Filomat 38:24 (2024), 8623–8633
https://doi.org/10.2298/FIL2424623D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A new conformable fractional variational iteration method to singular perturbation conformable fractional Cauchy problems

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Abstract. In this paper, we first introduce the perturbation conformable fractional differential equations and present a new variational iteration method for solving and investigating their solutions. In the new variational iteration method for the perturbation conformable fractional differential equations, we obtain the exact solution of the equation in the first iteration or several subsequent iterations. Then, by presenting some applied examples and finding their solutions with the new method presented, and plotting the graph of the obtained solutions and comparing them with the corresponding ordinary differential equations, we discuss the efficiency and importance of this method.

1. Introduction

In recent years, fractional differential equations have made a lot of progress and different definitions for fractional derivative have been presented. One of the most appropriate definitions of fractional derivative is conformable fractional derivative [18], and some of the preliminary properties of this derivative have been studied and investigated [1]. In this work, we obtain our goals by choosing the conformable fractional derivative and focusing on the features it has with the ordinary derivative, and considering the very good works and results that have been done in modeling mathematical problems based on it [6, 15, 19, 27, 35].

The main advantages of fractional derivatives are flexibility and non-locality. Since these derivatives are of fractional order, they can approximate real data with more flexibility than classical derivatives. Furthermore, they also take into consideration non-locality, something that classical derivatives cannot do.

Fractional differential equations have received much attention from scientists due to their many applications in mathematical modeling of natural phenomena such as fluid mechanics, biology, chemistry, acoustics, control theory, rheology and viscoelasticity in various branches of applied sciences. Therefore, many scientists have proposed various methods to solve these problems [3, 4, 6, 8, 15–17, 22–24, 26, 31, 32]. Singular fractional perturbation initial value problems often arise in many scientific and engineering fields [5, 25], that characterized by a small parameter such as ε multiplying the derivatives, and there exist initial boundary layers where the solutions change.

The variational iteration method (VIM) or the Lagrange multiplier techniqueis is one of the well-known semianalytical methods and widely used to solve for linear, nonlinear, homogeneous, and inhomogeneous ordinary as well as partial differential equations. This method was first proposed by He [10, 13].

2020 *Mathematics Subject Classification*. Primary 34D15; Secondary 34A08, 97M10.
Keywords. Conformable fractional derivative, Fractional perturbation, Variational iteration method.
Received: 15 July 2023; Revised: 18 February 2024; Accepted: 14 May 2024
Communicated by Miodrag Spalević
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Advantage of the method lies in its flexibility and ability to solve nonlinear equations easily and so by this method we can find the convergent successive approximations of the exact solution of the problem if such a solution exists. In [21], Salkuyeh has studied the convergence of VIM for solving linear system of ordinary differential equations with constant coefficients. In [33], Zhao and Xiao studied the singular variational initial value problems by VIM. In [30],Wu solved approximately two fractional differential equations with the fractional variational iteration method.

Omer Acan and et al, introduced conformable variational iteration method to solve two simple conformable fractional differential equations in [2]. The key feature of the presented method is that we obtain the exact solution of the equation in the first approximation.

In this paper we will investigate two cases of linear and non-linear singular perturbation Cauchy problems with conformable fractional derivative as following

$$\begin{cases} T_\alpha(x)(t) = f(t, x(t), y(t)), \\ \varepsilon T_\alpha(y)(t) = g(t, x(t), y(t)), \\ x(0) = x_0, y(0) = y_0, \end{cases} \tag{1}$$

and

$$\begin{cases} T_\alpha(x)(t) = Ax(t) + F(t, x(t), y(t)) = f(t, x(t), y(t)), \\ \varepsilon T_\alpha(y)(t) = By(t) + G(t, x(t), y(t)) = g(t, x(t), y(t)), \\ x(0) = x_0, y(0) = y_0, \end{cases} \tag{2}$$

where $t \in [0, T]$, $0 < \varepsilon < 1$ and $\alpha \in (0, 1]$. At first, we need definitions of the conformable fractional derivative and some of its properties, which we will mention in the next section.

2. Preliminaries

In this section, we first introduce conformable fractional derivative definition and some important properties of conformable fractional calculus.

Definition 2.1. *[18] Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. Then the conformable fractional derivative of f of order α, $0 < \alpha \leq 1$ is defined by*

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{3}$$

for all $t > 0$.

Every real function that is satisfied in 3, and the corresponding limit that exists, is called the α-differentiable function. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}$ exists, then define

$$T_\alpha(f)(0) = \lim_{t \to 0^+} T_\alpha(f)(t).$$

The relationship between the conformable derivative and the first derivative can be represented

$$T_\alpha(f)(t) = t^{1-\alpha} f'(t). \tag{4}$$

We can see that T_α satisfies in the following theorem.

Theorem 2.2. *[18] Let $0 < \alpha \leq 1$ and f, g be α-differentiable at a point $t > 0$. Then we have*

1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all $a, b \in \mathbb{R}$.*
2) $T_\alpha(t^p) = pt^{p-\alpha}$, *for all $p \in \mathbb{R}$.*
3) $T_\alpha(c) = 0$, *for all constant functions $f(t) = c$.*
4) $T_\alpha(fg) = fT_\alpha(g) + bT_\alpha(f)$.
5) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) + fT_\alpha(g)}{g^2}$.

Now we define the α-fractional integral of a function f.

Definition 2.3. *[18] Let $0 < \alpha \leq 1$ and $f : [a, b] \longrightarrow \mathbb{R}$ be continous function, then*

$$I_\alpha(f)(t) = \int_0^t x^{1-\alpha} f(x)dx, \tag{5}$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

3. Conformable fractional variational itertation method

In this section we will introduce new conformable fractional variational itertation method (CFVIM) for linear and non-linear conformable fractional differential equation system with a small parameter. For this purpose, we write the non-linear conformable fractional differential equation in the operator form as follows

$$T_\alpha u(t) + L(u(t)) + N(u(t)) = f(t), \tag{6}$$

where L is a linear operator, N is a non-linear operator, f is a non-homogeneous part and T_α is conformable fractional derivative of order α that $\alpha \in (0, 1]$. Now by the use of (4) we write the relation (6) in the following form

$$t^{1-\alpha} \frac{d}{dt} u(t) + L(u(t)) + N(u(t)) = f(t). \tag{7}$$

According to variational iteration method (VIM), we can construct the modified correction functional as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t s^{\alpha-1} \lambda(s)[T_\alpha u_n(s) + L(u_n(s)) + N(\tilde{u}_n(s)) - f(s)]ds, \tag{8}$$

or

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)[\frac{d}{ds} u_n(s) + s^{\alpha-1}(L(u_n(s)) + N(\tilde{u}_n(s)) - f(s))]ds. \tag{9}$$

To the analysis of the method we know, the varaiational iteration method chenges the fractional differential equation to a recurrence sequnce of functions and the limit of sequence is considered as the solution of the problem. Obviously the successive approximations $u_n, n \geq 0$, can be stablished by determining λ, that is general Lagrange's multipilier, which can be identifed optimally via the varaiational theory. Function $\tilde{u}_n$ denote the restrictive varaiation, i.e., $\delta\tilde{u}_n = 0$. In fact the solution of the (6) is considered as the fixed point of the following functional by starting with initial condidtion $u_0(t)$.

Theorem 3.1. *(Banach's fixed point theorem)[28] Assume that X is a Banach space and $A : X \longrightarrow X$ is a nonlinear mapping, and suppose that*

$$\| A[u] - A[v] \| \leq \gamma \| u - v \|, \qquad u, v \in X,$$

for some constant $\gamma < 1$. Then A has a unique fixed point, or the sequence

$$u_{n+1} = A[u_n],$$

with an arbitary choice of $u_0 \in X$, convergence to the fixed point of A, and also

$$\| u_m - u_n \| \leq \| u_1 - u_0 \| \sum_{i=n-1}^{m-2} \gamma^i$$

.

Now for the nonlinear mappling

$$A[u] = u(t) + \int_0^t \lambda(s)[\frac{d}{ds}u_n(s) + s^{\alpha-1}(L(u_n(s)) + N(u_n(s)) - f(s)))]ds$$

a sufficient condition for convergence of the VIM is contraction of A, and also the the sequence (9) converges to the fixed point f A which also is the solution of the (6).

Now we first determine the Lagrangian multiplier $\lambda(s)$ that will be identified optimally via integration by parts. The successive approximations $u_{j+1}, j \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(t)$. So we will use the initial value for selecting the zeroth approximation $u_0(t)$. Hence the exact solution may be obtained by using $u(t) = \lim_{n\longrightarrow\infty} u_n(t)$.

4. Illustrative examles

In this section, we give some examples to illustrate the effect of the proposed method.

Example 4.1. *We consider the singular conformable fractional equation as follows*

$$\varepsilon T_\alpha(y)(t) + y(t) = 0, \quad y(0) = 1, \tag{10}$$

that $0 < \varepsilon < 1$ *and* $\alpha \in (0, 1]$.

The exact solution of this problem is $y(t) = \exp(-\frac{t^\alpha}{\alpha\varepsilon})$. *By the using of the CFVIM in the previous section, we construct the following correction functionals*

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s)s^{\alpha-1}[T_\alpha(y)(s) + \frac{1}{\varepsilon}y(s)]ds. \tag{11}$$

To finde the optimal value of λ, *we have*

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_0^t \lambda(s)s^{\alpha-1}[s^{1-\alpha}\frac{d}{ds}y_n(s) + \frac{1}{\varepsilon}y_n(s)]ds. \tag{12}$$

So the general Lagrang multipliers can be readily identified by

$$\lambda(s,t)|_{s=t} = -\exp(-\frac{(s-t)^\alpha}{\alpha\varepsilon}). \tag{13}$$

As a result we obtain the following iteration formula

$$y_{n+1}(t) = y_n(t) - \int_0^t \exp(-\frac{(s-t)^\alpha}{\alpha\varepsilon})[\frac{d}{ds}y_n(s) + \frac{s^{\alpha-1}}{\varepsilon}y_n(s)]ds. \tag{14}$$

To get iterate sequence by the use of $y_0(t) = y(0) = 1$ *as an approximate solution and means the formula (14), we have*

$$y_1(t) = y_0 - \int_0^t \exp(-\frac{(s-t)^\alpha}{\alpha\varepsilon})[\frac{d}{ds}y_0(s) + \frac{s^{\alpha-1}}{\varepsilon}y_0(s)]ds = \exp(-\frac{t^\alpha}{\alpha\varepsilon}).$$

Which is the exact solution of (10). *Thus, in the first approximation, we could obtain the exact solution of the problem. Approximate solutions are plotted in figure (1).*

Example 4.2. *We consider the singular conformable fractional equation as follows*

$$\varepsilon T_\alpha(y)(t) = (t-1)y(t), \quad y(0) = 0.2, \tag{15}$$

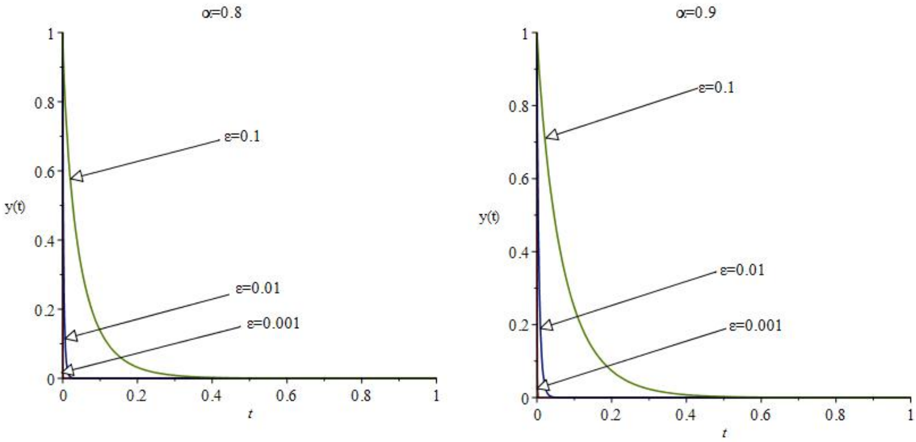

Figure 1: Results of $y(t)$ for Example 4.1 for $\alpha = 0.8$ and $\alpha = 0.9$ and different values of ε.

that $0 < \varepsilon < 1$ *and* $\alpha \in (0, 1]$. *One can see that the exact solution of this problem is* $y(t) = 0.2\exp(-\frac{1}{\varepsilon}(\frac{t^\alpha}{\alpha} - \frac{t^{\alpha-1}}{\alpha-1}))$. *By the using of new variational iteration method in the previous section, we construct the following correction functionals*

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s)[T_\alpha(y)(s) - \frac{s-1}{\varepsilon}y(s)]ds. \tag{16}$$

To finde the optimal value of λ, *we have*

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta \int_0^t \lambda(s)s^{\alpha-1}[s^{1-\alpha}\frac{d}{ds}y_n(s) - \frac{s-1}{\varepsilon}y_n(s)]ds. \tag{17}$$

So the general Lagrang multipliers can be readily identified by

$$\lambda(s,t)|_{s=t} = -\exp(-\frac{1}{\varepsilon}(\frac{(s-t)^\alpha}{\alpha} - \frac{(s-t)^{\alpha+1}}{\alpha+1})). \tag{18}$$

As a result we obtain the following iteration formula

$$y_{n+1}(t) = y_n(t) - \int_0^t \exp(-\frac{1}{\varepsilon}(\frac{(s-t)^\alpha}{\alpha} - \frac{(s-t)^{\alpha+1}}{\alpha+1})[\frac{d}{ds}y_n(s) - \frac{s-1}{\varepsilon}y_n(s)]ds. \tag{19}$$

To get iterate sequence by the use of $y_0(t) = y(0) = 0.2$ *as an approximate solution and means the formula (19), we have* $y(t) = 0.2\exp(-\frac{1}{\varepsilon}(\frac{t^\alpha}{\alpha} - \frac{t^{\alpha+1}}{\alpha+1}))$, *which is the exact solution of (15). For* $\alpha = 1$, *we have* $y(t) = 0.2\exp(-\frac{t}{\varepsilon}(1 - \frac{t^2}{2}))$ *[33]. As in the previous example, in the first approximation, we were able to obtain the exact solution of the equation, which shows the efficiency and accuracy of this method.*

Example 4.3. *Consider the system of the two linear singular fractional differential equations with a small parameter*

$$\begin{cases} T_\alpha(x)(t) = x(t) + y(t), & x(0) = 1, \\ \varepsilon T_\alpha(y)(t) = 2x(t) - y(t), & y(0) = 3, \end{cases} \tag{20}$$

that $0 < \varepsilon < 1$ *and* $\alpha \in (0, 1]$. *Now we can use the CFVIM in the previous section and construct the following correction functionals*

$$x_{n+1}(t) = x_n(t) + \int_0^t s^{\alpha-1}\lambda_1(s)[s^{1-\alpha}\frac{d}{ds}x_n(s) - y_n(s) - x_n(s)]ds, \tag{21}$$

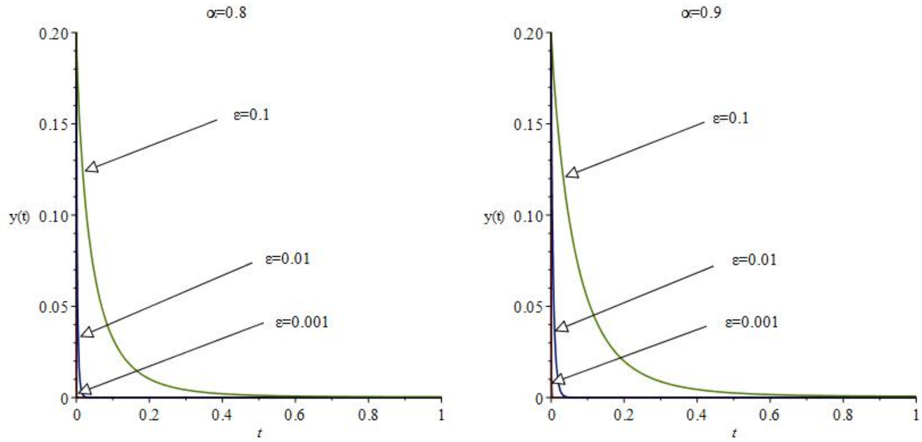

Figure 2: Results of $y(t)$ for Example 4.2 for $\alpha = 0.8$ and $\alpha = 0.9$ and different values of ε.

and

$$y_{n+1}(t) = y_n(t) + \int_0^t s^{\alpha-1}\lambda_2(s)[s^{1-\alpha}\frac{d}{ds}y_n(s) - \frac{1}{\varepsilon}(2x_n(s) + y_n(s))]ds. \quad (22)$$

To find the optimal value of Lagrang multipliers λ*, we have*

$$\delta x_{n+1}(t) = \delta x_n(t) + \delta\int_0^t s^{\alpha-1}\lambda_1(s)[s^{1-\alpha}\frac{d}{ds}x_n(s) - y_n(s) - x_n(s)]ds, \quad (23)$$

and

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta\int_0^t s^{\alpha-1}\lambda_2(s)[s^{1-\alpha}\frac{d}{ds}y_n(s) - \frac{1}{\varepsilon}(2x_n(s) + y_n(s))]ds. \quad (24)$$

So the stationary conditions are obtained as

$$\lambda_1(s)|_{s=t} = -\exp(-\frac{(s-t)^\alpha}{\alpha}), \quad \lambda_2(s)|_{s=t} = -\exp(-\frac{(s-t)^\alpha}{\alpha\varepsilon}). \quad (25)$$

As a result we obtain the following iteration formulas

$$x_{n+1}(t) = x_n(t) - \int_0^t s^{\alpha-1}\exp(-\frac{(s-t)^\alpha}{\alpha})[s^{1-\alpha}\frac{d}{ds}x_n(s) - y_n(s) - x_n(s)]ds, \quad (26)$$

and

$$y_{n+1}(t) = y_n(t) - \int_0^t s^{\alpha-1}\exp(-\frac{(s-t)^\alpha}{\alpha\varepsilon})[\varepsilon s^{1-\alpha}\frac{d}{ds}y_n(s) - \frac{1}{\varepsilon}(2x_n(s) + y_n(s))]ds. \quad (27)$$

Here we use x_2 *and* y_2 *as the approximate solution as follows*

$$\begin{cases} x_0 = 1, \\ x_1 = 5 - 4\exp(-\frac{t^\alpha}{\alpha}), \\ x_2 = 8 - 7\exp(-\frac{t^\alpha}{\alpha}) - \frac{\varepsilon}{1+\varepsilon}\exp(-\frac{(1+\varepsilon)t^\alpha}{\alpha\varepsilon}) + \frac{\varepsilon}{1+\varepsilon}, \\ \vdots \end{cases} \quad (28)$$

and

$$\begin{cases} y_0 = 3, \\ y_1 = 2 + \exp(-\frac{t^\alpha}{\alpha\varepsilon}), \\ y_2 = 10 + \exp(-\frac{t^\alpha}{\alpha\varepsilon}) - 8\exp(-\frac{t^\alpha}{\alpha}) + \frac{8}{1+\varepsilon}\exp(-\frac{(1+\varepsilon)t^\alpha}{\alpha\varepsilon}) - \frac{8}{1+\varepsilon}, \\ \vdots \end{cases} \tag{29}$$

The approximate solutions are plotted in Figure 3. This figures shows that the method gives a good approximation of the solution.

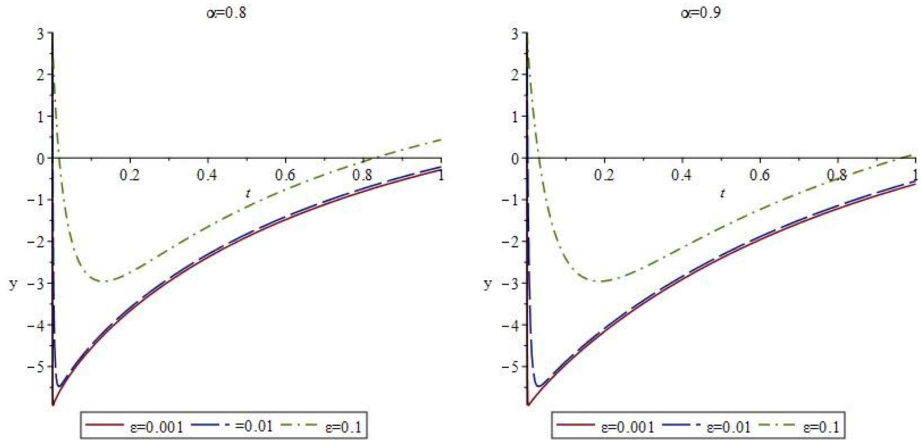

Figure 3: Results of $y(t)$ for example (4.3) for different values of ε and α.

Example 4.4. *Consider the system of non-linear singular conformable fractional differential equations from enzyme kinetics*

$$\begin{cases} T_\alpha(x)(t) = -x(t) - (x(t) + \kappa + \mu)y(t), & x(0) = 1, \\ \varepsilon T_\alpha(y)(t) = x(t) - (x(t) + \kappa)y(t), & y(0) = 0, \end{cases} \tag{30}$$

where κ, μ *are bounded non-negative constants which satisfy* $\kappa < \mu$ *with* $0 < \varepsilon < 1$ *and* $\alpha \in (0, 1]$. *We construct the following correction functionals by the using the MCFVIM in the previous section, so we have*

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda_1(s)s^{\alpha-1}[s^{1-\alpha}\frac{d}{ds}x_n(s) - x_n(s) - (x_n(s) + \kappa + \mu)y_n(s)]ds, \tag{31}$$

and

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda_2(s)s^{\alpha-1}[s^{1-\alpha}\frac{d}{ds}y_n(s) - \frac{1}{\varepsilon}(x_n(s) - (x_n(s) + \kappa)y_n(s))]ds. \tag{32}$$

To find the optimal value of Lagrang multipliers λ, *we have*

$$\delta x_{n+1}(t) = \delta x_n(t) + \delta\int_0^t [\lambda_1(s)\frac{d}{ds}x_n(s) - \lambda_1(s)s^{\alpha-1}(x_n(s) + (x_n(s) + \kappa + \mu)y_n(s))]ds, \tag{33}$$

and

$$\delta y_{n+1}(t) = \delta y_n(t) + \delta\int_0^t [\lambda_2(s)\frac{d}{ds}y_n(s) - \lambda_2(s)s^{\alpha-1}\frac{1}{\varepsilon}(x_n(s) - (x_n(s) + \kappa)y_n(s))]ds. \tag{34}$$

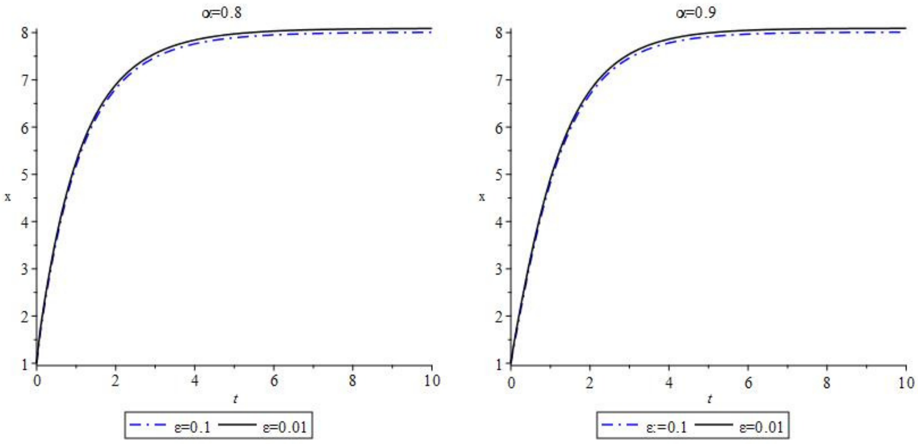

Figure 4: Results of $x(t)$ for example (4.4) for different values of ε and α.

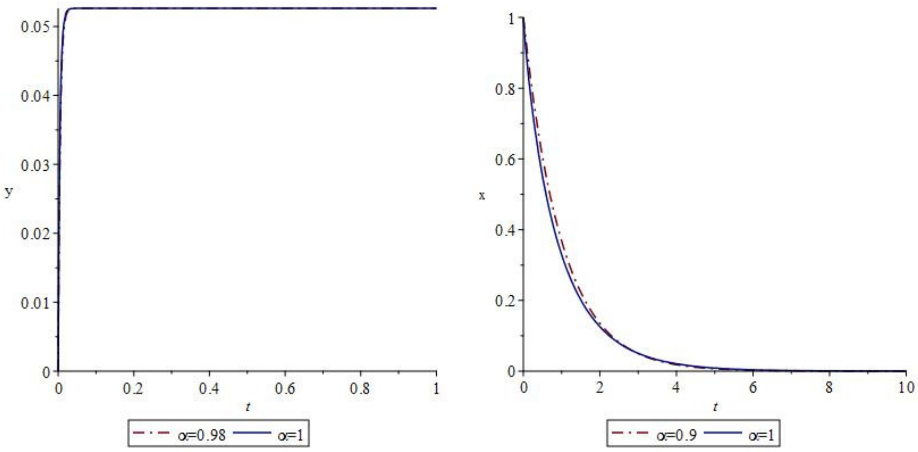

Figure 5: Results of $y(t)$ and $x(t)$ for example (4.4) for different values of ε and α.

Now we obtain the stationary conditions as follows

$$\lambda_1(s)|_{s=t} = -\exp(-\frac{(s-t)^\alpha}{\alpha}), \quad \lambda_2(s)|_{s=t} = -\exp(-\frac{\kappa(s-t)^\alpha}{\alpha\varepsilon}). \tag{35}$$

So we construct the following correction functionals

$$x_{n+1}(t) = x_n(t) - \int_0^t [e^{(-\frac{(s-t)^\alpha}{\alpha})}\frac{d}{ds}x_n(s) - e^{(-\frac{(s-t)^\alpha}{\alpha})}s^{\alpha-1}(x_n(s) + (x_n(s) + \kappa + \mu)y_n(s))]ds, \tag{36}$$

and

$$y_{n+1}(t) = y_n(t) + \int_0^t [e^{(-\frac{\kappa(s-t)^\alpha}{\alpha\varepsilon})}\frac{d}{ds}y_n(s) - e^{(-\frac{\kappa(s-t)^\alpha}{\alpha\varepsilon})}s^{\alpha-1}\frac{1}{\varepsilon}(x_n(s) - (x_n(s) + \kappa)y_n(s))]ds. \tag{37}$$

To get the conformable variational iterate sequences, we start with initial approximations $x_0(t) = x(0) = 1$, $y_0(t) =$

$y(0) = 0$, and $\varepsilon = 0.1$. For $\kappa = 19$ and $\mu = 20$ we have

$$\begin{cases} x_0 = 1, \\ x_1 = e^{-\frac{t^\alpha}{\alpha}}, \\ \vdots \end{cases} \tag{38}$$

and

$$\begin{cases} y_0 = 0, \\ y_1 = \frac{1}{19}(1 - e^{(-\frac{190t^\alpha}{\alpha})}), \\ \vdots \end{cases} \tag{39}$$

Here we used x_1 and y_1 as the approximate solutions of the system (30) and the approximate solutions are plotted in Figure 4 and Figure 5.

It is not without grace to state this point in the following that the described method can be used well for non-singular linear and non-linear systems of conformable partial differential equations(CPDEs) as well. In the following, we will show by investigating some problems that this new conformable variational iteration method can be used well for linear and non-linear CPDEs and has high accuracy. we assume that T_t^α represents the conformable fractional derivative operator with respect to time.

Example 4.5. *We consider the linear system*

$$\begin{cases} T_t^\alpha u - v_x - (u - v) = -2, & u(x,0) = 1 + e^x, \\ T_t^\alpha v + u_x - (u - v) = -2, & v(x,0) = -1 + e^x. \end{cases} \tag{40}$$

By the use of (4), *we obtain*

$$\begin{cases} t^{1-\alpha}u' - v_x - (u - v) = -2, & u(x,0) = 1 + e^x, \\ t^{1-\alpha}v' + u_x - (u - v) = -2, & v(x,0) = -1 + e^x. \end{cases} \tag{41}$$

As a result of new conformable variational iteration method we obtain the following iteration formulas

$$\begin{cases} u_{n+1} = u_n - \int_0^t s^{\alpha-1} e^{\frac{(s-t)^\alpha}{\alpha}} (s^{1-\alpha}u_{ns} - v_{nx} - u_n + v_n + 2)ds \\ v_{n+1} = u_n - \int_0^t s^{\alpha-1} e^{\frac{-(s-t)^\alpha}{\alpha}} (s^{1-\alpha}v_{ns} + u_{nx} - u_n + v_n + 2)ds, \end{cases} \tag{42}$$

we start with initial approximations $u_0 = 1 + e^x$ and $v_0 = -1 + e^x$. Now, using (42), *we obtain the first approximation of the system (40) as follows $u_1(x,t) = 1 + e^{x+\frac{t^\alpha}{\alpha}}$ and $v_1(x,t) = -1 + e^{x-\frac{t^\alpha}{\alpha}}$. These solutions are the exact solutions of the system (40) that are obtained in the first approximation of conformable variational iteration method [29].*

Example 4.6. *We consider the non-linear system*

$$\begin{cases} T_t^\alpha u + vu_x + u = 1, & u(x,0) = e^x, \\ T_t^\alpha v - uv_x - v = 1, & v(x,0) = e^{-x}. \end{cases} \tag{43}$$

By using (4), *we have*

$$\begin{cases} t^{1-\alpha}u' + vu_x + u = 1, & u(x,0) = e^x, \\ t^{1-\alpha}v' - uv_x - v = 1, & v(x,0) = e^{-x}. \end{cases} \tag{44}$$

By the use of new conformable variational iteration method we have the following iteration formulas

$$\begin{cases} u_{n+1} = u_n - \int_0^t s^{\alpha-1} e^{\frac{-(s-t)^\alpha}{\alpha}} (s^{1-\alpha} u_{ns} - v_n u_{nx} - u_n - 1) ds \\ v_{n+1} = u_n - \int_0^t s^{\alpha-1} e^{\frac{(s-t)^\alpha}{\alpha}} (s^{1-\alpha} v_{ns} + u_n v_{nx} - v_n - 1) ds, \end{cases} \tag{45}$$

using initial approximations $u_0 = e^x$ and $v_0 = e^{-x}$. Now, using (45), we obtain the first approximation of the system (43) as follows $u_1(x,t) = e^{x-\frac{t^\alpha}{\alpha}}$ and $v_1(x,t) = e^{x+\frac{t^\alpha}{\alpha}}$. The u_1 and v_1 are the exact solutions of the system (43) that are obtained in the first approximation of new method [29].

So we were able to obtain the solutions of linear and non-linrear non-singular conformable fractional PDEs system in the first approximation with the presented method.

It is possible to compare the solutions of the proposed method provided to obtain the exact solutions of these equations with the differential transform method (DTM),variational iteration method (VIM), etc. In [29], the obtained approximations converge to the exact solutions, while in the method presented, in the first approximation, we obtain the exact solution of the equations, which showed the accuracy and speed of obtaining the solutions of the assumed equations.

5. Discussion

So far, several analytical-approximate methods have been presented to solve singular and non-singular fractional differential equations. One of these methods is the method of variational iteration method, which is well used to solve the above problems. But in order to find the solutions of singular and non-singular conformable fractional differential equations, no suitable variational iteration method was provided, but in this paper, by obtaining a new variational iteration method for this type of equations, we were able to get an accurate and better solution than the differential transformation method(DTM), the homotopy analysis method(HAM), and other numerical methods provided. The importance of the presented method compared to these methods is to obtain the exact solution of the equation quickly and accurately. To see this, refer to the [2, 21, 29, 32–34].

6. Conclusion

We have considered two cases of linear and non-linear singular perturbation Cauchy problems with conformable fractional derivative. We developed a new variational itertation method with useful and effective solutions for presented problems. We implement this new variational itertation method to solve the some applicable examples like enzyme kinetics. We have shown the successful implementation of this method to solve the linear and non-linear singular perturbation conformable fractional systems. The behavior of solutions in the examples with the variation in α and ε is also shown by figures.

7. Acknowledgements

The authors would like to thank the referee for carefully reading the paper. His/her valuable comments and suggestions made many useful improvements throughout the paper.

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