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On Meir-Keeler condensing operators and solvability of a system of integral equations in the Banach space $BC(\mathbb{R}_+)$

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Abstract. In this paper, we establish the generalization of Meir-Keeler condensing operators using the concept of $L-$functions in Banach spaces. We prove some coupled fixed point theorems, and in application we use the obtained results to study the existence of solution of a coupled system of functional integral equations in Banach space $BC(\mathbb{R}_+)$.

1. Introduction and Preliminaries

Fixed point argument is very useful in the study of existence of solutions of functional equations, which has captured the attention of numerous researchers over an extensive period of time. Several approaches have been used to establish the existence of solutions of functional equations such as functional integral equations, differential equations, integro-differential equations, and fractional integro-differential equations. For some of the recent studies on fixed point theorems and application to functional equatons one can refer [10, 28, 29]. Due to the fact that fixed point methods requires for the conditions of compactness and the Lipstchitz condition to be satisfied, researchers came up with another technique which is effective for non-compact operators, that is the technique of measures of non-compactness. Measure of non-compactness is the function that determines the degree of non-compactness of a bouded set. The concept was firstly introduced in 1930 by Kuratowski [18]. It was Darbo [11] who generalized the classical Schauder's fixed point theorem and the Banach's contraction principle using the concept of Kuratowki measure of non-compactness. His theorem became very famous in studying the existence of solution of functional equations. Several researchers generalized Darbo's fixed point theorem and applied it to study the existence of solutions of differential and integral equations. Another important measure of non-compactness is the Hausdorff or ball measure of non-compactness introduced by Goldenstein *et al.* (1957) [12]. Following the work done by Darbo, many other researchers came up with generalizations of his theorem. In 1969, Meir and Keeler [17]

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introduced the concept of Meir-Keeler contraction in metric space. Later on Aghajani *et al.* [1] applied this concept to Banach spaces and came up with the so called Meir-Keeler condensing operators, and proved several fixed point theorems which are useful in studying the existence of solution for functional integral equations, in the same work of Aghajani *et al.* [1], the concept of L-functions was generalized and some fixed point theorems was proved. It was Lim [19] and Suzuki [26] who studied L-functions and proved fixed point theorems for Meir-Keeler contractive maps, their work act as a bench mark for the work done by Aghajani *et al.* [1].

Wairojjana *et al.* [27] extended the work done by Aghajani *et al.*[1] and proved fixed point theorems for Meir-Keeler condensing operators in partially ordered Banach spaces. Furthermore, Matani and Rezaei [20] proved fixed point theorems for Multivariate generalized Meir-Keeler condensing operators and used it to study the existence of solution for a system of integral equations of Volterra type in three variables.

The measures of non-compactness are convenient in the study of single and multi-valued fixed point theory, we refer [1, 4, 15, 20], for some studies which involve multi-valued fixed point theorems. Together with some algebraic considerations measures of non-compactness are useful in examining the existence of solutions to certain problems under specific conditions.

Recently, several authors [16, 21–25] studied the problems of existence of solutions of differential equations, fractional differential equations and integral equations in various spaces by using the techniques of measures of noncompactness.

In this paper, we extend and generalize the work done by Aghajani *et al.*[1]. We prove some coupled fixed point theorems and a theorem for the existence of solution for a coupled system of functional integral equations in Banach space $BC(\mathbb{R}_+)$.

Consider $\mathbb{R}_+=[0,\infty)$, suppose that $(M, \|.\|)$ is a real Banach space, if X is a non empty subset of M then by $\bar{X}$ and ConvX, we denote the closure and convex closure of X respectively. Let Q_M denote the family of all non empty and bounded subsets of M and N_M denote its subfamily consisting of all relatively compact sets.

Definition 1.1. *[8] A function $\lambda : Q_M \to \mathbb{R}_+$ is called a measure of non-compactness if it satisfies the following conditions:*

i. the family $\ker \lambda = Y \in Q_M : \lambda(Y) = 0$ is nonempty and $\ker \lambda \subset N_M$,
ii. $Y \subset Z \implies \lambda(Y) \leq \lambda(Z)$,
iii. $\lambda(\bar{Y}) = \lambda(Y)$,
iv. $\lambda(\mathrm{Conv}Y) = \lambda(Y)$,
v. $\lambda(kY + (1-k)Z) \leq k\lambda(Y) + (1-k)\lambda(Z)$ for $k \in [0, 1]$,
vi. if (Y_n) is a sequence of closed sets from Q_M such that $Y_{n+1} \subset Y_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} \lambda(Y_n)=0$ then $\bigcap_{n=1}^{\infty} Y_n \neq \emptyset$.

If a measure of non-compactness satisfies the following additional conditions, then it is called a regular measure.

vii. $\lambda(Y_1 \cup Y_2) = \max\{\lambda(Y_1), \lambda(Y_2)\}$,
viii. $\lambda(Y_1 + Y_2) \leq \lambda(Y_1) + \lambda(Y_2)$,
ix. $\lambda(kY) = |k|\lambda(Y)$,
x. $\ker \lambda = N_M$.

The family *ker*λ is said to be the *kernel* of measure λ.

Darbo [11], in his work introduced the following definition of condensing operators and proved a very famous Darbo's fixed point theorem.

Definition 1.2. *[5] Let M_1 and M_2 be two Banach spaces and let λ_1 and λ_2 be arbitrary measures of non-compactness on M_1 and M_2 respectively. An operator f from M_1 to M_2 is called a (λ_1, λ_2)-condensing operator if it is continuous and $\lambda_2(f(D)) < \lambda_1(D)$ for every set $D \in M_1$ with compact closure.*

Remark : if $M_1 = M_2$ and $\lambda_1 = \lambda_2 = \lambda$, then f is called a λ-condensing operator.

Theorem 1.3. *[11] Let H be a nonempty, closed, bounded and convex subset of a Banach space M and* $f : H \to H$ *be a continuous mapping such that there exists a constant* $k \in [0, 1)$ *with the property* $\lambda_2(f(H)) < k\lambda_1(H)$. *Then f has a fixed point in H.*

In 1969 [17], Meir and Keeler generalized the Banach contraction principle and proved an interesting fixed point theorem. Again the results of Meir and Keeler's work were generalized to Banach space by Aghajan *et al.* [1] where they proved some fixed points theorems which guarantees the existence of solution for functional equations.

Definition 1.4. *[1] Let C be a nonempty subset of a Banach space M and let* λ *be an arbitrary measure of non-compactness on M. We say that an operator* $T : C \to C$ *is a Meir-Keeler condensing operator if for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\epsilon \leq \lambda(X) < \epsilon + \delta \implies \lambda(T(X)) < \epsilon$$

for any bounded subset X of C.

Theorem 1.5. *[1] Let C be a nonempty subset of a Banach space M and let* λ *be an arbitrary measure of non-compactness on M. If* $T : C \to C$ *is a continuous and Meir-Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in C is compact.*

Lim [19] introduced the notion of $L-$functions and characterized Meir-Keeler contractions in metric spaces

Definition 1.6. *[19] A function* ϕ *from* $\mathbb{R}_+$ *into itself is called an L-function if* $\phi(0) = 0$, $\phi(s) > 0$ *for* $s \in (0, \infty)$, *and for every* $s \in (0, \infty)$ *there exists* $\delta > 0$ *such that* $\phi(t) \leq s$, *for all* $t \in [s, s + \delta]$.

Example 1.7. *[1] If we define* $\phi(t) = kt$, *for* $0 < k \leq 1$, *then* ϕ *is an* $L - function$.

Definition 1.8. *[19] We say that* $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ *is a strictly L-function if* $\theta(0) = 0, \theta(s) > 0$ *for* $s \in (0, \infty)$, *and for every* $s \in (0, \infty)$ *there exists* $\delta > 0$ *such that* $\theta(t) < s$, *for all* $t \in [s, s + \delta]$.

Example 1.9. *The function* $\theta(b) = \ln(1 + kb)$ *where* $0 < k \leq 1$ *is a strict* $L - function$.

Using $L - functions$ Aghajani *et al.* [1, 2] proved several fixed point theorems, these theorems will serve as guidelines for our main results

Theorem 1.10. *[1] Let C be a nonempty and bounded subset of a Banach space M,* λ *be an arbitrary measure of non-compactness on M and* $T : C \to C$ *be a continuous operator. Then T is a Meir-Keeler condensing operator if and only if there exists an L-function* ϕ *such that*

$$\lambda(T(X)) < \phi(\lambda(X)),$$

for all $X \in Q_M$ *with* $\lambda(X) \neq 0$.

Theorem 1.11. *[1] Let C be a nonempty, bounded, closed and convex subset of a Banach space M and let* $T : C \to C$ *be a continuous operator such that*

$$\lambda(T(X)) \leq \theta(\lambda(X)),$$

for each $X \subset C$ *where* λ *is an arbitrary measure of non-compactness and* θ *is a strictly L-function. Then T has at least one fixed point and the set of all fixed points of T in C is compact.*

Aghajani *et al.* [1] obtained the following results for the compact operators.

Definition 1.12. *Let M be the Banach space. An operator F is said to be compact if the closure of* $F(Y)$ *is compact whenever* $Y \subset M$ *is bounded.*

From Theorem 1.11 we consider the following Corollary.

Corollary 1.13. *[1] Suppose that E is a nonempty, bounded, closed, and convex subset of a Banach space M and let* $F : E \to M$ *be an operator such that* $\|Fx - Fy\| \leq \theta(\|x - y\|)$, *where* θ *is a nondecreasing and right continuous strictly L-function. Assume that* $G : E \to M$ *is compact continuous operator, we define* $T(x) = F(x) + G(x)$ *and assume that* $T(x) \in E$ *for* $x \in E$. *Then, T has fixed point in E and the set of all fixed points of T in E is compact.*

2. Coupled fixed point theorem

Definition 2.1. *[9] Let X be a subset of a Banach space M, an element $(x, y) \in X \times X$ is called coupled fixed point of a mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.*

Example 2.2. *A map defined by $T(x, y) = x^2 + y^2$ has a unique coupled fixed point $(0, 0)$.*

Theorem 2.3. *[7] Suppose λ_1, $\lambda_2 ... \lambda_n$ are the measures of non-compactness in Banach spaces $M_1, M_2, ...M_n$, respectively. Moreover assume that $F : [0, \infty)^n \to [0, \infty)$ is convex and $F(x_1, x_2, ..., x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, ..., n$, then $\tilde{\lambda}(X) = F(\lambda_1(X_1), \lambda_2(X_2), ..., \lambda_n(X_n))$ defines a measure of non-compactness in $M_1, M_2, ..., M_n$, where X_i denotes the natural projection of X into M_i, for $i = 1, 2, ..., n$.*

Example 2.4. *[3] let λ be a measure of non-compactness on a Banach space M, then if we take $F_1(x, y) = x + y$ and $F_2(x, y) = max\{x, y\}$ for $(x, y) \in \mathbb{R}^2_+$, conditions of Theorem 2.3 are satisfied therefore*

$$\tilde{\lambda}_1 = \lambda(X_1) + \lambda(X_2)$$

and

$$\tilde{\lambda}_2 = max\{\lambda(X_1), \lambda(X_2)\}$$

defines measures of non-compactness in the space $M \times M$, where $X_i = 1, 2$ denote the natural projections of X.

Definition 2.5. *Let E be a nonempty, bounded, closed, and convex subset of a Banach space M and λ be an arbitrary measure of non-compactness. An opertor $T : E \times E \to E$ is a Meir-Keeler condensing operator if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq \lambda(X_1) + \lambda(X_2) < \epsilon + \delta \implies \lambda(T(X_1 \times X_2)) < \epsilon.$$

Theorem 2.6. *Let E be a nonempty, bounded, closed, and convex subset of a Banach space M and λ an arbitrary measure of non-compactness on E. If $T : E \times E \to E$ is a continuous Meir-Keeler condensing operator, then T has at least one coupled fixed point.*

Proof. *Note from Example 2.4 that $\tilde{\lambda} = \lambda(X_1) + \lambda(X_2)$ is a measure of non-compactness on $E \times E$, for any bounded subset X of $E \times E$, where X_i, $i = 1, 2$ denotes the natural projections of X.*
We define an operator $H : E \times E \to E \times E$ by

$$H(x, y) = (T(x, y)), (T(y, x))$$

is clearly continuous on $E \times E$. Now we claim that H satisfies conditions of Theorem 1.5.
Now, let $\epsilon > 0$ and $\delta(\epsilon) > 0$ be as in Definition 2.5. If X is a bounded subset of $E \times E$ such that

$$\epsilon \leq \lambda(X) < \epsilon + \delta$$

then

$$\epsilon \leq \lambda(X_1) + \lambda(X_2) < \epsilon + \delta(\epsilon),$$

where X_i, $i = 1, 2$, denotes the natural projection of X. By axiom (2) and Theorem 2.3 we have

$$\begin{aligned} \tilde{\lambda} H(X) &\leq \tilde{\lambda}(T(X_1 \times X_2)) \times T(X_2 \times X_1) \\ &= \lambda(T(X_1 \times X_2)) + \lambda(T(X_2 \times X_1)) \\ &< \epsilon. \end{aligned}$$

Thus from Theorem 1.5 H has at least one coupled fixed point in $E \times E$ and the fixed point of H is also a fixed point of T.

□

Theorem 2.7. *Let E be a nonempty, bounded, closed and convex subset of a Banach space M , λ an arbitrary measure of non-compactness on E, and suppose that ϕ an L-function. If $T : E \times E \to E$ is an operator satisfying*

$$\lambda(T(X_1 \times X_2)) < \frac{1}{2}\phi(\lambda(X_1) \times \lambda(X_2)), \tag{1}$$

then T has atleast one coupled fixed point.

Proof. Let $H : E \times E \to E \times E$ defined by

$$H(x, y) = (T(x, y), T(y, x))$$

be a continuous map, we have the fact that

$$\tilde{\lambda}_1 = \lambda(X_1) + \lambda(X_2)$$

defines a measure of non-compactness on $M \times M$, X_i $i = 1, 2$ denotes the natural projections of X.
Now, let $X \subset E \times E$ be any nonempty subset. Then by condition (2) of Definition 1.1 and Equation (2.7) we obtain

$$\begin{aligned}\tilde{\lambda}(H(X)) &\leq \tilde{\lambda}(T(X_1 \times X_2) \times T(X_2 \times X_1)) \\ &= \lambda(T(X_1 \times X_2)) + \lambda(T(X_2 \times X_1)) \\ &< \phi(\lambda(X_1) + \lambda(X_2)) \\ &\leq \phi(\lambda(X)).\end{aligned}$$

From Theorem 1.10 H has a fixed point and it is equivalent to the fixed point of T. □

Theorem 2.8. *Let E be a nonempty, bounded, closed and convex subset of a Banach space M , λ an arbitrary measure of non-compactness on E, and suppose that θ a strict L-function. If $T : E \times E \to E$ is an operator satisfying*

$$\lambda(T(X_1 \times X_2)) \leq \frac{1}{2}\theta(\lambda(X_1) \times \lambda(X_2)),$$

then T has atleast one coupled fixed point.

The proof is similar to that of Theorem 2.7

Corollary 2.9. *Suppose that E is a nonempty, bounded, closed, and convex subset of a Banach space M and let $F : E \to M$ be an operator such that*

$$\left\|F(x, y) - F(u, v)\right\| \leq \frac{1}{2}\theta(\|x - u\| + \left\|y - v\right\|), \tag{2}$$

where θ is a nondecreasing and right continuous strictly L-function. Assume that $H : E \times E \to M$ is compact continuous operator, we define $T(x, y) = F(x, y) + H(x, y)$ and assume that $T(x, y) \in E$ for all $x, y \in E$. Then, T has atleast a coupled fixed point.

Proof. Let $\lambda : Q_M \to \mathbb{R}_+$ be the Kuratowski measure of non-compactness defined in Definition 1.1, furthermore assume that X_1 and X_2 are nonempty subsets of E. Since θ is nondecreasing and from (2), we have

$$\begin{aligned}\left\|F(x, y) - F(u, v)\right\| &\leq \frac{1}{2}\theta(\|x - u\| + \left\|y - v\right\|) \\ &\leq \frac{1}{2}\theta(diam\,\|x - u\| + diam\left\|y - v\right\|)\end{aligned}$$

and

$$diam(F(X_1 \times X_2)) \leq \frac{1}{2}\theta(diam(X_1) + diam(X_2)).$$

Using the fact that θ is a right continuous and by definition of Kuratowski measure of non-compactness we have,

$$\lambda(F(X_1 \times X_2)) \leq \frac{1}{2}\theta(\lambda(X_1) + \lambda(X_2)),$$

and since G is compact, we obtain

$$\begin{aligned}\lambda(T(X_1 \times X_2)) &= \lambda((F+H)(X_1 \times X_2))\\ &\leq \lambda(F(X_1 \times X_2) + H(X_1 \times X_2))\\ &\leq \lambda(F(X_1 \times X_2)) + \lambda(H(X_1 \times X_2))\\ &\leq \frac{1}{2}\theta(\lambda(X_1) + \lambda(X_2)).\end{aligned}$$

Applying Theorem 2.8, the proof is complete. □

3. Applications and Examples

Let $BC(\mathbb{R}_+)$ denote the space of all real valued functions defined, continuous, and bounded on $\mathbb{R}_+$ with the standard supremum norm

$$\left\|y\right\| = sup\left\{\left|y(t)\right| : t \geq 0\right\}, y \in BC(\mathbb{R}_+).$$

Let $Y \in Q_{BC(\mathbb{R}_+)}$, suppose that $\epsilon, K > 0$ and $y \in Y$ be fixed. We define the following quantities

$$\omega^K(y,\epsilon) = sup\left\{\left|y(t,s) - y(u,v)\right| : t,s,u,v \in [0,K], |t-u| \leq \epsilon, |s-v| \leq \epsilon\right\}.$$

This quantity represents the modulus of continuity of the function y on the interval $[0,K]$, while the quantity

$$\omega^K(Y,\epsilon) = sup\left\{\omega^K(y,\epsilon), y \in Y\right\},$$

is the modulus of continuity of the set Y, we also define,

$$\omega_0^K(Y) = \lim_{\epsilon\to 0}\omega^K(Y,\epsilon),$$

$$\omega_0(Y) = \lim_{K\to\infty}\omega_0^K(Y),$$

and

$$\lambda(Y) = \frac{1}{2}(\omega_0(Y)) + \lim_{max(s,t)\to\infty}\sup diam Y(t,s), \tag{3}$$

where

$$\lim_{max(s,t)\to\infty}\sup A = \inf_{T>0}\left(\sup_{max(s,t)>T} A\right).$$

Banaś [6] proved that the function λ is a measure of non-compactness in the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ (in the sense of Definition 1.1).

The following lemma is useful for the proof of the next theorem.

Lemma 3.1. *Suppose that g satisfies the hypothesis (iii) of Theorem 3.2. Then $G : BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ given by*

$$G(x,y)(t) = \int_0^t g(t,s,x(s),y(s))ds$$

is compact and continuous.

Proof. First we show that $G(x, y)(t)$ is continuous for any $x, y \in BC(\mathbb{R}_+)$.
Let $x, y \in BC(\mathbb{R}_+)$ and $\epsilon > 0$. Take $u, v \in BC(\mathbb{R}_+)$ with $\left\|(x, y) - (u, v)\right\|_{BC(\mathbb{R}_+)^2} < \epsilon$, then by condition (ii) and (10), there exists $K > 0$ such that for $t > K$ we have

$$\left|G(x, y)(t) - G(u, v)(t)\right| \leq \int_0^t \left|g(t, s, x(s), y(s))ds - g(t, s, u(s), v(s))ds\right| \leq \epsilon \tag{4}$$

for any $x, y, u, v \in BC(\mathbb{R}_+)$. Also if $t \in [0, K]$, then the first inequality (4) implies that

$$\left|G(x, y)(t) - G(u, v)(t)\right| \leq K\Upsilon_K(\epsilon),$$

where, $\Upsilon_K(\epsilon)$

$$= \sup\left\{\left|g(t, s, x, y) - g(t, s, u, v)\right| : t \in [0, K], x, y, u, v \in [-b, b], \left\|(x, y) - (u, v)\right\|_{BC((\mathbb{R}_+)^2} \leq \epsilon\right\}$$

with $b = \|x\|_\infty + \left\|y\right\|_\infty + \epsilon$. By using the continuity of g on $[0, K] \times [0, K] \times [-b, b] \times [-b, b]$, we have $\Upsilon_K(\epsilon) \to 0$ as $\epsilon \to 0$. Thus G is a continuous function on $BC((\mathbb{R}_+) \times BC((\mathbb{R}_+)$.
Now, let X_1, X_2 be nonempty and bounded subset of $BC(\mathbb{R}_+)$ and assume that $K > 0$ and $\epsilon > 0$ are arbitrary constants. Let $t_1, t_2 \in [0, K]$ with $|t_1 - t_2| \leq \epsilon$ and $(x, y) \in X_1 \times X_2$. We have

$$\begin{aligned} \left|G(x, y)(t_1) - G(x, y)(t_2)\right| &\leq \left|\int_0^t g(t_1, s, x(s), y(s))ds - \int_0^t g(t_2, s, x(s), y(s))ds\right| \\ &\leq K\omega_r^K(g, \epsilon) + U_r^K\epsilon, \end{aligned}$$

where $r = \sup_{x,y \in X}\left\{\|x\|_\infty + \left\|y\right\|_\infty\right\}$,
$\omega_r^K = \sup\left\{\left|g(t_1, s, x, y) - g(t_2, s, x, y)\right| : t_1, t_2 \in [0, K], x, y \in [-r, r], |t_1 - t_2| \leq \epsilon\right\}$,
$U_r^K = \sup\left\{\left|g(t, s, x, y)\right| : t_1, t_2 \in [0, K], x, y \in [-r, r]\right\}$
Since (x, y) was arbitrary, we obtain

$$\omega^K(G(X_1 \times X_1), \epsilon) \leq K\omega_r^K(g, \epsilon) + U_r^K\epsilon. \tag{5}$$

On the other hand, by the uniform continuity of g on $[0, K] \times [0, K] \times [-r, r] \times [-r, r]$, we have $\omega_r^K(g, \epsilon) \to 0$ as $\epsilon \to 0$.
Therefore we obtain

$$\omega_0^K(G(X_1 \times X_2)) = 0$$

and finally

$$\omega_0(G(X_1 \times X_2)) = 0 \tag{6}$$

In addition, for arbitrary $(x, y), (u, v) \in X_1 \times X_2$ and $t \in \mathbb{R}_+$ we have

$$\left|G(x, y)(t) - G(u, v)\right| \leq \int_0^t \left|g(t, s, x(s), y(s))ds - g(t, s, u(s), v(s))ds\right| \leq \beta(t)$$

where

$$\beta(t) = \sup\left\{\left|g(t, s, x(s), y(s)) - g(t, s, u(s), v(s))\right| : t, s \in [0, K], x, y, u, v \in BC((\mathbb{R}_+)\right\}.$$

Thus, we have

$$diam G(X_1 \times X_2)(t) = 0. \tag{7}$$

Taking the limit as $t \to \infty$ in the inequality (7) and using (iii) we get

$$\limsup_{t \to \infty} diam G(X_1, X_2)(t) = 0. \tag{8}$$

Further, combining (6) and (7) we get

$$\limsup_{t\to\infty} diam G(X_1 \times X_2) + \omega_0(G(X_1 \times X_2)) = 0. \tag{9}$$

Or equivalently

$$\lambda(G(X_1 \times X_2)) = 0.$$

Thus, G is compact and the proof is complete. □

Theorem 3.2. *Assume that the following conditions are satisfied:*

i. The function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a nondecreasing and upper semicontinuous strictly L-function θ such that

$$\left|f(t, x_1, x_2) - f(t, y_1, y_2)\right| \leq \frac{1}{2}\theta(\left|x_1 - y_1\right| + \left|x_2 - y_2\right|), \tag{10}$$

for all $(x_1, x_2), (y_1, y_2) \in M \times M, t \geq 0$.

ii. $H := \sup\left\{\left|f(t, 0, 0, 0)\right| : t \in \mathbb{R}_+\right\} < \infty$.

iii. The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive constant N such that

$$N = \sup\left\{\left|\int_0^t g(t, s, x(s), y(s))ds\right| : t, s \in \mathbb{R}_+, x, y \in BC(\mathbb{R}_+)\right\}. \tag{11}$$

Moreover,

$$\lim_{t\to\infty} \int_0^t \left|g(t, s, x(s), y(s)) - g(t, s, u(s), v(s))\right| ds = 0 \tag{12}$$

for all $x, y, u, v \in BC(\mathbb{R}_+)$.

iv. There exists a positive solution r_0 of the inequality

$$\frac{1}{2}\theta(2r) + H + N \leq r.$$

Then the coupled system of functional integral equations

$$\begin{cases} x(t) = f(t, x(t), y(t)) + \int_0^t g(t, s, x(s), y(s))ds \\ y(t) = f(t, y(t), x(t)) + \int_0^t g(t, s, y(s), x(s))ds \end{cases}, \tag{13}$$

has at least one solution on the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.
For any $x, y \in BC(\mathbb{R}_+)$, let

$$\left\|(x, y)\right\|_{BC(\mathbb{R}_+)^2} = \|x\|_\infty + \left\|y\right\|_\infty.$$

We have to prove that the solution of (13) in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ is equivalent to the coupled fixed point of G.

Proof. We define the operator $F, T : BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \to BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$ as follows:

$$F(x, y)(t) = f(t, X(t), y(t))$$

and

$$T(x, y)(t) = f(t, x(t), y(t)) + \int_0^t g(t, s, x(s), y(s))ds.$$

Applying condition (i)-(iv) for arbitrary fixed $t \in \mathbb{R}_+$, we have

$$\begin{aligned} G(x,t)(t) &\leq \left| f(t,x(t),y(t)) + \int_0^t g(t,s,x(s),y(s))ds - f(t,0,0,0) \right| + \left| f(t,0,0,0) \right| \\ &\leq \frac{1}{2}\theta(|x(t)| + \left|y(t)\right| + \left| \int_0^t g(t,s,x(s),y(s))ds \right| + \left| f(t,0,0,0) \right|) \\ &\leq \frac{1}{2}\theta(|x(t)| + \left|y(t)\right|) + H + N. \end{aligned}$$

Thus, keeping in mind assumption (iv) we infer that T is a self mapping of the ball $\bar{B_{r_0}}$. Next, by condition (ii) of Theorem 3.2 it is obvious that F and G for any $x, y \in BC(\mathbb{R}_+)^2$ are continuous functions, and

$$\left\| F(x,y) - F(u,v) \right\| < \theta(\left\| (x,y) - (u,v) \right\| BC(\mathbb{R}_+)^2).$$

Let $\lambda : Q_M \to BC(\mathbb{R}_+)$ be the Kuratowski measure of non-compactness defined by Definition 1.1, and using Theorem 2.8, we get

$$\lambda(F(x)) \leq \theta(\lambda(x)).$$

Thus, F is a Meir-Keeler condensing operator. Finally, since $T(x,y) = F(x,y) + G(x,y), G$ is a compact and continuous operator and F is a continuous Meir-Keeler condensing operator. Therefore, by Corollary 2.9, T has a fixed point.

Example 3.3. *Consider the following system of integral equations.*

$$\begin{cases} x(t) = \frac{1}{8}e^{-t^2} + \dfrac{t^2 \ln(1+|x(t)|)}{6(2+t^2)} + \dfrac{e^{-t}\ln(1+|y(t)|)}{4} + \ln\left(1 + \dfrac{1}{3}\displaystyle\int_0^t \dfrac{\sin(1+sy(s)+\cos^2\{sx(s)\})}{e^{t^2}}\right)ds \\ y(t) = \frac{1}{8}e^{-t^2} + \dfrac{t^2 \ln(1+|y(t)|)}{6(2+t^2)} + \dfrac{e^{-t}\ln(1+|x(t)|)}{4} + \ln\left(1 + \dfrac{1}{3}\displaystyle\int_0^t \dfrac{\sin(1+sx(s)+\cos^2\{s(s)\})}{e^{t^2}}\right)ds \end{cases}$$

We have, $f(t,x(t),y(t)) = \frac{1}{8}e^{-t^2} + \dfrac{t^2\ln(1+|x(t)|)}{6(2+t^2)} + \dfrac{e^{-t}\ln(1+|y(t)|)}{4}$,
$g(t,s,x(s),y(s)) = \dfrac{\sin(1+sy(s)+\cos^2\{sx(s)\})}{e^{t^2}}$.
It is obvious that the function f is continuous.
Now from condition (ii) we have

$$\left| f(t,0,0,0) \right| = \frac{1}{8}e^{-t^2}.$$

Implying that

$$H = \sup\left\{ \left| f(t,0,0,0) \right| \right\} = \frac{1}{8}$$

is bounded.

Let $t \in \mathbb{R}_+, x_1, x_2, y_1, y_2 \in \mathbb{R}$. Suppose that $|x_1| \geq \left|y_1\right|, |x_2| \geq \left|y_2\right|$. Then we have

$$
\begin{aligned}
\left|f(t,x_1,x_2) - f(t,y_1,y_2)\right| &\leq \frac{t^2}{6(2+t^2)}|\ln(1+|x|) - \ln(1+y_1)| + \frac{e^{-t}}{4}|\ln(1+|x_2|) - \ln(1+|y_2|)|, \\
&\leq \frac{t^2}{6(2+t^2)}\left|\ln\left(\frac{1+|x_1|}{1+|y_1|}\right)\right| + \frac{e^{-t}}{4}\left|\ln\left(\frac{1+|x_2|}{1+|y_2|}\right)\right|, \\
&\leq \left|\ln\left(\frac{1+|x_1|}{1+|y_1|}\right)\right| + \frac{1}{4}\left|\ln\left(\frac{1+|x_2|}{1+|y_2|}\right)\right|, \\
&= \frac{1}{4}\left|\ln\left(1+\frac{|x_1|-|y_1|}{1+|y_1|}\right)\right| + \frac{1}{4}\left|\ln\left(1+\frac{|x_2|-|y_2|}{1+|y_2|}\right)\right|, \\
&\leq \frac{1}{4}\ln(1+|x_1-y_1|) + \frac{1}{4}\ln(+|x_2-y_2|), \\
&\leq \frac{1}{2}\ln\left(1+\frac{|x_1-y_1|+|x_2-y_2|}{2}\right) \\
&= \frac{1}{2}\theta(|x_1-y_1|+|x_2-y_2|).
\end{aligned}
$$

Therefore,

$$\left|f(t,x_1,x_2) - f(t,y_1,y_2)\right| \leq \frac{1}{2}\theta(\left|x_1-y_1\right| + \left|x_2-y_2\right|).$$

The function g is continuous, furthermore for each $t, s \in \mathbb{R}_+$, and $x, y, u, v \in \mathbb{R}$

$$\left|g(t,s,x(s),y(s) - g(t,s,u(s),v(s))\right| \leq \frac{4}{e^{t^2}}.$$

Therefore,

$$\int_0^t \left|g(t,s,x(s),y(s) - g(t,s,u(s),v(s))\right| ds \leq \frac{4}{e^{t^2}}.$$

and

$$\lim_{t\to\infty}\int_0^t \left|g(t,s,x(s),y(s) - g(t,s,u(s),v(s))\right| ds \leq \lim_{t\to\infty}\frac{4}{e^{t^2}} \quad = 0.$$

for all $x, y, u, v \in BC(\mathbb{R}_+)$.
Further,

$$\left|g(t,s,x(s),y(s)\right| \leq \frac{2}{e^{t^2}}.$$

Therefore,

$$\left|\int_0^t g(t,s,x(s),y(s)ds\right| \leq \frac{2}{e^{t^2}}$$

for any $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.
Thus,

$$N \leq \sup\left\{\frac{2t}{e^{t^2}} : t \geq 0\right\} = \frac{1}{\sqrt{2e}}.$$

Now substituting the values of H and N in condition (iv) we have the following inequality

$$\frac{1}{2}\ln(1+r) + \frac{1}{8} + \frac{1}{\sqrt{2e}} < r,$$

for any $r \geq 1$ *we have*

$$r - \frac{1}{2}\ln(1+r) - \frac{1}{8} - \frac{1}{\sqrt{2e}} > 0.$$

We can choose $r_0 = 1$. *All conditions of Theorem 3.2 are satisfied, hence the system (13) has atleast one solution in the space* $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.

□

4. Conclusion

We have used the technique of measures of non-compactness to study the existence of solution of a coupled system of integral equations,

$$\begin{aligned} x(t) &= f(t, x(t), y(t)) + \int_0^t g(t, s, x(s), y(s))ds \\ y(t) &= f(t, y(t), x(t)) + \int_0^t g(t, s, y(s), x(s))ds. \end{aligned} \tag{14}$$

in Banach space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$. Meir-Keeler condensing operators and $L-$functions have been used in our work to obtain a generalized coupled fixed point theorem which guarantees for the given system to have a solution in $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$. Also the method of modulus of continuity is used to define measure of non-compactness in this space. Moreover, we have provided an example to support our results.

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