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Approximation of fuzzy numbers using Bernstein-Kantorovich operators of max-product kind

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Abstract. In this paper, in order to study the approximation of a fuzzy number with support $[x, y]$ by Bernstein-Kantorovich operator of max-product kind, we first extend these operators from interval [0,1] to a compact interval $[x, y]$. We evaluate their orders of uniform approximation to a function $\mathcal{F}$ and prove that they preserve quasi-concavity of function $\mathcal{F}$. Besides studying their approximation properties with regard to a fuzzy number, we also prove that these sequence of operators preserve the core and support of the fuzzy number. Finally, we present some graphical representations in order to show approximation of a non continuous fuzzy number using these operators.

1. Introduction

In approximation theory, nonlinear positive operators using discrete linear operators were proposed by B. Bede et al. [1, 2]. Then, S.G. Gal [3] introduced the max product Bernstein operators and studied order of approximation and some shape preserving properties of these nonlinear operators. In [4] Lucian Coroianu and S. G. Gal introduced and studied the Kantorovich Variants of certain well known max product type approximation operators. Recently, the Bézier type Kantorovich q-Baskakov operators via wavelets are studied in [19].

Approximation of fuzzy numbers using triangular and trapezoidal fuzzy members are investigated in [5–7, 9, 10, 14] . Also approximation of a fuzzy number by non-linear side functions are discussed in [8, 11, 16, 20, 24]. For other similar works one can see [14–16]. Moreover, the recent textbooks [22, 23] devoted to summability theory and the classical sets of fuzzy valued sequences, and related topics can be consulted for recent developments in the area and can provide a scope for extension of the present study.

The main motivation of this study is the paper [25] in which in order to approximate a continuous fuzzy number, the max product Bernstein operators were extended from unit interval $[0, 1]$ to a compact interval. We extend Bernstein Kantarovich operator of max product kind[4] to a compact interval and present certain advantages that these operators have in approximating a fuzzy number which may not be continuous.

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2. Essential preliminaries

We recall some definitions and results from [4]. A positive linear operator associated with a function $\mathcal{F}: I \to [0,+\infty)$ can be defined in its general form by

$$\mathcal{D}_\eta(\mathcal{F})(u) = \sum_{i\in I_\eta} p_{\eta,i}(u)\mathcal{F}(u_{\eta,i}), \quad u \in I, \eta \in \mathbb{N} \tag{1}$$

where $p_{\eta,i}(u)$ are basis functions defined on interval I with $\sum_{i\in I_\eta} p_{\eta,i}(u) = 1$. I_η represents either finite or infinite sets of indices and $u_{\eta,i}; i \in I_\eta$ denotes a division of I.
For the sequence of operators $\mathcal{D}_\eta(\mathcal{F})(u)$, its max-product type version is defined by

$$(\mathcal{L}_\eta^M)(\mathcal{F}(u)) = \frac{\bigvee_{i\in I_\eta} p_{\eta,i}(u).\mathcal{F}(u_{\eta,i})}{\bigvee_{i\in I_\eta} p_{\eta,i}(u)}, \quad u \in I, \eta \in \mathbb{N} \tag{2}$$

Here $\bigvee_{i\in A} a_i = sup_{i\in A} a_i$
Now, for the max-product operator $\mathcal{L}_\eta^M$, we can define its Kantorovich variant by

$$(\mathcal{LK}_\eta^M)(\mathcal{F}(u)) = \frac{\bigvee_{i\in I_\eta} p_{\eta,i}(u).\frac{1}{(u_{\eta,i+1}-u_{\eta,i})}\int_{u_{\eta,i}}^{u_{\eta,i+1}} \mathcal{F}(t)dt}{\bigvee_{i=0}^{\eta} p_{\eta,i}(u)}, \tag{3}$$

Let us choose $p_{\eta,i}(u) = \binom{\eta}{i} u^i(1-u)^{\eta-i}$, $I = [0,1]$, $I_\eta = 0,1,...,\eta-1$ and $u_{\eta,i} = \frac{i}{\eta+1}$. In this case, $\mathcal{L}_\eta^M$ are called as the max-product type Bernstein Operators. We can define their Kantorovich variant by the expression

$$(\mathcal{BK}_\eta^M)(\mathcal{F}(u)) = \frac{\bigvee_{i=0}^{\eta} p_{\eta,i}(u)(\eta+1)\int_{\frac{i}{\eta+1}}^{\frac{(i+1)}{\eta+1}} \mathcal{F}(t)\,\mathrm{d}t}{\bigvee_{i=0}^{\eta} p_{\eta,i}(u)}. \tag{4}$$

Definition 2.1. *A continuous function $g : [x,y] \to \mathbb{R}$ is called quasi convex if*

$$g(\delta u + (1-\delta)v) \le max\{g(u), g(v)\},$$

for all $u, v \in [x,y]$, $\delta \in [0,1]$. Also, g is quasi-concave if -g is quasi-convex.

Remark 2.2. *Definition 2.1 can be equivalently stated as the following[25];*
If g is continuous and quasi-convex on $[x,y]$ then there exists a number $w \in [x,y]$ such that g is nonincreasing in the interval $[x,w]$ and nondecreasing in the interval $[w,y]$. Also, if g is continuous and quasi-concave on $[x,y]$ then there exists a number $w \in [x,y]$ such that g is nondecreasing in the interval $[x,w]$ and nonincreasing in the interval $[w,y]$.

Definition 2.3. *The characterisation of fuzzy number k is given by an upper semicontinuous function $\mu_k : \mathbb{R} \to [0,1]$ which satisfies the following statements:*

(i) There exists an interval $[x,y]$ such that $\mu_k(u) = 0$ for all u outside $[x,y]$.
(ii) There exists $w, z \in \mathbb{R}$, $w \le z$ such that:
  - (a) *μ_k is nondecreasing on $[x,w]$;*
  - (b) *$\mu_k(u) = 1$ for all $u \in [w,z]$;*
  - (c) *μ_k is nonincreasing on $[z,y]$.*

Remark 2.4. *The collection $\{u \in \mathbb{R} : \mu_k(u) = 1\}$ is known as core of k usually denoted as core(k). Closure of $\{u \in \mathbb{R} : \mu_k(u) > 0\}$ is known as support of k and usually denoted as supp(k). It is easy to understand that supp(k) is a compact interval.*
Moreover, if μ_k is a continuous function and supp(k) $= [x,y]$ and core(k) $= [w,z]$, then $x < w \le z < y$. For simplicity, we use the same notation for fuzzy number and the characterisation function throughout the paper.

3. Construction of Bernstein-Kantorovich max-product operators on arbitrary compact interval

Let $\mathcal{F} \in C_+([x,y])$, the space of all positive continuous functions defined on $[x,y]$. We define Bernstein Kantorovich max product Operator on $[x,y]$ in the following way:

$$(\mathcal{BK}_\eta^M)(\mathcal{F}(u)) = \frac{\bigvee_{i=0}^{\eta} p_{\eta,i}(u)\frac{\eta+1}{y-x}\int_{x+\frac{i(y-x)}{\eta+1}}^{x+\frac{(i+1)(y-x)}{\eta+1}} \mathcal{F}(t)\,\mathrm{d}t}{\bigvee_{i=0}^{\eta} p_{\eta,i}(u)}, u \in [x,y] \tag{5}$$

where $p_{\eta,i}(u) = \binom{n}{k}(\frac{u-x}{y-x})^i(\frac{y-u}{y-x})^{\eta-i}$. We show that the operators defined by 5 are well defined.
In order to show that the operators are well defined we need to show that if $\mathcal{F} \in C_+([x,y])$ then $\mathcal{BK}_\eta^{(M)}(\mathcal{F}) \in C_+([x,y])$. It is easy to notice that $\bigvee_{k=0}^{\eta} p_{\eta,i}(u) > 0$ for all $u \in [x,y]$, now it is sufficient to prove $\int_{x+\frac{i(y-x)}{\eta+1}}^{x+\frac{(i+1)(y-x)}{\eta+1}} \mathcal{F}(t)dt > 0$ for $\mathcal{F} \in C_+([x,y])$ for all $\eta \in \mathbb{N}$ and $i \in \{0,1,2,\cdots,\eta\}$. As $[x+\frac{i(y-x)}{\eta+1}, x+\frac{(i+1)(y-x)}{\eta+1}]$ are subintervals of $[x,y]$ for all $\eta \in \mathbb{N}$ and $i \in \{0,1,2,\cdots,\eta\}$. Since $\mathcal{F} \in C_+([x,y])$, it implies that $\mathcal{F} \in C_+([x+\frac{i(y-x)}{\eta+1}, x+\frac{(i+1)(y-x)}{\eta+1}])$, *i.e*, $\mathcal{F}$ is positive and continuous on each subinterval $[x+\frac{i(y-x)}{\eta+1}, x+\frac{(i+1)(y-x)}{\eta+1}]$. So there exists $x_0 \in [x+\frac{i(y-x)}{\eta+1}, x+\frac{(i+1)(y-x)}{\eta+1}]$ such that $\mathcal{F}(x_0) > 0$. Let $\epsilon = \frac{\mathcal{F}(x_0)}{3}$ and by continuity of $\mathcal{F}$, there exists a $\delta > 0$ such that for $x \in (x_0-\delta, x_0+\delta)$ implies that $\mathcal{F}(x) \in (\mathcal{F}(x_0)-\epsilon, \mathcal{F}(x_0)+\epsilon)$. Hence $\mathcal{F}(x) > \mathcal{F}(x_0)-\epsilon$ and $\mathcal{F}(x_0)-\epsilon = \frac{2\mathcal{F}(x_0)}{3} > 0$, we have $\mathcal{F}(x) > 0$ for each subinterval $[x+\frac{i(y-x)}{\eta+1}, x+\frac{(i+1)(y-x)}{\eta+1}]$. Now $\int_{x+\frac{i(y-x)}{\eta+1}}^{x+\frac{(i+1)(y-x)}{\eta+1}} \mathcal{F}(t)dt \geq \int_{x_0-\delta}^{x_0+\delta} \mathcal{F}(t)dt \geq \int_{x_0-\delta}^{x_0+\delta} \frac{2\mathcal{F}(x_0)}{3}dt = \frac{4\mathcal{F}(x_0)}{3}\delta > 0$. Hence $\int_{x+\frac{i(y-x)}{\eta+1}}^{x+\frac{(i+1)(y-x)}{\eta+1}} \mathcal{F}(t)dt \geq 0$ for $t \in [x+\frac{i(y-x)}{\eta+1}, x+\frac{(i+1)(y-x)}{\eta+1}]$, for all $\eta \in \mathbb{N}$ and $i \in \{0,1,2,\cdots,\eta\}$. Since the maximum of a finite number of continuous functions is a continuous function. Now, it is easily seen that $\mathcal{BK}_\eta^{(M)}(\mathcal{F}) \in C_+([x,y])$, which means that $\mathcal{BK}_\eta^{(M)}(.)$ is well defined. Also, even if $\mathcal{F} : [x,y] \to \mathbb{R}_+$ is Lebesgue integrable and not necessarily continuous it can be shown that $\mathcal{BK}_\eta^{(M)}(\mathcal{F}) \in C_+([x,y])$.

In the results that follow, we evaluate the order of uniform approximation of the newly defined operator $\mathcal{BK}_\eta^{(M)} : C_+([x,y]) \to C_+([x,y])$. We further prove that it preserves the quasi-concavity too.

Theorem 3.1. *[4] Let $\mathcal{F} : [0,1] \to \mathbb{R}_+$ be a continuous function then we have the following estimate*

$$\left|\mathcal{BK}_\eta^{(M)}(\mathcal{F})(u) - \mathcal{F}(u)\right| \leq 24\omega_1\Big(\mathcal{F}; \frac{1}{\sqrt{\eta+1}}\Big) + 2\omega_1\Big(\mathcal{F}; \frac{1}{\eta+1}\Big)$$

for all $\eta \in \mathbb{N}$, $x \in [0,1]$.

Theorem 3.2. *[4] Let $\mathcal{F} : [0,1] \to \mathbb{R}_+$ be a function and $\eta \in \mathbb{N}$, be fixed. Also, let there exists a $w \in [0,1]$ such that $\mathcal{F}$ is nondecreasing in the interval $[0,w]$ and nonincreasing in the interval $[w,1]$. Then, there exists $w' \in [0,1]$ such that $\mathcal{BK}_\eta^{(M)}(\mathcal{F})$ is nondecreasing in the interval $[0,w']$ and nonincreasing in the interval $[w',1]$. Moreover, $|w-w'| \leq \frac{1}{\eta+1}$ and*

$$\left|\mathcal{BK}_\eta^{(M)}(\mathcal{F})(w) - \mathcal{F}(w)\right| \leq \omega_1\Big(\mathcal{F}; \frac{1}{\eta+1}\Big).$$

Theorem 3.3. *Let $\mathcal{F} : [x,y] \to \mathbb{R}_+$ be a continuous function, then we have the following estimate*

$$\left|\mathcal{BK}_\eta^{(M)}(\mathcal{F})(u) - \mathcal{F}(u)\right| \leq 24([y-x]+1)\omega_1\Big(\mathcal{F}; \frac{1}{\sqrt{\eta+1}}\Big) + 2([y-x]+1)\omega_1\Big(\mathcal{F}; \frac{1}{\eta+1}\Big)$$

for all $n \in \mathbb{N}$, $u \in [x,y]$. Here ω_1 represents modulus of continuity of $\mathcal{F}$ on $[x,y]$.

Proof. Let $u \in [x, y]$ be mapped to a $v \in [0, 1]$ through the map given by

$$u = x + (y - x)v \tag{6}$$

Clearly,

$$v = (u - x)/(y - x) \tag{7}$$

and

$$1 - v = (y - u)/(y - ax). \tag{8}$$

Now, let $g : [0, 1] \to \mathbb{R}$, be a function such that $g(v) = \mathcal{F}(x + (y - x)v)$. Using (3.2) , (3.3) and the expression for $g(i/\eta)$ we get by simple calculations,

$\mathcal{BK}_\eta^{(M)}(\mathcal{F})(u) = \mathcal{BK}_\eta^{(M)}(g)(v)$, which further gives,

$$\left|\mathcal{BK}_\eta^{(M)}(\mathcal{F})(u) - \mathcal{F}(u)\right| = \left|\mathcal{BK}_\eta^{(M)}(g)(v) - g(v)\right| \leq 24\omega_1\left(g; \frac{1}{\sqrt{\eta + 1}}\right) + 2\omega_1\left(g; \frac{1}{\eta + 1}\right)$$

Using the properties of modulus of continuity, $\omega_1\left(g; \frac{1}{\sqrt{\eta+1}}\right) \leq \omega_1\left(\mathcal{F}; \frac{y-x}{\sqrt{\eta+1}}\right)$ and $\omega_1\left(g; \frac{1}{\eta+1}\right) \leq \omega_1\left(\mathcal{F}; \frac{y-x}{\eta+1}\right)$. Also, $\omega_1(\mathcal{F}; \lambda\delta) \leq ([\lambda] + 1)\omega_1(\mathcal{F}; \delta)$ we obtain $\omega_1\left(g; \frac{1}{\sqrt{\eta+1}}\right) \leq ([y - x] + 1)\omega_1\left(\mathcal{F}; \frac{1}{\sqrt{\eta+1}}\right)$ and $\omega_1\left(g; \frac{1}{\eta+1}\right) \leq ([y - x] + 1)\omega_1\left(\mathcal{F}; \frac{1}{\eta+1}\right)$ and the theorem is proved.

Theorem 3.4. *Let $\mathcal{F} : [x, y] \to \mathbb{R}_+$ be a function and let $\eta \in \mathbb{N}$, be fixed. Also suppose that there exists $w \in [x, y]$ such that $\mathcal{F}$ is nondecreasing in the interval $[x, w]$ and nonincreasing in the interval $[w, y]$. Then, there exists $w' \in [x, y]$ such that $\mathcal{BK}_\eta^{(M)}(\mathcal{F})$ is nondecreasing in the interval $[x, w']$ and nonincreasing in the interval $[w', y]$. Moreover, $|w - w'| \leq \frac{y-x}{\eta+1}$ and*

$$\left|\mathcal{BK}_\eta^{(M)}(\mathcal{F})(w) - \mathcal{F}(w)\right| \leq ([y - x] + 1)\omega_1\left(\mathcal{F}; \frac{1}{\eta + 1}\right)$$

Proof. Consider the function g constructed in Theorem 3.3, above. Suppose, $g(w_1) = w$ where $w_1 \in [0, 1]$. Because g is obtained by the composition of $\mathcal{F}$ and the linear nondecreasing map given by (3.1) in previous theorem, it can be concluded that g is nondecreasing in the interval $[0, w_1]$ and nonincreasing in the interval $[w_1, 1]$. By Theorem 3.2, there exists $w_1' \in [0, 1]$ such that $\mathcal{BK}_\eta^{(M)}(g)$ is nondecreasing in the interval $[0, w_1']$, nonincreasing in the interval $[w_1', 1]$ and in addition we have $\left|\mathcal{BK}_\eta^{(M)}(g)(w_1) - g(w_1)\right| \leq \omega_1\left(g; \frac{1}{\eta+1}\right)$ and $|w_1 - w_1'| \leq \frac{1}{\eta+1}$. Let $w' = x + (y - x)w_1'$. If $u_1, u_2 \in [x, w']$ with $u_1 \leq u_2$ then let $v_1, v_2 \in [0, w_1']$ be such that $u_1 = x + (y - x)v_1$ and $u_2 = x + (y - x)v_2$. Then, it follows that $\mathcal{BK}_\eta^{(M)}(\mathcal{F})(u_1) = \mathcal{BK}_\eta^{(M)}(g)(v_1)$ and $\mathcal{BK}_\eta^{(M)}(\mathcal{F})(u_2) = \mathcal{BK}_\eta^{(M)}(g)(v_2)$. The monotonicity of $\mathcal{BK}_\eta^{(M)}(g)$ implies $\mathcal{BK}_\eta^{(M)}(g)(v_1) \leq \mathcal{BK}_\eta^{(M)}(g)(v_2)$, that is $\mathcal{BK}_\eta^{(M)}(\mathcal{F})(u_1) \leq \mathcal{BK}_\eta^{(M)}(\mathcal{F})(u_2)$. Thus, it follows that $\mathcal{BK}_\eta^{(M)}(\mathcal{F})$ is nondecreasing in the interval $[x, w']$. Similarly, $\mathcal{BK}_\eta^{(M)}(\mathcal{F})$ is nonincreasing in the interval $[w', y]$. Moreover, note that $|w_1 - w'| \leq \frac{1}{\eta+1}$ which gives $|w - w'| = |(y - x)(|w_1 - w'|) \leq \frac{y-x}{\eta+1})| \leq \frac{1}{\eta+1}$. Also, $\left|\mathcal{BK}_\eta^{(M)}(g)(w_1) - g(w_1)\right| \leq \omega_1\left(g; \frac{1}{\eta+1}\right)$. Considering,

$\left(g; \frac{1}{\eta+1}\right) \leq ([b-a]+1)\omega_1\left(\mathcal{F}; \frac{1}{\eta+1}\right)$, we have,

$$\begin{aligned}\left|\mathcal{BK}_\eta^{(M)}(\mathcal{F})(w) - \mathcal{F}(w)\right| &= \left|\mathcal{BK}_\eta^{(M)}(g)(w_1) - g(w_1)\right| \\ &\leq 12\omega_1\left(g; \frac{1}{\eta+1}\right) \\ &\leq ([y-x]+1)\omega_1\left(\mathcal{F}; \frac{1}{\eta+1}\right)\end{aligned}$$

and thus the proof of the theorem is completed.

Remark 3.5. *We can state Theorem 3.4, above, in the following way as well ; if $\mathcal{F} : [x, y] \to \mathbb{R}_+$ is a continuous function and is quasi-concave then the operator $\mathcal{BK}_\eta^{(M)}(\mathcal{F})$ is quasi-concave.*

4. Applications to the approximation of fuzzy numbers

We first see the following auxiliary results of this section.

Lemma 4.1. *[25] Let x, y be real numbers such that $x < y$. For the natural number η, $i, j \in \{0, 1, ..., \eta\}$ and $u \in (x + j \cdot \frac{y-x}{\eta+1}, x + (j+1) \cdot \frac{y-x}{\eta+1})$, let*

$$m_{i,\eta,j}(u) = \frac{p_{\eta,i,}(u)}{p_{\eta,j,}(u)}.$$

Then $m_{i,\eta,j}(u) < 1$, $j \in \{0, 1, ..., \eta\}$ and $i \in \{0, 1, ..., \eta\}\backslash\{j\}$.

Lemma 4.2. *Let $x, y \in \mathbb{R}$, $x < y$ and $\mathcal{F} : [x, y] \to \mathbb{R}_+$ is a bounded function then, for $j \in \{0, 1, ..., \eta\}$, we have $\mathcal{BK}_\eta^{(M)}(\mathcal{F})(x + j(y-x)/\eta) \geq \frac{\eta+1}{y-x} \int_{x+\frac{i(y-x)}{\eta+1}}^{x+\frac{(i+1)(y-x)}{\eta+1}} \mathcal{F}(t)\, \mathrm{d}t$*

Proof. Since, $(x+j(y-x))/\eta \in (x+j(y-x)/(\eta+1), x+(j+1)(y-x)/(\eta+1))$ and $m_{i,\eta,j}(x+j(y-x)/\eta) = \frac{p_{\eta,i}(x+j(y-x)/\eta)}{p_{\eta,j}(x+j(y-x)/\eta)}$ for all $k \in \{0, 1, ..., \eta\}$, from Lemma 4.1, it follows that $\bigvee_{i=0}^{\eta} p_{\eta,i}(x + j(y-x)/\eta) = p_{\eta,j}(x + j(y-x)/\eta)$. Further,

$$\begin{aligned}\mathcal{BK}_\eta^{(M)}(\mathcal{F})(x + j(y-x)/\eta) &= \frac{\bigvee_{i=0}^{\eta} p_{\eta,i}(x + j(y-x)/\eta)\frac{\eta+1}{y-x}\int_{x+\frac{i(y-x)}{\eta+1}}^{x+\frac{(i+1)(y-x)}{\eta+1}} \mathcal{F}(t)\, \mathrm{d}t}{p_{\eta,j}(x + j(y-x)/\eta)} \\ &\geq \frac{p_{\eta,j}(x + j(y-x)/\eta)\frac{\eta+1}{y-x}\int_{x+\frac{i(y-x)}{\eta+1}}^{x+\frac{(i+1)(y-x)}{\eta+1}} \mathcal{F}(t)\, \mathrm{d}t}{p_{\eta,j}(x + j(y-x)/\eta)} \\ &= \frac{\eta+1}{y-x}\int_{x+\frac{i(y-x}{\eta+1}}^{x+\frac{(i+1)(y-x)}{\eta+1}} \mathcal{F}(t)\, \mathrm{d}t\end{aligned}$$

which proves the lemma.

. Now, let us consider a fuzzy number k having $supp(k) = [x, y]$ and $core(k) = [w, z]$. Let $\eta \in \mathbb{N}$ then we consider the function $\widetilde{\mathcal{BK}}_\eta^{(M)}(k) : \mathbb{R} \to [0, 1]$, such that, $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(u) = 0$ for all u outside $[x, y]$ and $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(u) = \mathcal{BK}_\eta^{(M)}(k)(u)$ for all $u \in [x, y]$. Let us now prove the following main theorem.

Theorem 4.3. *Let k be a fuzzy number having support, $supp(k) = [x, y]$ and core, $core(k) = [w, z]$ such that $x \leq w < z \leq y$. Then $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)$ is also a fuzzy number such that :*

(i) $supp(k) = supp(\widetilde{\mathcal{BK}}_\eta^{(M)}(k))$;

(ii) *If* $core(\widetilde{\mathcal{BK}}_\eta^{(M)}(k)) = [w_\eta, z_\eta]$, *then* $|w - w_\eta| \leq \frac{y-x}{\eta}$ *and* $|z - z_\eta| \leq \frac{y-x}{\eta}$

(iii) *Moreover, if k is continuous on $[x, y]$, then*

$$\left|\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(u) - k(u)\right| \leq 12([y - x] + 1)\omega_1\Big(k; \frac{1}{\sqrt{\eta + 1}}\Big)$$

for all $u \in \mathbb{R}$.

Proof. Let η be a natural number such that $\frac{y-x}{\eta} < z - w$. From Theorem 3.1, there exists $w' \in [x, y]$ such that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)$ is nondecreasing on $[x, w']$ and nonincreasing on $[w', y]$. Using the definition of $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)$, it follows that $\|\widetilde{\mathcal{BK}}_\eta^{(M)}(k)\| \leq \|k\|$. Since $\|k\| = 1$, we have $\|\widetilde{\mathcal{BK}}_\eta^{(M)}(k)\| \leq 1$. (where $\|\cdot\|$ is the uniform norm on the space of all bounded functions defined on $[x, y]$ generally written as $B([x, y])$). So, in order to prove that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)$ is a fuzzy number, we prove that there exists a number $\alpha \in [x, y]$ such that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k) = 1$. Let us suppose $\alpha = x + j(y - x)/\eta$ where we choose j in a way that $w < \alpha < z$. The existence of such j is guaranteed by the fact that $\frac{y-x}{\eta} < z - w$. Since $\alpha \in core(k)$, it follows that $k(\alpha) = 1$. Now, by Lemma 4.2 it follows that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)$ is a fuzzy number. We now prove the remaining theorem.

(i) Using the definitions of k and $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)$, it results that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(u) = 0$ outside $[x, y]$. Now, since $k(u) > 0$ and $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(u) = \mathcal{BK}_\eta^{(M)}(k)(u)$ for all $u \in (x, y)$, we find that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(u) > 0$ for all $u \in (x, y)$, which proves (i).

(ii) Because k is nondecreasing in the interval $[x, w]$ and nonincreasing in the interval $[w, z]$, by Theorem 3.1 it follows that there exists $w'(\eta) \in [x, y]$ such that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)$ is nondecreasing in the interval $[x, w'(\eta)]$ and nondecreasing in the interval $[w'(\eta), y]$. Also, $|w - w'(\eta)| \leq (y - x)/(\eta + 1)$. Moreover, k is nondecreasing in the interval $[x, z]$ and nonincreasing in the interval $[z, y]$. Therefore there exists $z'(\eta) \in [x, y]$ such that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)$ is nonincreasing in the interval $[x, z'(\eta)]$ and nondecreasing in the interval $[z'(\eta), y]$. We also have, $|z - z'(\eta)| \leq (y - x)/(\eta + 1)$. For $\eta \in \mathbb{N}$ satisfying $(y - x)/\eta < (z - w)/2$, it follows that $w'(\eta) < z'(\eta)$ which implies that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)$ is constant on $[w'(\eta), z'(\eta)]$, that is $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(u) = 1$ for all $u \in [w'(\eta), z'(\eta)]$. This implies $[w'(\eta), z'(\eta)] \subseteq core(\widetilde{\mathcal{BK}}_\eta^{(M)}(k))$, or $w(\eta) < w'(\eta)$ and $z'(\eta) \leq z(\eta)$. But we have $w(\eta) > x + j_1(y - x)/\eta$ and $z(\eta) < x + j_2(y - x)/\eta$ where j_1 and j_2 are chosen in a way that $x + j_1(y - x)/\eta < w \leq x + (j_1 + 1)(y - x)/\eta$ and $x + (j_2 - 1)(y - x)/\eta \leq z < x + j_2(y - x)/\eta$. To prove statement(i), we observe that (from the proof of Lemma 4.2) $\bigvee_{i=0}^{\eta} p_{\eta,i}(x + j_1(y - x)/\eta) = p_{\eta,j_1}(x + j(y - x)/\eta)$, which means that

$$\widetilde{\mathcal{BK}}_\eta^{(M)}(x + j_1(y - x)/\eta) = \frac{\bigvee_{i=0}^{\eta} p_{\eta,i}(x+j_1(y-x)/\eta)k(x+i(y-x)/\eta)}{p_{\eta,j_1}(x+j_1(y-x)/\eta)}.$$

Let $i_1 \in \{0, 1, ..., \eta\}$ be chosen in a way that $\bigvee_{i=0}^{\eta} p_{\eta,i}(x + j_1(y - x)/\eta)k(x + i(y - x)/\eta) = \bigvee_{i=0}^{\eta} p_{\eta,i_1}(x + j_1(y - x)/\eta)k(x + i_1(y - x)/\eta)$. When $i_1 = j_1$, we get that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(x + j_1(y - x)/\eta) = k(x + j_1(y - x)/\eta)$ and since $x + j_1(y - x)/\eta \in core(k)$, we get that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(x + j_1(y - x)/\eta) < 1$. Also, when $i_1 = j_1$ then using Lemma 4.1, it follows that $m_{i_1,\eta,j_1}(x + j_1(y - x)/\eta) < 1$ and since $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(x + j_1(y - x)/\eta) = m_{i_1,\eta,j_1}(x + j_1(y - x)/\eta) \cdot k(x + i_1(y - x)/\eta)$ we can again conclude that $\widetilde{\mathcal{BK}}_\eta^{(M)}(k)(x + j_1(y - x)/\eta) < 1$. Now, since $x + j_1(y - x)/\eta,\ w(\eta) \in [x, z'(\eta)]$ which follows immediately because for $(y-x)\eta < (z-w)/2$ we have $w < z^{'}(\eta)$ and since $\widetilde{\mathcal{BK}}_{\eta}^{(M)}(k)$ is nondecreasing in the interval $[x, z^{'}(\eta)]$ and using the fact that $\widetilde{\mathcal{BK}}_{\eta}^{(M)}(k)(x + j_1(y-x)/\eta) < \widetilde{\mathcal{BK}}_{\eta}^{(M)}(k)(w_{\eta})$, it results that we necessarily have $w(\eta) > x + j_1(y-x)/\eta$. The statement $z(\eta) < x + j_2(y-x)/\eta$ has a similar proof. Keeping the above inequalities in mind, we get that $w - (y-x)/\eta < w(\eta) \leq w^{'}(\eta) \leq w + (y-x)/(\eta+1)$. This clearly means that $|w(\eta) - w| \leq (y-x)/\eta$. By using the similar reasoning we get that $|z(\eta) - z| \leq (y-x)/\eta$ which completes the proof of statement (ii).

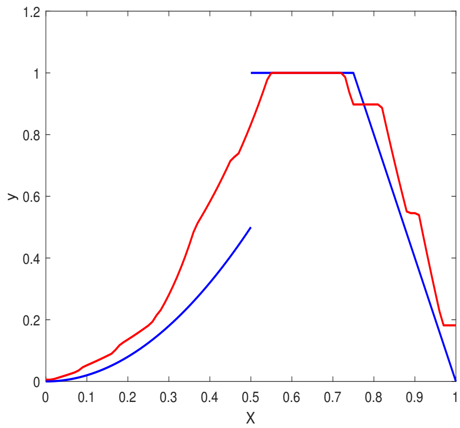
(iii) The statement (iii) immediately follows from Theorem 3.3, considering the continuity of k.

5. Graphical analysis

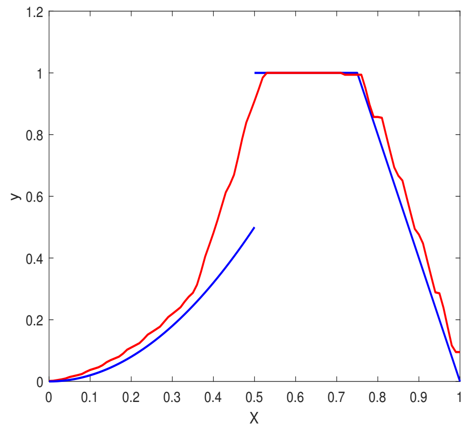
For graphical demonstration we have taken the following discontinuous function

$$F(u) = \begin{cases} 2u^2, & 0 \leq u \leq 0.5 \\ 1, & 0.5 < u < 0.75 \\ 4-4u & 0.75 \leq u \leq 1 \,. \end{cases}$$

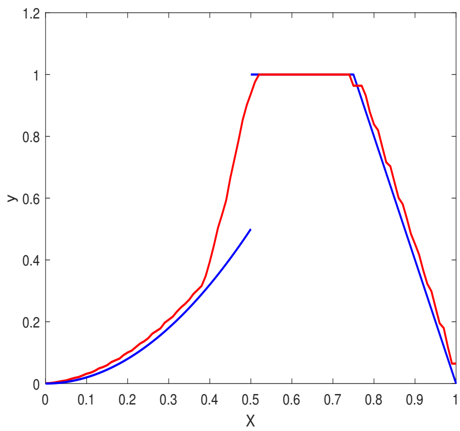
We have presented the approximation of the above fuzzy number by Max product Bernstein Kantorovich operators for different values of η. Blue color represent the graph of the function while red one represent the operator. Support of above fuzzy number is $[0, 1]$ and core is $[0.5, 0.75]$. One can observe that if we increase the value of η the approximation becomes better. It is easy to notice from figure 1(a) to figure 1(d), when we increase the value of η from 10 to 45 the approximation in each case going to be better.



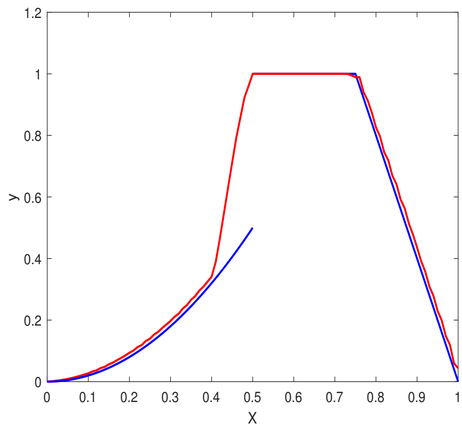
(a) $\eta = 10$



(b) $\eta = 20$



(c) $\eta = 30$



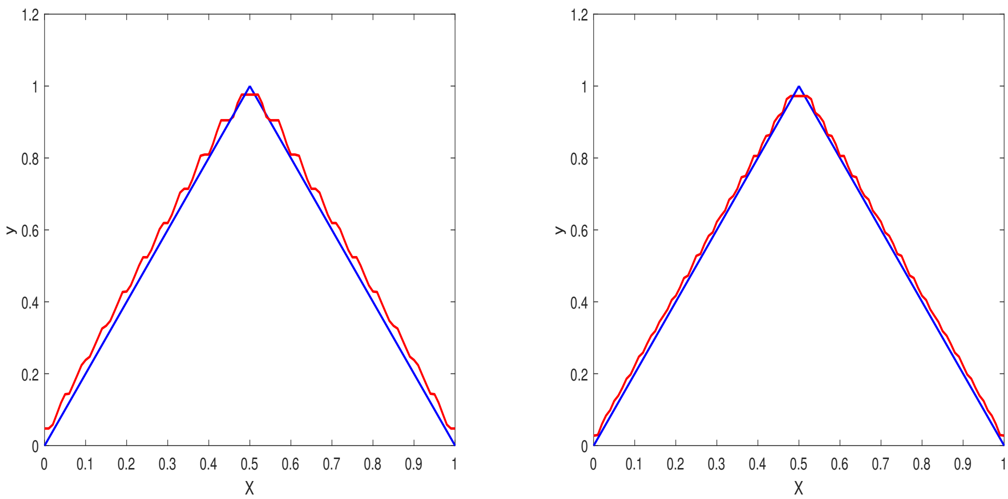
(d) $\eta = 45$

Figure 1: Approximation of discontinuous Fuzzy Number for different values of η.

For Figure 2 we have taken the following unimodal fuzzy number defined as follows

$$G(u) = \begin{cases} 2u, & 0 \leq u \leq 0.5 \\ 2 - 2u & 0.5 \leq u \leq 1\,. \end{cases}$$

One can observe from figure 2 that the operator approximate better the functions(or the unimodal fuzzy number in the present case) when we increase the value of η from 20 to 35. One of the advantage of our operator is that it can be used to approximate discontinuous function as well(in particular discontinuous fuzzy number).



(a) $\eta = 20$ (b) $\eta = 35$

Figure 2: Approximation of a unimodal Fuzzy Number for different values of η.

6. Conclusion

The approximation of a fuzzy number by max product type Bernstein operator was studied in [25]. However, Bernstein max product operator can be considered only for a fuzzy number which is continuous whereas its Kantorovich variant is defined for a bigger class of functions i.e. functions which are positive and integrable in Lebesgue sense generally denoted by the space $L_{+}[0,1]$. Its generalization on $[x, y]$ is of important significance in approximating a fuzzy number which is not necessarily continuous as illustrated in the graphics presented in section 5.

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Competing interests

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Authors contributions

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