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Lower characteristic, demicompact linear operators, and essential spectra

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Abstract. In this paper, we present some results on the ”lower” characteristic involving demicompact operators. They are used to establish a fine description of the B-Weyl spectrum, and to investigate some perturbation results. Finally, some results concerning the Schechter and Jeribi essential spectra are given.

1. Introduction

Let X and Y be two Banach spaces. By a bounded operator T from X into Y, we mean a linear operator with domain X and range $R(T) \subseteq Y$. By $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X into Y and by $\mathcal{K}(X,Y)$ the subspace of all compact operators of $\mathcal{L}(X,Y)$. If $T \in \mathcal{L}(X,Y)$ then $\rho(T)$ denotes the resolvent set of T, $\alpha(T)$ the dimension of the kernel $N(T)$ of T and $\beta(T)$ the codimension of the range $R(T)$ in Y of T. The classes of upper semi-Fredholm from X into Y is defined by

$$\Phi_+(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } R(T) \text{ closed in } Y\},$$

and the classes of lower semi-Fredholm from X into Y is defined by

$$\Phi_-(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } R(T) \text{ closed in } Y\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \bigcap \Phi_-(X,Y)$ is the set of Fredholm operators from X into Y, and $\Phi_\pm(X,Y) := \Phi_+(X,Y) \bigcup \Phi_-(X,Y)$ is the set of semi-Fredholm operators from X into Y. If $X = Y$, the sets $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, and $\Phi_-(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, and $\Phi_-(X)$, respectively. The index of an operator $T \in \Phi_\pm(X)$ is $i(T) := \alpha(T) - \beta(T)$.

An operator $F \in \mathcal{L}(X,Y)$ is called an upper semi-Fredholm perturbation if $T + F \in \Phi_+(X,Y)$ whenever $T \in \Phi_+(X,Y)$. The set of upper semi-Fredholm perturbations is denoted by $\mathcal{F}_+(X,Y)$. These classes of operators were introduced and investigated by Gohberg et al. in [5]. It was shown in [1], that $\mathcal{F}_+(X,Y)$ is closed subsets of $\mathcal{L}(X,Y)$ (see also [9–11, 20–23]).

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An operator $T \in \mathcal{L}(X)$ is called a B-Fredholm operator, $T \in B\mathcal{F}(X)$, if there exists an integers $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T_n = T_{|R(T^n)}$ is Fredholm, where $R(T^n)$ is the range of the operator T^n. If for some integer n the range space $R(T^n)$ is closed and $T_n := T_{|R(T^n)}$ is an upper semi-Fredholm operator, then T is called an upper semi B-Fredholm operator and we write $T \in B\mathcal{F}_+(X)$. The B-Fredholm spectrum $\sigma_{BF}(T)$ and upper semi B-Fredholm spectrum $\sigma_{uBF}(T)$ of T, are respectively defined by:

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin B\mathcal{F}(X)\}$$

and

$$\sigma_{uBF}(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin B\mathcal{F}_+(X)\}.$$

The operator T is said to be B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by:

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \text{ is not a } B\text{-Weyl operator}\}.$$

Now, let A be a unitary algebra. It is well known that an element x of A is Drazin invertible of degree k if there is an element b of A such that $x^k bx = x^k$, $bxb = b$, $xb = bx$ (see [15]). The Drazin invertible spectrum $\sigma_D(a)$ of an element a in A is defined by:

$$\sigma_D(a) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - a \text{ is not a Drazin invertible operator}\}.$$

Note that, the concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. As resulted in [13], for $T \in \mathcal{L}(X)$, we have $\sigma_{BW}(T) \subset \sigma_D(T)$.

Theorem 1.1. *[15] Let X be a Banach space and $T \in \mathcal{L}(X)$ be such that 0 is isolated in the spectrum $\sigma(T)$ of T. Then T is Drazin invertible if and only if T is a B-Weyl operator.* ◊

Now, an operator $T \in \mathcal{L}(X)$ is said to be semi-regular if $R(T)$ is closed and $N(T) \subset R(T^n)$, for all $n \geq 0$. Recall their an operator $T \in \mathcal{L}(X)$ is said to be quasi-nilpotent if $\sigma(T) = \{0\}$. T admits a generalized Kato decomposition, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, where $T_{|M}$ is semi-regular and $T_{|N}$ is quasi-nilpotent. For $T \in \mathcal{L}(X)$, the ascent $a(T)$ and the descent $d(T)$ of T are provided by

$$a(T) = \inf\{n \in \mathbb{N} \text{ such that } N(T^n) = N(T^{n+1})\},$$
$$d(T) = \inf\{n \in \mathbb{N} \text{ such that } R(T^n) = R(T^{n+1})\},$$

where $\inf \emptyset = \infty$. We denote by

$$L\mathcal{D}(X) := \{T \in \mathcal{L}(X) \text{ such that } a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed in } X\}$$

and the left Drazin spectrum $\sigma_{lD}(T)$ of T is defined by:

$$\sigma_{lD}(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin L\mathcal{D}(X)\}.$$

Definition 1.2. *[17] Let D be a bounded subset of X. We define $\gamma(D)$, the Kuratowski measure of noncompactness of D, to be* $\inf\{d > 0$ *such that D can be covered by a finite number of sets of diameter less than or equal to $d\}$.* ◊

The following proposition gives somes properties of the Kuratowski measure of noncompactness which are frequently used.

Proposition 1.3. *Let D and D' be two bounded subsets of X then we have the following properties*

(i) *$\gamma(D) = 0$ if and only if D is relatively compact.*
(ii) *If $D \subseteq D'$, then $\gamma(D) \leq \gamma(D')$.*
(iii) *$\gamma(D + D') \leq \gamma(D) + \gamma(D')$.*
(iv) *For every $\alpha \in \mathbb{C}$, $\gamma(\alpha D) = |\alpha|\, \gamma(D)$.* ◊

Definition 1.4. *[18] Let $T \in \mathcal{L}(X, Y)$, $\gamma(.)$ be the Kuratowski measure of noncompactness in X. Let $k \geq 0$, T is said to be k-set-contraction if, for any bounded subset B of X, $T(B)$ is a bounded subset of X and $\gamma(T(B)) \leq k\gamma(B)$. T is said to be condensing if, for any bounded subset B of X such that $\gamma(B) > 0$, $T(B)$ is a bounded subset of X and $\gamma(T(B)) < \gamma(B)$.* ♢

Definition 1.5. *Let X be a Banach space and let $T : X \longrightarrow X$ be a bounded linear operator. The operator T is said to be demicompact (or relative demicompact), if for every bounded sequence $(x_n)_n \in X$ such that $x_n - Tx_n \to x \in X$, then there exists a convergent subsequence of $(x_n)_n$.* ♢

Remark 1.6. *It is well known that*

(*i*) *Every k-set-contraction operator such that $k < 1$ is condensing.*
(*ii*) *Every condensing operator is* 1*-set-contraction.*
(*iii*) *Every condensing operator is demicompact.* ♢

Definition 1.7. *Let $T \in \mathcal{L}(X)$. We define $\bar{\gamma}(T)$ by*

$$\bar{\gamma}(T) := \inf\{k \text{ such that } T \text{ is } k\text{-set-contraction}\}.$$ ♢

In the following proposition, we give some properties of $\bar{\gamma}(.)$ that we will need in the sequel.

Proposition 1.8. *[2, 4] Let X be a Banach space and $T \in \mathcal{L}(X)$, then we have the following properties*

(*i*) $\bar{\gamma}(T) = 0$ *if and only if T is compact.*
(*ii*) *If $T, S \in \mathcal{L}(X)$, then $\bar{\gamma}(ST) \leq \bar{\gamma}(S)\bar{\gamma}(T)$.*
(*iii*) *If $K \in \mathcal{K}(X)$, then $\bar{\gamma}(T + K) = \bar{\gamma}(T)$.*
(*iv*) *If B is a bounded subset of X, then $\gamma(T(B)) \leq \bar{\gamma}(T)\gamma(B)$.* ♢

The paper is organized in the following way. In Section 2, we present the main results of this paper. We prove some result concerning the "lower" characteristic. In Section 3, we present a new characterization of the B-Weyl spectrum and we establish some perturbation results. Finally, we give some results concerning the Jeribi and Schechter essential spectra.

2. Main results

Definition 2.1. *For $T \in \mathcal{L}(X, Y)$, we define the "lower" characteristic*

$$[T]_a = \sup\{k \;:\; k > 0,\; \gamma(T(M)) \geq k\gamma(M) \text{ for all bounded } M \subset X\} \tag{1}$$

as elements of $[0, \infty]$. ♢

Note that in finite dimensional spaces we have $[T]_a = \infty$. In infinite dimensional spaces, where this characteristic is of more use, we get

$$[T]_a = \inf_{0<\gamma(M)<\infty} \frac{\gamma(T(M))}{\gamma(M)}.$$

Sets with $\gamma(M) = 0$ can be left out here, since the continuity of T assures that also $\gamma(T(M)) = 0$. This can be seen by considering $\gamma(T(M)) \leq \gamma(T(\overline{M}))$.

Proposition 2.2. *[24] Let X, Y, Z be three Banach spaces, $T \in \mathcal{L}(X, Y)$ and $R \in \mathcal{L}(Y, Z)$. Then $[R]_a[T]_a \leq [RT]_a$.* ♢

Theorem 2.3. *[14] Let $T \in \mathcal{L}(X, Y)$. Then $[T]_a > 0$ if and only if T is upper semi-Fredholm.* ♢

The set of semi-Weyl operators is defined by

$$\mathcal{W}_+(X) = \{T \in \mathcal{L}(X) \text{ such that } [T]_a > 0 \text{ and } i(T) \leq 0\}.$$

Remark 2.4. (*i*) *If T is compact or nilpotent, i.e., there exists $n \in \mathbb{N}^*$ such that $T^n = 0$, then $[I-T]_a > 0$.*
(*ii*) *Let T be a bounded linear operator, and let $p \in \mathbb{N}^*$. If $[I-T^p]_a > 0$, then $[I-T]_a > 0$.*
(*iii*) *The converse of (ii) is false. In fact, let X be an infinite dimensional Banach space and T be a bounded linear operator such that $[I-T]_a > 0$ and $T^2 = I$. Then $[I-T^2]_a = 0$.* ◊

Theorem 2.5. Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then T is demicompact if and only if $[I-T]_a > 0$.◊

Proof. We first show that $N(I-T)$ is finite dimensional. Let $S := \{x \in X$ such that $(I-T)x = 0$ and $\|x\| = 1\}$ and $(x_n)_n$ be a bounded sequence of S. Since T is demicompact, there exists a subsequence $(x_{n_i})_i$ of $(x_n)_n$ which converges to $x \in X$. Thus, it follows from the continuity of the norm and the boundness of T that $x \in X$, $x - Tx = 0$ and $\|x\| = 1$. Hence $\alpha(I-T)$ is finite. Now, we claim that $R(I-T)$ is closed. Applying Lemma 5.1 in [19], we can write $X = N(I-T) \oplus X_0$, where X_0 is a closed subspace of X, then it is a Banach space. In view of Theorem 3.12 in [19], it suffices to prove that there is a constant $\lambda > 0$ such that for every $x \in X_0$, $\|Tx\| \geq \lambda \|x\|$. If not, there exists a sequence $(x_n)_n$ of X_0 such that $\|x_n\| = 1$ and $\|(I-T)x_n\| \to 0$. Since T is demicompact, there exists a subsequence $(x_{n_i})_i$ of $(x_n)_n$ which converges to $x \in X$. Moreover, $I-T$ is closed and $(I-T)x = 0$, hence $x = 0$ which contradicts the continuity of the norm. Since $\dim N(I-T) < \infty$, we may find a closed subspace X_0 of X with $X = X_0 \oplus N(I-T)$. The projection $P : X \longrightarrow X_0$ satisfies $[P]_a = 1$, since $I-P$ is compact. Consider the canonical isomorphism $\widetilde{L} : X_0 \longrightarrow R(I-T)$. Since $I-T = \widetilde{L}P$, $[\widetilde{L}]_a > 0$ and in view of Proposition 2.2, we conclude that also

$$[I-T]_a \geq [\widetilde{L}]_a [P]_a > 0.$$

Inversely, suppose that $[I-T]_a > 0$ and fix $k \in (0, [I-T]_a)$. Since the set $M = N(I-L) \bigcap B_X$ is mapped into $(I-T)(M) = \{0\}$, we get

$$\gamma(M) \leq \frac{1}{k}\gamma((I-T)(M)) = 0,$$

which show that $\overline{M}$ is compact, and hence $N(I-T)$ is finite dimensional. We prove now that the range $R(I-T)$ of $I-T$ is closed. Since $\dim N(I-T) < \infty$, there exists a closed subspace $X_0 \subset X$ such that $X = X_0 \oplus N(I-T)$. Let $(y_n)_n$ be a sequence in $R(I-T)$ converging to some $y \in Y$, and choose $(x_n)_n$ in X with $(I-T)x_n = y_n$. Now, we distinguish two cases. First, suppose that $(x_n)_n$ is bounded. With $k > 0$ as before we get then

$$\gamma(\{x_1, x_2, \cdots, x_n, \cdots\}) \leq \frac{1}{k}\gamma(\{y_1, y_2, \cdots, y_n, \cdots\}) = 0,$$

and hence $x_{n_k} \to x$ for some subsequence $(x_{n_k})_k$ of $(x_n)_n$ and suitable $x \in X$. By continuity we see that $(I-T)x = y$, and so $y \in R(I-T)$. On the other hand, suppose that $\|x_n\| \to \infty$. Set $e_n = \frac{x_n}{\|x_n\|}$ and $E = \{e_1, e_2, \cdots, e_n, \cdots\}$. Then clearly $E \subset \{x \in X : \|x\| = 1\}$ and

$$(I-T)e_n = \frac{(I-T)x_n}{\|x_n\|} = \frac{y_n}{\|x_n\|} \to 0 \text{ as } n \to \infty.$$

Hence, $\gamma((I-T)(E)) = 0$. On the other hand, $\gamma((I-T)(E)) \geq k\gamma(E)$, by (1), and thus $\gamma(E) = 0$. Whithout loss of generality, we may assume that the sequence $(e_n)_n$ converge to some element $e \in \{x \in X_0 : \|x\| = 1\}$. So, $(I-T)e = 0$, contradicting the fact that $X_0 \bigcap N(I-T) = \{0\}$. Thus, $I-T \in \Phi_+(X)$. By $\alpha(I-T) < \infty$, we deduce that there exists a closed subspace C of X such that

$$N(I-T) \oplus C = X.$$

We deduce that

$$(I-T)_{|C} : (C, \|\cdot\|) \longrightarrow (R(I-T), \|\cdot\|)$$

is invertible with bounded inverse on $R(I-T)$. Now, take a bounded sequence $(x_n)_n$ of X such that $((I-T)x_n)_n$ converges to y. Obviously, $y \in R(I-T)$. Using the boundedness of $((I-T)_{|C})^{-1}$ on $R(I-T)$, we deduce that $(x_n)_n$ converges to $((I-T)_{|C})^{-1}(y) = z$. Hence, $(x_n)_n$ converges to z. So, T is demicompact and the proof is achieved. □

By using Theorems 2.3 and 2.5, we have the following.

Corollary 2.6. *Let $T \in \mathcal{L}(X,Y)$. Then T is demicompact if and only if $I - T$ is upper semi-Fredholm.*

Corollary 2.7. *Let $T \in \mathcal{L}(X)$. If T is a 1-set-contraction, then $[I - \mu T]_a > 0$ for each $\mu \in [0,1)$.*

Proof. If T is a 1-set-contraction then by using [16, Lemma 2.6], we have μT is demicompact for each $\mu \in [0,1)$. The result follows from Theorem 2.5. □

Theorem 2.8. *Let $p \in \mathbb{N}^*$ and $T \in \mathcal{L}(X)$. If $[T]_a > 0$, then $[T^p]_a > 0$.* ◊

Proof. Since $[T]_a > 0$, by using Theorem 2.3, it follows that $R(T)$ is closed and $\alpha(T) < \infty$. Now, we will prove by induction that for all $p \in \mathbb{N}^*$, $[T^p]_a > 0$. The case $p = 1$ follows from the hypothesis. Assume that $[T^p]_a > 0$ and take $(x_n)_n$ be a bounded sequence of X such that $T^{p+1}x_n \to y$, $y \in X$. Put $z_n := T^p x_n$. Then the sequence $(z_n)_n$ is bounded. Indeed, since $\alpha(T) < \infty$, then there exists a closed subspace X_0 of X such that

$$X = N(T) \oplus X_0.$$

Hence, the mapping $T : X_0 \longrightarrow R(T)$ is bijective. As $R(T)$ is a closed subspace of X, it follows $T^{-1} : R(T) \longrightarrow X_0$ is bounded. Thus,

$$\|z_n - T^{-1}y\| = \|T^{-1}(T^{p+1}x_n - y)\| \to 0.$$

So, the sequence $(z_n)_n$ is convergent. Next, since $T \in \mathcal{L}(X)$, $Tz_n \to y$ and $[T]_a > 0$, then there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ such that $T^p x_{\varphi(n)}$ converges. Now, the result follows from the hypothesis of induction $[T^p]_a > 0$ and Theorem 2.5. □

Corollary 2.9. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If $T \in \Phi_+(X)$, then for all $n \in \mathbb{N}$, $T^n \in \Phi_+(X)$.* ◊

Proof. The proof follows from Theorems 2.8 and 2.3. □

Lemma 2.10. *[6, Lemma 3.2] Let X be a Banach space and T will be a linear operator with domain $\mathcal{D}(T)$ in the linear space X. For $k = 0,1,2,\cdots$ and $i = 0,1,2,\cdots$, we have*

$$\frac{R(T^i)}{R(T^{i+k})} \simeq \frac{\mathcal{D}(T^i)}{\{R(T^k) + N(T^i)\} \bigcap \mathcal{D}(T^i)}.$$ ◊

Lemma 2.11. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed in X, codim$(R(T) + N(T^n)) < \infty$, and $[T]_a > 0$, then $T \in B\mathcal{F}(X)$.* ◊

Proof. Since $[T]_a > 0$ and $R(T^n)$ is closed in X, then $[T_{|R(T^n)}]_a > 0$. By appliying Theorem 2.3, we infer that $T_{|R(T^n)}$ is upper semi-Fredholm. By using Lemma 2.10, we have

$$\frac{R(T^n)}{R(T^{n+1})} \simeq \frac{X}{R(T) + N(T^n)}.$$

Since $\text{codim}(R(T) + N(T^n)) < \infty$, then $\dim \frac{R(T^n)}{R(T^{n+1})} < \infty$, which implies that $T_{|R(T^n)}$ is a Fredholm operator and $R(T^n)$ is closed. Thus, T is B-Fredholm. □

Lemma 2.12. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed in X, and $\dim \frac{R(T^n)}{N(I-T)} < \infty$, then*

$$\frac{\frac{R(T^n)}{N(I-T)}}{\frac{R(T^{n+1})}{N(I-T)}} \simeq \frac{R(T^n)}{R(T^{n+1})}.$$ ◊

Proof. Since $\dim \frac{R(T^n)}{N(I-T)} < \infty$, and

$$N(I-T) \subset R(T^{n+1}) \subset R(T^n),$$

then

$$\frac{\frac{R(T^n)}{N(I-T)}}{\frac{R(T^{n+1})}{N(I-T)}} \simeq \frac{R(T^n)}{R(T^{n+1})}.$$

This completes the proof. □

Lemma 2.13. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed in X, $\dim \frac{R(T^n)}{N(I-T)} < \infty$, and $[T]_a > 0$, then $T \in B\mathcal{F}(X)$.* ◊

Proof. By using Lemma 2.12, we infer that

$$\frac{\frac{R(T^n)}{N(I-T)}}{\frac{R(T^{n+1})}{N(I-T)}} \simeq \frac{R(T^n)}{R(T^{n+1})}.$$

Thus, $\dim \frac{R(T^n)}{R(T^{n+1})} < \infty$. It follows that $T_{|R(T^n)}$ is a Fredholm operator and $R(T^n)$ is closed. This is equivalent to T is B-Fredholm. □

A consequence of Lemmas 2.11 and 2.13, we have the following:

Corollary 2.14. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Assume that there exists $n \in \mathbb{N}$ such that*

(i) $R(T^n)$ is closed in X, and
(ii) $\dim \frac{R(T^n)}{N(I-T)} < \infty$ or $codim(R(T) + N(T^n)) < \infty$.

If $[T]_a > 0$, then $T \in B\mathcal{F}(X)$. ◊

Definition 2.15. *Let $T \in \mathcal{L}(X)$. T is said power compact operator if there exists $m \in \mathbb{N}^*$ satisfying $T^m \in \mathcal{K}(X)$.* ◊

Corollary 2.16. *Let $T \in \mathcal{L}(X)$ be a power compact operator then for every $\mu \in (0,1]$, $[I-\mu T]_a > 0$.* ◊

Proof. Since T is a power compact operator, then by using [16, Lemma 2.8], we have μT is demicompact for each $\mu \in (0,1]$. The result follows from Theorem 2.5. □

Corollary 2.17. *Let $T \in \mathcal{L}(X)$. If $\bar{\gamma}(T^m) < 1$, for some $m \in \mathbb{N}^*$, then for every $\mu \in (0,1]$, $[I-\mu T]_a > 0$.* ◊

Theorem 2.18. *Let $T \in \mathcal{L}(X)$. If $[I-T]_a > 0$, then $I-T$ is an upper semi-Fredholm operator.* ◊

Proof. Consequence direct of the proof of Theorem 2.5. □

Theorem 2.19. *Let $T \in \mathcal{L}(X)$. If $[I-\mu T]_a > 0$ for each $\mu \in [0,1]$, then $I-T$ is a Fredholm operator of index zero.* ◊

Proof. Since $[I-\mu T]_a > 0$ for each $\mu \in [0,1]$, then by using Theorem 2.3, we have $I-\mu T$ is an upper semi-Fredholm operator on X for each $\mu \in [0,1]$. By the stability results for semi-Fredholm operators of Kato [12], the index $i(I-\mu T)$ is continuous in μ. Since it is an integer, including infinite value, it must be constant for every $\mu \in [0,1]$, then $i(I-\mu T) = i(I-T) = i(I) = 0$. So, $I-T$ is a Fredholm operator of index zero. □

Corollary 2.20. *Let $T \in \mathcal{L}(X)$. If $[I-\mu T]_a > 0$ for each $\mu \in [0,1]$, then $I-\lambda T$ is a Fredholm operator of index zero for every $\lambda \in (0,1]$.* ◊

Corollary 2.21. *Let $T \in \mathcal{L}(X)$. If $[I-T]_a > 0$ and T is 1-set-contraction, then $I-T$ is a Fredholm operator of index zero.*

Proof. The proof follows from Lemma 2.7 and Theorem 2.19. □

Proposition 2.22. *Let $T \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$, where X, Y and Z are three Banach spaces. If $TB \in \Phi_+(X, Z)$, then $B \in \Phi_+(X, Y)$.* ◊

A consequence of Theorem 2.3 and Proposition 2.22, we have the following.

Proposition 2.23. *Let $T \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$, where X, Y and Z are three Banach spaces. If $[TB]_a > 0$, then $[B]_a > 0$.* ◊

Lemma 2.24. *Let X be a Banach space and let A, $B \in \mathcal{L}(X)$. Let $\mathbb{C}[z]$ be the set of polynomials with coefficients in $\mathbb{C}$ and consider $P(z) = \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]$. Then*

$$P(AB) - P(BA) = \sum_{k=0}^{n} a_k \sum_{i=0}^{k-1} (AB)^i (AB - BA)(BA)^{k-i-1}. \quad ◊$$

Proof. For $P(z) = \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]$, we have

$$\begin{aligned} P(AB) - P(BA) &= \sum_{k=0}^{n} a_k[(AB)^k - (BA)^k] \\ &= \sum_{k=0}^{n} a_k \sum_{i=0}^{k-1} (AB)^i (AB - BA)(BA)^{k-i-1}. \end{aligned}$$

This complete the proof. □

Theorem 2.25. *Let X be a Banach space and let A, $B \in \mathcal{L}(X)$ such that $AB - BA \in \mathcal{F}_+(X) := \mathcal{F}_+(X, X)$. Then for every complex polynomial $P(\cdot)$, we have $[I - P(AB)]_a > 0$ if and only if $[I - P(BA)]_a > 0$.* ◊

Proof. Consider $P(z) = \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]$. By using Lemma 2.24, we have

$$P(AB) - P(BA) = \sum_{k=0}^{n} a_k \sum_{i=0}^{k-1} (AB)^i (AB - BA)(BA)^{k-i-1}.$$

Since $[I - P(AB)]_a > 0$, then by applying Theorem 2.3, we infer that $I - P(AB) \in \Phi_+(X)$. Now, let us state

$$I - P(BA) = I - P(AB) + P(AB) - P(BA). \tag{2}$$

Since $AB - BA \in \mathcal{F}_+(X)$ and in view of $\mathcal{F}_+(X)$ is an ideal of $\mathcal{L}(X)$, then $P(AB) - P(BA) \in \mathcal{F}_+(X)$. Consequently, in view of Eq. (2), we have $I - P(BA) \in \Phi_+(X)$. The result follows from Theorem 2.3. □

Theorem 2.26. *Let X be a Banach space. Let $T \in \mathcal{L}(X)$, w be an n-th root of the unit with $n \in \mathbb{N}^*$. Then $[I - T^n]_a > 0$ if and only if $[I - w^k T]_a > 0$ for all $0 \le k \le n - 1$.* ◊

Proof. We have the following equality

$$I - T^n = \prod_{0 \le k \le n-1} (I - w^k T). \tag{3}$$

If $[I - T^n]_a > 0$, then by using the commutativity of $I - w^k T$ and $I - w^l T$ $k \neq l$ and both Eq. (3) and Proposition 2.23, we infer that $[I - w^k T]_a > 0$ for all $0 \le k \le n - 1$. Inversely, by using both Eq. (3) and Proposition 2.2, we have

$$[I - T^n]_a \ge \prod_{0 \le k \le n-1} [I - w^k T]_a. \tag{4}$$

If $[I - w^k T]_a > 0$ for all $0 \le k \le n - 1$, then by using Eq. (4), we have $[I - T^n]_a > 0$. □

Let C_w be the space of continuous w-periodic functions $x : [0,t] \longrightarrow \mathbb{R}$, equipped with the maximum norm

$$\|x\|_\infty = \sup_{s\in[0,t]} |x(s)|$$

and C'_w be the space of continuously differentiable w-periodic functions $x : [0,t] \longrightarrow \mathbb{R}$, equiped with the norm

$$\|f\|_1 = \|f\|_\infty + \|f'\|_\infty.$$

The spaces $(C_w, \|\cdot\|_\infty)$ and $(C'_w, \|\cdot\|_1)$ are Banach spaces. Let T be the operator defined by

$$Tx(t) = Ix(t) - \left[{}_{GL}D^{\frac{1}{q}} x(t) - \frac{x(0)t^{-\frac{1}{q}}}{\Gamma\left(1-\frac{1}{q}\right)} \right]^{\frac{1}{2}},$$

$q \in \mathbb{N} \setminus \{0,1\}$, and $H : C'_w \longrightarrow C_w$ be provided by the formula

$$(Hx)(t) = x'(t),$$

where ${}_{GL}D^{\frac{1}{q}}$ is the Grünwald-Letnikov fractional derivative with fractional order $\frac{1}{q}$. It is clear that H is a bounded linear operator and

$${}_{GL}D^{\frac{1}{q}} x(t) = \frac{x(0)t^{-\frac{1}{q}}}{\Gamma\left(1-\frac{1}{q}\right)} + D^{-\left(1-\frac{1}{q}\right)} x'(t),$$

where $D^{-\left(1-\frac{1}{q}\right)}$ is the fractional integral with fractional order $1-\frac{1}{q}$. Notice that

$${}_{GL}D^{\frac{1}{q}} x(t) = \frac{x(0)t^{-\frac{1}{q}}}{\Gamma\left(1-\frac{1}{q}\right)} +_C D^{\left(\frac{1}{q}\right)} x(t),$$

where ${}_C D^{\left(\frac{1}{q}\right)}$ is the Caputo derivative of fractional order $\frac{1}{q}$, implying that

$$\begin{aligned} (I-T)^{2q} x(t) &= \left[{}_{GL}D^{\frac{1}{q}} x(t) - \frac{x(0)t^{-\frac{1}{q}}}{\Gamma\left(1-\frac{1}{q}\right)} \right]^q \\ &= \left[CD^{\left(\frac{1}{q}\right)} \right]^q x(t). \end{aligned}$$

Grounded on [7, Theorem 3.4], we infer that

$$(I-T)^{2q} x(t) = x'(t).$$

It is easy to see that $\|(I-T)^{2q}\| < 1$ and $(I-T)^{2q}$ is a demicompact operator. Hence, by using Theorem 2.5, we have $[I-(I-T)^{2q}]_a > 0$. By applying Theorem 2.26, we get $[I - w^k(I-T)]_a > 0$ for all $0 \le k \le 2q-1$. For $k=0$, we have $[I-(I-T)]_a = [T]_a > 0$. By applying again Theorem 2.3, we deduce that

$$Tx(t) = x(t) - \left[{}_{GL}D^{\frac{1}{q}} x(t) - \frac{x(0)t^{-\frac{1}{q}}}{\Gamma\left(1-\frac{1}{q}\right)} \right]^{\frac{1}{2}}$$

is an upper semi-Fredholm operator.

3. Essential spectra

Let $T \in \mathcal{L}(X)$. We define the Schechter essential spectrum [8] by

$$\sigma_{e_5}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma(T+K).$$

The following propositions give a characterization of the Schechter essential spectrum by means of Fredholm operators.

Proposition 3.1. *[19] Let $T \in \mathcal{L}(X)$ then $\lambda \notin \sigma_{e_5}(T)$ if and only if $\lambda \in \Phi^0_T(X)$, where $\Phi^0_T(X) := \{\lambda \in \mathbb{C}$ such that $\lambda - T \in \Phi(X)$ and $i(\lambda - T) = 0\}$.* ◊

In this section, we will give a refinement of the Schechter essential spectrum. For this, let X be a Banach space and $T \in \mathcal{L}(X)$. We define these sets Λ_X, $\Upsilon_T(X)$ and $\Psi_T(X)$ by:

$$\begin{aligned}
&\Lambda_X = \{J \in \mathcal{L}(X) \text{ such that } [I - \mu J]_a > 0 \text{ for every } \mu \in [0,1]\},\\
&\Upsilon_T(X) = \{K \in \mathcal{L}(X) \text{ such that } \forall \lambda \in \rho(T+K), -(\lambda - T - K)^{-1}K \in \Lambda_X\},\\
&\Psi_T(X) = \{K \text{ is } T\text{-bounded such that } \forall \lambda \in \rho(T+K), -K(\lambda - T - K)^{-1} \in \Lambda_X\}.
\end{aligned}$$

We define the Jeribi essential spectra by

$$\sigma_r(T) = \bigcap_{K \in \Upsilon_T(X)} \sigma(T+K)$$

$$\sigma_l(T) = \bigcap_{K \in \Psi_T(X)} \sigma(T+K).$$

We have the following result.

Theorem 3.2. *[3] For each $T \in \mathcal{L}(X)$,*

$$\sigma_{e5}(T) = \sigma_r(T) = \sigma_l(T).$$

Corollary 3.3. *Let $T \in \mathcal{L}(X)$, and let $\mathcal{E}(X)$ be a subset of $\Upsilon_T(X)$ (resp. $\Psi_T(X)$) containing $\mathcal{K}(X)$. Then*

$$\sigma_{e5}(T) = \bigcap_{K \in \mathcal{E}(X)} \sigma(T+K).$$

Moreover, if for all $K, K' \in \mathcal{E}(X)$, we have $K \pm K' \in \mathcal{E}(X)$, then, for every $K \in \mathcal{E}(X)$,

$$\sigma_{e5}(T) = \sigma_{e5}(T+K).$$ ◊

Theorem 3.4. *[13] Let X be a Banach space and $T \in \mathcal{L}(X)$. Then*

(i) $\sigma_{ID}(T) = \sigma_{uBF}(T) \bigcup S(T)$, where $S(T)$ is the set of all $\lambda \in \mathbb{C}$ such that T does not have the single-valued extension property at λ.
(ii) $\sigma_D(T) = \sigma_{BF}(T) \bigcup [S(T) \bigcup S(T^*)]$, where T^* denotes the adjoint of T. ◊

Theorem 3.5. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed in X and $\dim \frac{R(T^n)}{N(I-T)} < \infty$ or $codim(R(T)+N(T^n)) < \infty$, then $\sigma_{BW}(T) \subset \sigma_r(T) \bigcap \sigma_1(T)$, where $\sigma_1(T) = \{\lambda \in \mathbb{C}$ such that $\lambda - T \notin \mathcal{W}_1(X)\}$ and $\mathcal{W}_1(X) = \{T \in \Phi_+(X)$ such that $i(T) = 0\}$.* ◊

Proof. Let $\lambda \notin \sigma_r(T)$. Then there exists $K \in \Upsilon_T(X)$ such that $-(\lambda - T - K)^{-1}K \in \Lambda_X$, whenever $\lambda \in \rho(T+K)$. Hence, $[I + \mu(\lambda - T - K)^{-1}K]_a > 0$ for every $\mu \in [0,1]$. Now, appliying Theorem 2.19, we infer that $I + (\lambda - T - K)^{-1}K \in \Phi(X)$ and $i(I + (\lambda - T - K)^{-1}K) = 0$. Moreover, we have

$$\lambda - T = (\lambda - T - K)(I + (\lambda - T - K)^{-1}K).$$

Hence, by using [19, Theorem 5.7], we infer that $\lambda - T \in \Phi(X)$ and $i(\lambda - T) = 0$. One gets $\lambda - T \in B\mathcal{F}(X)$ and $i(\lambda - T) = 0$. So, $\lambda \notin \sigma_{BW}(T)$. Now, let $\lambda \in \sigma_{BW}(T)$, then $\lambda - T \notin B\mathcal{F}(X)$ or $i(\lambda - T) \neq 0$. Grounded on Corollary 2.14, we have $[\lambda - T]_a = 0$. Hence, $\lambda - T \notin \mathcal{W}_+(X)$. So, $\lambda \in \sigma_1(T)$. We get $\sigma_{BW}(T) \subset \sigma_r(T)$ and $\sigma_{BW}(T) \subset \sigma_1(T)$. Thus, $\sigma_{BW}(T) \subset \sigma_r(T) \bigcap \sigma_1(T)$. □

Corollary 3.6. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed in X and $\dim \frac{R(T^n)}{N(I-T)} < \infty$ or $codim(R(T) + N(T^n)) < \infty$, then $\sigma_{BW}(T) \subset \sigma_l(T)$.* ◊

Proof. By Theorems 3.2 and 3.5, we have $\sigma_{BW}(T) \subset \sigma_r(T)$ and $\sigma_r(T) = \sigma_l(T)$. Hence, $\sigma_{BW}(T) \subset \sigma_l(T)$. □

Corollary 3.7. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed in X and $\dim \frac{R(T^n)}{N(I-T)} < \infty$ or $codim(R(T) + N(T^n)) < \infty$, then $\sigma_{BW}(T) \subset \bigcap_{K \in \mathcal{E}(X)} \sigma(T+K)$, where $\mathcal{E}(X)$ is a subset of $\Upsilon_T(X)$ containing $\mathcal{K}(X)$.* ◊

Proof. Departing from Corollary 3.3 and Theorem 3.2, we have

$$\sigma_{e5}(T) = \bigcap_{K \in \mathcal{E}(X)} \sigma(T+K),$$

where $\mathcal{E}(X)$ is a subset of $\Upsilon_T(X)$ containing $\mathcal{K}(X)$ and $\sigma_{e5}(T) = \sigma_r(T)$. Then by Theorem 3.5, one gets

$$\sigma_{BW}(T) \subset \bigcap_{K \in \mathcal{E}(X)} \sigma(T+K). \quad \square$$

Corollary 3.8. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed in X and $\dim \frac{R(T^n)}{N(I-T)} < \infty$ or $codim(R(T) + N(T^n)) < \infty$, then $\sigma_{BW}(T) \subset \sigma_r(T) \bigcap \sigma_D(T)$.* ◊

Proof. According to Theorem 3.5, we infer that $\sigma_{BW}(T) \subset \sigma_r(T)$ and we obtain $\sigma_{BW}(T) \subset \sigma_D(T)$, which implies $\sigma_{BW}(T) \subset \sigma_r(T) \bigcap \sigma_D(T)$. □

Corollary 3.9. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed in X and $\dim \frac{R(T^n)}{N(I-T)} < \infty$ or $codim(R(T) + N(T^n)) < \infty$, then $\sigma_D(T) \subset \sigma_r(T) \bigcup \sigma_{g_k}(T)$.* ◊

Proof. Clearly

$$\sigma_D(T) = \sigma_{BW}(T) \bigcup \sigma_{g_k}(T)$$

and $\sigma_{BW}(T) \subset \sigma_r(T)$. We get $\sigma_D(T) \subset \sigma_r(T) \bigcup \sigma_{g_k}(T)$. □

Theorem 3.10. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If there exists $n \in \mathbb{N}$ such that:*

(i) $R(T^n)$ is closed in X, and
(ii) $\dim \frac{R(T^n)}{N(I-T)} < \infty$ or $codim(R(T) + N(T^n)) < \infty$.

Then,

$$\sigma_D(T) \subset [S(T) \bigcup S(T^*)] \bigcup [\sigma_{e1}(T) \bigcap \sigma_r(T)],$$

where $S(T)$ is the set of all $\lambda \in \mathbb{C}$ such that T does not have the single-valued extension property at λ. ◊

Proof. According to Theorem 3.5, we have $\sigma_{BF}(T) \subset \sigma_{e_G}(T)$. It's obvious that $\sigma_{BF}(T) \subset \sigma_{BW}(T)$. Therefore, we conclude that $\sigma_{BF}(T) \subset \sigma_{e_G}(T) \bigcap \sigma_{BW}(T)$. Thus,

$$\sigma_{BF}(T) \bigcup [S(T) \bigcup S(T^*)] \subset [S(T) \bigcup S(T^*)] \bigcup [\sigma_{e_G}(T) \bigcap \sigma_{BW}(T)].$$

Now, by using Theorem 3.5 and Theorem 3.4, we conclude that

$$\sigma_D(T) \subset [S(T) \bigcup S(T^*)] \bigcup [\sigma_{e_G}(T) \bigcap \sigma_r(T)]. \quad \square$$

Theorem 3.11. *Let X be a Banach space and $T, K \in \mathcal{L}(X)$. If $KT = TK$, then*

$$\sigma_{BW}(T) = \bigcap_{K \in \Upsilon_T(X)} \sigma_D(T + K). \quad \diamond$$

Proof. Let $\lambda \notin \bigcap_{K \in \Upsilon_T(X)} \sigma_D(T + K)$, then there exists $K \in \Upsilon_T(X)$ such that $\lambda - T - K$ is Drazin invertible. From Theorem 1.1, $\lambda - T - K$ is a B-Fredholm operator and $i(\lambda - T - K) = 0$. Hence, applying Theorem 2.19, we get $[I + (\lambda - T - K)^{-1}K] \in B\mathcal{F}(X)$ and $i[I + (\lambda - T - K)^{-1}K] = 0$. Moreover, we have

$$\lambda - T = (\lambda - T - K)[I + (\lambda - T - K)^{-1}K].$$

As $KT = TK$, then $K(\lambda - T - K) = (\lambda - T - K)K$, for all $\lambda \in \rho(T + K)$. Applying Theorem 1.2.5, we conclude that $\lambda \notin \sigma_{BW}(T)$. On the other side, we have $\mathcal{F}_0(X) \subset \Upsilon_T(X)$, where $\mathcal{F}_0(X)$ is the ideal of finite rank operators in the algebra $\mathcal{L}(X)$. Thus,

$$\bigcap_{K \in \Upsilon_T(X)} \sigma_D(T + K) \subset \sigma_{BW}(T). \quad \square$$

Corollary 3.12. *Let X be a Banach space and $T, K \in \mathcal{L}(X)$. Let $\mathcal{E}(X)$ be a subset of $\Upsilon_T(X)$ containing $\mathcal{F}_0(X)$ and assume that $KT = TK$. Then*

$$\sigma_{BW}(T) = \bigcap_{K \in \mathcal{E}(X)} \sigma_D(T + K). \quad \diamond$$

Proof. Since $\mathcal{E} \subset \Upsilon_T(X)$, one gets

$$\bigcap_{K \in \Upsilon_T(X)} \sigma_D(T + K) \subset \bigcap_{K \in \mathcal{E}(X)} \sigma_D(T + K).$$

Applying Theorem 3.11, we obtain

$$\sigma_{BW}(T) \subset \bigcap_{K \in \mathcal{E}(X)} \sigma_D(T + K).$$

On the other side, we have $\mathcal{F}_0(X) \subset \mathcal{E}(X)$. Hence,

$$\bigcap_{K \in \mathcal{E}(X)} \sigma_D(T + K) \subset \sigma_{BW}(T). \quad \square$$

Corollary 3.13. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Let $\mathcal{H}_T(X)$ be a subset of $\Upsilon_T(X)$ containing $\mathcal{F}_0(X)$. If for all $K, K' \in \mathcal{H}_T(X)$, $K \pm K' \in \mathcal{H}_T(X)$, and $KT = TK$. Then for every $K \in \mathcal{H}_T(X)$,*

$$\sigma_{BW}(T) = \sigma_{BW}(T + K). \quad \diamond$$

Proof. We let

$$\sigma'(T) = \bigcap_{K \in \mathcal{H}_T(X)} \sigma_D(T + K).$$

Applying Corollary 3.12, we obtain $\sigma_{BW}(T) = \sigma'(T)$. Since, for each $K \in \mathcal{H}_T(X)$, we have $\mathcal{H}_T(X) + K = \mathcal{H}_T(X)$, it follows that $\sigma'(T + K) = \sigma'(T)$. Therefore, $\sigma_{BW}(T) = \sigma_{BW}(T + K)$. $\square$

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