]_a = \infty$. In infinite dimensional spaces, where this characteristic is of more use, we get the equivalence representations

$$[T]_A = \sup_{0<\omega(M)<\infty} \frac{\omega(T(M))}{\omega(M)}$$

and

$$[T]_a = \inf_{0<\omega(M)<\infty} \frac{\omega(T(M))}{\omega(M)}.$$

Sets with $\omega(M) = 0$ can be left out here, since the continuity of $T$ assures that also $\omega(T(M)) = 0$. This can be seen by considering $\omega(T(M)) \leq \omega(T(\overline{M}))$. From this representation it is also clear that an operator $T$ with $[T]_A < \infty$ maps bounded sets into bounded sets.

**Definition 3.2.** *Suppose that X and Y are Banach spaces. Let $T \in C(X, Y)$.*

*(i) The operator T is called weakly compact if $T(M)$ is a relatively weakly compact subset of Y whenever M is a bounded subset of X.*

*(ii) The operator T is called weakly proper, if the preimage $T^{-1}(N)$ is weakly compact for every weakly compact set $N \subset Y$.*

**Proposition 3.3.** *Let X, Y and Z be Banach spaces. For $T$, $S \in C(X, Y)$ and $R \in C(Y, Z)$ the following assertions holds true:*

*(i)* *T is weakly compact if, and only if,* $[T]_A = 0$.

*(ii)* *If* $[T]_a > 0$ *and* $[T]_q > 0$, *then T is weakly proper.*

*(iii)* $[T]_a > 0$ *implies that T is weakly proper on closed bounded sets.*

*(iv)* *One of the quantities on the left being finite,*

$$[T]_a - [S]_A \leq [T+S]_a \leq [T]_a + [S]_A.$$

*(v)* *One of the quantities on the left being finite*

$$|[T]_a - [S]_a| \leq [T-S]_A.$$

*In particular,* $[T-S]_A = 0$ *implies* $[T]_a = [S]_a$.

*(vi)* $[T^{-1}]_A = [T]_a^{-1}$ *if T is a homeomorphism and either T is linear or X and Y are finite dimensional.*

*(vii)* $[R]_a[T]_a \leq [R \circ T]_a \leq [R]_A[T]_a$, *where the second inequality holds if* $[R]_A < \infty$.

*Proof.* (*i*) This assertion follows from the definition of a weakly compact operator and the definition of the measure of weak noncompactness of an operator.
(*ii*) Since $[T]_a > 0$, we may find a $k > 0$ such that $\omega(T(M)) \geq k\omega(M)$ for each bounded $M \in X$. As $[T]_q > 0$, Lemma 2.3 shows that $T$ is coercive. Therefore, for any weakly compact set $N \in Y$, $T^{-1}(N)$ is bounded and

$$\omega(T^{-1}(N)) \leq \frac{1}{k}\omega(T(T^{-1}(N))) \leq \frac{1}{k}\omega(N) = 0.$$

Thus, $\overline{T^{-1}(N)}^{w} \in \mathcal{K}^w$. Since $T$ is continuous, $T^{-1}(N)$ is also closed and therefore weakly compact.
(*iii*) The same reasoning shows that $T$ is weakly proper on closed bounded sets if only $[T]_a > 0$.
(*iv*) We have

$$\begin{aligned}
[T+S]_a &= \inf_{0<\omega(M)<\infty} \frac{\omega((T+S)(M))}{\omega(M)} \\
&= \inf_{0<\omega(M)<\infty} \frac{\omega((T(M)+S(M))}{\omega(M)} \\
&\leq \inf_{0<\omega(M)<\infty} \left\{ \frac{\omega(T(M)}{\omega(M)} + \frac{\omega(S(M)}{\omega(M)} \right\} \\
&\leq \inf_{0<\omega(M)<\infty} \frac{\omega(T(M)}{\omega(M)} + \sup_{0<\omega(M)<\infty} \frac{\omega(S(M)}{\omega(M)} \\
&\leq [T]_a + [S]_A.
\end{aligned}$$

For the second inequality replace $T$ by $T+S$ and $S$ by $-S$.

($v$) Similarly, we have

$$\begin{aligned}
[T-S]_A &= \sup_{0<\omega(M)<\infty} \frac{\omega((T-S)(M))}{\omega(M)} \\
&= \sup_{0<\omega(M)<\infty} \frac{\omega((T(M)-S(M))}{\omega(M)} \\
&\geq \sup_{0<\omega(M)<\infty} \left\{ \frac{\omega(T(M)}{\omega(M)} - \frac{\omega(S(M)}{\omega(M)} \right\} \\
&\geq \sup_{0<\omega(M)<\infty} \frac{\omega(T(M)}{\omega(M)} - \inf_{0<\omega(M)<\infty} \frac{\omega(S(M)}{\omega(M)} \\
&\geq \inf_{0<\omega(M)<\infty} \frac{\omega(T(M)}{\omega(M)} - \inf_{0<\omega(M)<\infty} \frac{\omega(S(M)}{\omega(M)} \\
&\geq [T]_a - [S]_a,
\end{aligned}$$

which for symmetry reasons proves ($v$).
($vi$) The two assumptions that $T$ is linear and that $X$ and $Y$ are finite dimensional both assure that $\|x\| \to \infty$ if, and only if, $\|T(x)\| \to \infty$. We therefore can consider the chain of equalities

$$\begin{aligned}
[T^{-1}]_A &= \sup_{0<\omega(N)<\infty} \frac{\omega(T^{-1}(N))}{\omega(N)} \\
&= \sup_{0<\omega(M)<\infty} \frac{\omega(M)}{\omega(T(M))} \\
&= \inf_{0<\omega(M)<\infty} \left( \frac{\omega(T(M)))}{\omega(M} \right)^{-1} \\
&= \frac{1}{[T]_a}.
\end{aligned}$$

($vii$) We have

$$\begin{aligned}
[R\circ T]_a &= \inf_{0<\omega(M)<\infty} \frac{\omega(R(T(M)))}{\omega(M)} \\
&= \inf_{0<\omega(M)<\infty} \frac{\omega(R(T(M)))}{\omega(T(M))} \frac{\omega(T(M))}{\omega(M)} \\
&\geq \inf_{0<\omega(M)<\infty} \frac{\omega(R(T(M)))}{\omega(T(M))} \inf_{0<\omega(M)<\infty} \frac{\omega(T(M))}{\omega(M)} \\
&\geq \inf_{0<\omega(N)<\infty} \frac{\omega(R(N))}{\omega(N)} \inf_{0<\omega(M)<\infty} \frac{\omega(T(M))}{\omega(M)} \\
&\geq [R]_a [T]_a.
\end{aligned}$$

The last inequality holds true, because $\omega(T(M)) \not\to \infty$, then $[T]_a = 0$ and the inequality holds and if $\omega(T(M)) \to \infty$ we can set $T(M) = N$ to see that the inequality holds. □

**Lemma 3.4.** *Let $T : X \longrightarrow Y$ be linear and bounded. If $T$ is bijective, then $[T]_q > 0$ and $[T]_a > 0$.*

*Proof.* Since $T$ is linear, bijective, and bounded, then it is invertible and its inverse is also a linear and bounded operator. Using Lemma 2.3 ($iv$), we get

$$\begin{aligned}
[T]_q &= [T^{-1}]_Q^{-1} \\
&= \|T^{-1}\|^{-1} \\
&= \frac{1}{\|T^{-1}\|} > 0,
\end{aligned}$$

and using Proposition 3.3 (*vi*), we get

$$\begin{aligned} [T]_a &= \frac{1}{[T^{-1}]_A} \\ &= \frac{1}{\|T^{-1}\|} > 0. \end{aligned}$$

This completes the proof. □

The following result is an analog version of Dugundji's theorem was proved in [31].

**Theorem 3.5.** *[31, Theorem 2.12] Let $X$ and $Y$ be Banach spaces and let $T : C \longrightarrow K$ be a continuous map, where $C \subset X$ is closed and $K \subset Y$ is convex. Then, there exists a continuous mapping $\widetilde{T} : X \longrightarrow K$ such that $\widetilde{T}(x) = T(x)$ for $x \in C$.*

**Proposition 3.6.** *Let $X$ and $Y$ be Banach spaces and let $T : C \longrightarrow Y$ be weakly compact, where $C \subset X$ is closed and bounded. Then, there exists a weakly compact extension $\widetilde{T}$ of $T$ to $X$ with $\widetilde{T}(X) \subset co(T(C))$.*

*Proof.* Since $T$ is weakly compact and $C$ is bounded, then $T(C)$ is weakly compact subset. By using Lemma 2.4, so is $\mathrm{co}(T(C))$. Therefore, an extension $\widetilde{T}$ of $T$ as in Theorem 3.5 is also weakly compact. □

## 4. Weak Stable Solvability of Nonlinear Operators

In this section we introduce the notion of stable solvability with respect to the weak topology and we analyse the important properties of this notion.

**Definition 4.1.** *Let $X$ be a Banach space. A continuous operator $T : X \to X$ is called weakly stably solvable (w-stably solvable for short), if given any sequentially weakly continuous and weakly compact operator $S : X \to X$ with $[S]_Q = 0$, the equation $T(x) = S(x)$ has a solution $x \in X$.*

**Example 4.2.** *Let $X$ be a Banach space. If $T : X \to X$ is an invertible operator with sequentially weakly continuous inverse, then $T$ is w-stably solvable. In fact, let $S : X \to X$ be an arbitrary sequentially weakly continuous and weakly compact operator for which the set $\{x \in X : S(x) \neq 0\}$ is bounded. Invoking Lemma 4.6, it suffices to prove that the equation $T(x) = S(x)$ has a solution $x \in X$. To this end, taking into account the facts that $S$ is weakly compact and $T^{-1}$ is sequentially weakly continuous, by applying the Eberlein-Šmulian theorem we can infer that $T^{-1}(S)$ maps bounded subsets of $X$ into relatively weakly compact ones, and consequently it defines a weakly compact operator. On the other hand, it is clear that $T^{-1}(S)$ is sequentially weakly continuous. Hence, from the Schauder–Tikhonov fixed point theorem (see [11, page 113]) it follows that the mapping $T^{-1}(S)$ has a fixed point, and consequently the equation $T(x) = S(x)$ has a solution $x \in X$.*

*In particular case, if $X$ is a finite-dimensional Banach space and $T : X \to X$ is an invertible operator with continuous inverse, then $T$ is w-stably solvable.*

**Example 4.3.** *Let $X$ be a Dunford Pettis Banach space, see Definition 2.11. If $T : X \to X$ is a contraction operator, then the operator $I - T$ is w-stably solvable. To prove this fact, let us take an arbirary sequentially weakly continuous and weakly compact operator $S : X \to X$ for which the set $\{x \in X : S(x) \neq 0\}$ is bounded. Notice that $(I - T)$ is invertible, and $(I - T)^{-1}$ is continuous, see Remark 4.1 [26]. Since $X$ has the Dunford Pettis property and $S$ is weakly compact, we deduce that $(I - T)^{-1}S$ is sequentially weakly continuous. Arguing as in the above example, we can prove that the mapping $(I - T)^{-1}S$ is weakly compact. Thus, by the Schauder–Tikhonov fixed point theorem, $(I - T)^{-1}S$ has a fixed point, and consequently the equation $S(x) = (I - T)(x)$ has a solution $x \in X$.*

**Lemma 4.4.** *Let $T \in C(X, X)$ be w-stably solvable operator. Then $T$ is surjective.*

*Proof.* For $y \in Y$ the operator $S(x) = y$ satisfies $[S]_A = [S]_Q = 0$. By hypothesis this equation has a solution, and consequently $T$ is surjective. □

**Definition 4.5.** *Let $T \in C(X, X)$.*

*(i) We call*

$$\mu(T) := \inf\{k : k \geq 0; T \text{ is not } k - w - \text{stably solvable}\}$$

*the measure of weak stable solvability of T.*

*(ii) We call T strictly w-stably solvable if $\mu(T) > 0$.*

**Lemma 4.6.** *Let $T \in C(X, X)$ with $[T]_q > 0$. Then T is w-stably solvable if and only if the equation $T(x) = S(x)$ has a solution $x \in X$ for every sequentially weakly continuous and weakly compact operator $S : X \to X$ for which the set $\{x \in X : S(x) \neq 0\}$ is bounded.*

*Proof.* Let $S : X \to X$ be a sequentially weakly continuous, compact operator with $[S]_Q = 0$. For $n \in \mathbb{N}$ define the operator $S_n(x) = d_n(\|x\|)S(x)$, where

$$d_n(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq n, \\ 2 - 1/nt, & \text{if } n \leq t \leq 2n, \\ 0, & \text{if else.} \end{cases}$$

Then $\{x, S_n(x) \neq 0\}$ is bounded and $S_n$ is weakly compact. Then, there exists a sequence $u_n$ such that $S_n(u_n) = u_n$. Without loss of generality we can suppose that if $\|u_n\| \leq 1$ for some $n$. Indeed, if $\|u_n\| > 1$ for all $n$, then we can prove that $[T]_q = 0$ and this is a contradiction So, if $\|u_n\| \leq 1$ for some $n$, then

$$T(x_n) = S_n(x_n) = S(x_n).$$

□

**Corollary 4.7.** *The identity operator is w-stably solvable.*

*Proof.* Using the above Lemma with $T = I$. Let $S : X \to X$ be an arbitrary sequentially weakly continuous, weakly compact operator with $\{x \in X : S(x) \neq 0\}$ is bounded. Then, there exists $r > 0$ such that $S(\Omega) \subset \Omega$. Here $\Omega = B_r$, the closed ball with center zero and radius $r$. Then, in view of Remark 2.7, an application of O'Regan's fixed point theorem, Theorem 2.8, implies that $S$ has a fixed point, or equivalently $S(x) = I(x)$. □

## 5. Application to Fixed Point Theory

This section is devoted to investigate new Leray-Schauder fixed point results using the notions mentioned in previous sections.

**Theorem 5.1.** *Let X be a Banach space, T be a w-stably solvable operator on X. Let $U \subset X$ be a weakly open set and such that $0 \in U$, and $S : \overline{U^w} \longrightarrow X$ be a sequentially weakly continuous with $[S]_A < 1$. Suppose also that S satisfies the Leray-Schauder boundary condition:*

$$y \notin \{\lambda S(y); \lambda \in (0, 1)\} \; y \in \partial_X(U).$$

*Then, either*

**($\mathcal{A}_1$)** *$T(x) = 0$ has a solution on $X \setminus U$, or*

**($\mathcal{A}_2$)** *there exists $x \in U$ such that $T(x) = S(x)$.*

*Proof.* Let
$$M = \left\{x \in \overline{U^w}, \lambda S(x) = x \text{ for some } \lambda \in [0,1]\right\}.$$
The set $M$ is nonempty since $0 \in M$, and $M \cap \partial_X(U) = \emptyset$. We first claim that $M$ is relatively weakly compact. If it is not the case, then $\omega(M) > 0$. Since $M \subset \overline{co}(S(M) \cup \{0\})$, by the properties of the De Blasi measure we get
$$\omega(M) \leq \omega\left(\overline{co}(S(M) \cup \{0\})\right) = \omega\left(S(M)\right) < \omega(M),$$
which is absurd. We next prove that $M$ is weakly closed. Let $x \in \overline{M^w}$. Taking into account the fact that $\overline{M^w}$ is weakly compact, in view of the Eberlein-Smulian Theorem, there exists a sequence $(x_n)_n$ such that $x_n \rightharpoonup x$. Notice that for each integer $n \in \mathbb{N}$, there is $\lambda_n \in [0,1]$ such that
$$x_n = \lambda_n S(x_n).$$
By extracting a subsequence, if necessary, we assume that
$$\lambda_n \to \lambda \in [0,1].$$
Taking into account the sequential weak continuity of $S$ and letting $n \longrightarrow \infty$, we get $\lambda S(x) = x$, and so $x \in M$, and consequently the set $M$ is weakly closed.
Keeping in mind that $M \cap \partial_X(U) = \emptyset$, $M$ is weakly compact, and $\partial_X(U)$ is weakly closed since $X$ endowed with the weak topology is a Tychonoff space, the Urysohn Theorem for the weak topology [20] ensures the existence of a weakly continuous mapping $\varphi : X \to [0,1]$ such that
$$\varphi(x) = \begin{cases} 1, & \text{if } x \in M; \\ 0, & \text{if } x \in \partial_X(U). \end{cases}$$
Consider the operator $S_1 : X \to X$ given by
$$S^*(x) = \begin{cases} \varphi(x)S(x), & \text{if } x \in \overline{U^w}; \\ 0, & \text{if } x \in X \setminus \overline{U^w}. \end{cases}$$
It is clear that
$$S^*(X) \subset \overline{co}(S(\overline{U^w}) \cup \{0\}). \tag{2}$$
Consequently, for all $M \subset X$ be a bounded set, we have
$$\omega(S^*(M)) \leq \omega(\overline{co}(S(\overline{U^w}) \cup \{0\})) \leq \omega(S(\overline{U^w}), \tag{3}$$
which implies that $S^*$ is weakly compact operator. Because $\partial_X(U) = \partial_X(\overline{U}^w)$, $\varphi$ is weakly continuous and $S$ is weakly sequentially continuous, we have that $S^*$ is weakly sequentially continuous. Since $T$ is w-stably solvable, there exists $x \in X$ such that $T(x) = S^*(x)$. If $x \in X \setminus \overline{U^w}$, then $T(x) = 0$, else $T(x) = S^*(x) = \varphi(x)S(x) = S(x)$. □

When $T = I$, we get the following fixed point theorem.

**Corollary 5.2.** *Let $X$ be a Banach space. Let $U \subset X$ be a weakly open set and such that $0 \in U$, and let $S : \overline{U^w} \longrightarrow X$ be a sequentially weakly continuous operator such that $[S]_A < 1$.*
*Then, either*

**($\mathcal{A}_1$)** *there exist $x \in \partial_X(U)$ and $\lambda \in (0,1)$ such that $x = \lambda S(x)$, or*

**($\mathcal{A}_2$)** *there exists $x \in U$ such that $S(x) = x$.*

According to Proposition 3.3, every weakly compact operator $S$ satisfies the condition $[S]_A = 0$. As a consequence of Theorem (5.1) we get the following result:

**Theorem 5.3.** *Let $X$ be a Banach space, $T$ be a $w$-stably solvable operator on $X$. Let $U \subset X$ be a weakly open set and such that $0 \in U$, and let $S : \overline{U^w} \longrightarrow X$ be a weakly completely continuous operator. Suppose also that $S$ satisfies the Leray-Schauder boundary condition:*

$$y \notin \{\lambda S(y); \lambda \in (0,1)\} \; y \in \partial_X(U).$$

*Then, either*

**($\mathcal{A}_1$)** *$T(x) = 0$ has a solution on $X \setminus U$, or*

**($\mathcal{A}_2$)** *there exists $x \in U$ such that $T(x) = S(x)$.*

**Corollary 5.4.** *Let $X$ be a Banach space. Let $U \subset X$ be a weakly open set and such that $0 \in U$, and let $S : \overline{U^w} \longrightarrow X$ be a weakly completely continuous operator.*
*Then, either*

**($\mathcal{A}_1$)** *there exists $x \in \partial_X(U)$ and $\lambda \in (0,1)$ such that $x = \lambda S(x)$, or*

**($\mathcal{A}_2$)** *there exist $x \in U$ such that $S(x) = x$.*

**Theorem 5.5.** *Let $X$ be a Banach space, $T$ be a $w$-stably solvable operator on $X$. Let $U \subset X$ be a weakly open set and such that $0 \in U$, and $S : \overline{U^w} \longrightarrow X$ be a sequentially weakly continuous with $[S]_A < 1$. If $\overline{U^w}$ is weakly compact and $S$ satisfies the Leray-Schauder boundary condition:*

$$y \notin \{\lambda S(y); \lambda \in (0,1)\} \; y \in \partial_X(U).$$

*Then, either*

**($\mathcal{A}_1$)** *$T(x) = 0$ has a solution on $X \setminus U$, or*

**($\mathcal{A}_2$)** *there exists $x \in U$ such that $T(x) = S(x)$.*

*Proof.* Let

$$M = \left\{x \in \overline{U^w}, \lambda S(x) = x \text{ for some } \lambda \in [0,1]\right\}.$$

The set $M$ is nonempty since $0 \in M$, and $M \cap \partial_X(U) = \emptyset$. It is clear that $M$ is relatively weakly compact. We next prove that $M$ is weakly closed. Let $x \in \overline{M^w}$. Taking into account the fact that $\overline{M^w}$ is weakly compact, in view of the Eberlein-Smulian Theorem, there exists a sequence $(x_n)_n$ such that $x_n \rightharpoonup x$. Notice that for each integer $n \in \mathbb{N}$, there is $\lambda_n \in [0,1]$ such that

$$x_n = \lambda_n S(x_n).$$

By extracting a subsequence, if necessary, we assume that

$$\lambda_n \to \lambda \in [0,1].$$

Taking into account the sequential weak continuity of $S$ and letting $n \longrightarrow \infty$, we get $\lambda S(x) = x$, and so $x \in M$, and consequently the set $M$ is weakly closed.
Keeping in mind that $M \cap \partial_X(U) = \emptyset$, $M$ is weakly compact, and $\partial_X(U)$ is weakly closed since $X$ endowed with the weak topology is a Tychonoff space, the Urysohn Theorem for the weak topology [20] ensures the existence of a weakly continuous mapping $\varphi : X \to [0,1]$ such that

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in M; \\ 0, & \text{if } x \in \partial_X(U). \end{cases}$$

Consider the operator $S_1 : X \to X$ given by

$$S^*(x) = \begin{cases} \varphi(x)S(x), & \text{if } x \in \overline{U^w}; \\ 0, & \text{if } x \in X \setminus \overline{U^w}. \end{cases}$$

It is clear that

$$S^*(X) \subset \overline{co}(S(\overline{U^w}) \cup \{0\}). \tag{4}$$

Consequently, for all $M \subset X$ be a bounded set, we have

$$\omega(S^*(M)) \leq \omega(\overline{co}(S(\overline{U^w}) \cup \{0\})) \leq \omega(S(\overline{U^w}), \tag{5}$$

which implies that $S^*$ is weakly compact operator. Because $\partial_X(U) = \partial_X(\overline{U}^w)$, $\varphi$ is weakly continuous and $S$ is weakly sequentially continuous, we have that $S^*$ is weakly sequentially continuous. Since $T$ is w-stably solvable, there exists $x \in X$ such that $T(x) = S^*(x)$. If $x \in X \setminus \overline{U^w}$, then $T(x) = 0$, else $T(x) = S^*(x) = \varphi(x)S(x) = S(x)$. □

When $T = I$, we get the Theorem 3.1 in [10].

**Corollary 5.6.** *Let $X$ be a Banach space. Let $U \subset X$ be a weakly open set and such that $0 \in U$, and $S : \overline{U^w} \longrightarrow X$ be a sequentially weakly continuous with $[S]_A < 1$. If $U$ is relatively weakly compact and $S$ satisfies the Leray-Schauder boundary condition:*

$$y \notin \{\lambda S(y); \lambda \in (0,1)\} \ y \in \partial_X(U).$$

*Then, there exists $x \in U$ such that $S(x) = x$.*

**Remark 5.7.** *The condition that $\overline{U^w}$ is weakly compact in the statement of Theorem 5.5 can be removed if we assume that $S(\overline{U^w})$ is relatively weakly compact.*

## 6. Volterra Integral Equations

Let $X$ be a Banach space and $C(J, X)$ be the Banach space of all continuous functions from $J := [0, 1]$ to $X$, endowed with the sup-norm $\|\cdot\|_\infty$, defined by

$$\|h\|_\infty = \sup_{t \in J} \|h(t)\|, h \in C(J, X).$$

Consider the following Volterra integral equation in $X$:

$$x(t) = f(x)(t) + \int_0^t k(t,s)g(s, x(s))\, ds,\ t \in J, \tag{6}$$

where $k : J \times J \to \mathbb{R}$ is continuous, $f$ and $g$ satisfy some suitable conditions.

This section is devoted to discuss existence results for the following Volterra integral equation in the Banach space $X$.

Let us consider the following assumptions:

($H_1$) The function $g : J \times X \to X$ is such that:

(a) $g$ is sequentially weakly continuous with respect to the second variable and there exists integrable function $\gamma$ on $J$ such that $\|g(s, x)\| \leq \gamma(s)\|x\|$ for a.e. $s \in J$ and all $x \in X$.

(b) There exists a constant $L > 0$ such that

$$\omega(g(J \times M(J))) \leq L\omega(M)$$

for all bounded subset $M$ of $C(J, X)$.

($H_2$) The operator $f : C(J, X) \longrightarrow C(J, X)$ is sequentially weakly continuous and

$$[f]_A < 1 - L\|k(\cdot,\cdot)\|_\infty.$$

To make lecture of the Volterra integral equation (1) easier, let us consider the operators $S$ defined on $C(J,X)$ by:

$$(Sx)(t) \quad = f(x)(t) + \int_0^t k(t,s)g(s,x(s))\,ds.$$

This means that Equation (1) is equivalent to:

$$x = S(x).$$

**Theorem 6.1.** *Assume that assumptions $(H_1)$ and $(A_2)$ hold. Let $U$ be a weakly open, bounded subset of $C(J,X)$ such that $0 \in U$. In addition, suppose that for any solution $x$ to the equation $x = \lambda \cdot S(x)$ for some $0 < \lambda < 1$, we have $x \neq \partial_X(U)$. Then the Volterra integral equation (1) has a solution in $U$.*

*Proof.* We will show that $S$ satisfy all conditions of Corollary 5.2. For this purpose, we begin by proving that $S$ maps $\overline{U^w}$ into $C(J,X)$. Let $x \in \overline{U^w}$ be arbitrary and $t, t' \in J$. According to the Hahn-Banach's theorem, there exists $\varphi \in X^*$ with $\|\varphi\| = 1$ and $\|S(x)(t) - S(x)(t')\| = \varphi(S(x)(t) - S(x)(t'))$. Then,

$$\begin{aligned}\|S(x)(t) - S(x)(t')\| &\leq \int_0^t |k(t,s) - k(t',s)|\,|\varphi(g(s,x(s)))|\,ds + \int_{t'}^t |k(t',s)|\,|\varphi(g(s,x(s)))|ds \\ &\leq \int_0^t |k(t,s) - k(t',s)|\,\|g(s,x(s))\|\,ds + \int_{t'}^t |k(t',s)|\,\|g(s,x(s))\|ds.\end{aligned}$$

Since $\|g(s,x(s))\| \leq \gamma(s)\|x(s)\|$ for a.e. $s \in J$, it follows that

$$\|S(x)(t) - S(x)(t')\| \leq \|x\|_\infty \left( \int_0^t |k(t,s) - k(t',s)|\,\gamma(s)ds + \|k(\cdot,\cdot)\|_\infty \int_{t'}^t \gamma(s)ds \right).$$

This implies that,

$$\|Sx(t) - Sx(t')\| \leq \|\overline{U^w}\| \left( \int_0^t \gamma(s)|k(t,s) - k(t',s))|ds + \int_{t'}^t |k(t',s)|\,\gamma(s)ds \right). \tag{7}$$

If we consider the continuity of the partial $t \mapsto k(t,s)$ on the compact interval $J$ and deploy the dominated convergence theorem, we obtain that $Sx \in C(J,X)$.

Second, we shall claim that the operator $S$ is sequentially weakly continuous. Indeed, let $(x_n)_n \subset \overline{U^w}$ such that $x_n \rightharpoonup x \in \overline{U^w}$. By using the Dobrakov's theorem, in view of assumption $(H_1) - (a)$ we get

$$x^*(g(s,x_n(s)) \longrightarrow x^*(g(s,x(s)) \text{ for all } x^* \in X^*.$$

Moreover, since $\|g(s,x_n(s))\| \leq \gamma(s)\|\overline{U^w}\|$, it follows from the dominated convergence theorem of the Pettis integral that since $\|g(s,x_n(s))\| \leq \gamma(s)\|\overline{U^w}\|$,

$$\int_0^t k(t,s)g(s,x_n(s))ds \rightharpoonup \int_0^t k(t,s)g(s,x(s))ds.$$

Taking into account the boundedness of $S(\overline{U^w})$, and deploy the Dobrakov theorem, we get $S(x_n) \rightharpoonup S(x)$. Now, we claim that $[S]_A < 1$. To this end, we need to decompose $S$ under the form:

$$S := S_1 + S_2,$$

where

$$S_1(x)(t) = f(x)(t) \text{ and } S_2(x)(t) = \int_0^t k(t,s)g(s,x(s))\,ds.$$

Let $M$ be a bounded subset of $\overline{U^w}$ such that $\omega(M) > 0$. By definition, for all $t \in J$, we have

$$S_2(\overline{U^w})(t) = \{S_2(x)(t), x \in M\}.$$

By Lemma 2.10, we obtain

$$\begin{aligned}\omega(S_2(M)(t)) &\leq \omega\Big(\int_0^t k(t,s)\, f(s, M(s))\Big)\\ &\leq \omega(\overline{co}(k(t,J)\, f(s, M(J))))\\ &\leq \|k(\cdot,\cdot)\|_\infty\, \omega(f(J \times M(J))).\end{aligned}$$

By Inequality (7), we deduce that $S_2(M)$ is equi-continuous subset of $C(J, X)$, and so, by Lemma 2.10, we get

$$\omega(S_2(M)) \leq \|k(\cdot,\cdot)\|_\infty\, \omega(f(J \times M(J))).$$

Then, using assumption $(H_1) - (b)$, we obtain

$$\omega(S_2(M)) \leq L\, \|k(\cdot,\cdot)\|_\infty\, \omega(M)).$$

Hence, from the subadditivity of $\beta$ we derive that

$$\begin{aligned}\omega(S(M)) &\leq \omega(S_1(M)) + \omega(S_2(M))\\ &\leq \omega(f(M)) + L\, \|k(\cdot,\cdot)\|_\infty\, \omega(M)).\end{aligned}$$

This implies that

$$\frac{\omega(S(M))}{\omega(M)} \leq \frac{\omega(f(M))}{\omega(M)} + \|k(\cdot,\cdot)\|\, L.$$

Since $[f]_A < (1 - L\, \|k(\cdot,\cdot)\|_\infty)$, therefore, we conclude that $[S]_A < 1$.

Hence, by Corollary 5.2, the operator $S$ has a fixed point on $U$, which is the solution of the Volterra integral equation (6). This completes the proof. □

## Conclusion

In this paper, we introduce the stable solvability notion of nonlinear operators within the context of the weak topology. Several interesting properties of this notion are discussed. In addition, we use the weak stable solvability to establish new variants of Leray Schauder fixed point results.

Our obtained results improve and extend different results in the literature. In particular, setting $T = I$ in Theorem 5.5, we obtain Theorem 3.1 in [10]. Moreover, Corollary 5.6 extend Theorem 2.9 and Theorem 2.10 in [6] and prove that the condition ws-compact in such statement is not necessary.

As an application, in the present work we have discussed the existence of solution of Volterra integral equations. However, other several nonlinear functional differential equations could also be studied for existence of the solutions using the same arguments in an analogous way with appropriate modifications.

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