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Approximation in the generalized Hölder space

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Abstract. In the present paper the Borel's exponential means are used to estimate a better degree of approximation of functions belonging to the generalized Hölder space. Also we have established some lemmas for proving our main theorem.

1. Introduction

In 2002 Das et al. [2] introduced the generalized Hölder space $H_p^{(\omega)}$. In that paper, they have estimated the degree of approximation by using T^* method. In recent years, many researchers have estimated degree of approximation in $H_p^{(\omega)}$ space and its generalized spaces by using different summation methods [3, 6, 7, 9–13]. The Borel summation method is also used by several researchers like Chandra [1], Das et al. [4], Padhy et al. [14], Volosivets [15] to estimate the rate of convergence in Hölder space, Besov space and weighted Lorentz space. Recently, Krasniqi [8] estimated the degree of approximation in the $H_p^{(\omega)}$ space by using deferred matrix means. We note that approximation using Borel's exponential means of Fourier series of functions belonging to the $H_p^{(\omega)}$ space have not been studied so far, which motivated us to study the problem further.

2. Definitions and Notations

Let $L_p[0, 2\pi]$ be the space of periodic functions with period 2π and the functions are integrable in the sense of Lebesgue.
The generalized Hölder space is denoted by $H_p^{(\omega)}$ and defined as follows [2]

$$H_p^{(\omega)} = \left\{ f \in L_p[0, 2\pi] : A(f, \omega) < \infty \text{ and } 0 < p \leq \infty \right\},$$

where ω is the integral modulus of continuity and

$$A(f, \omega) = \sup_{\zeta \neq 0} \frac{\|f(x+\zeta) - f(x)\|_p}{\omega(|\zeta|)}.$$

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The norm in the $H_p^{(\omega)}$ space is defined as

$$\|f\|_p^{(\omega)} = \left\{ \int_0^{2\pi} |f|^p \right\}^{1/p} + A(f, \omega), 1 < p < \infty$$

and

$$\|f\|_p^{(\omega)} = ess \sup(f) + A(f, \omega),\ p = \infty.$$

Note that $(H_p^{(\omega)}, \|.\|)$ is a Banach space [2].
Every $f \in H_p^{(\omega)}$ can be represented by its Fourier series as follows

$$f(x) \sim \frac{a_0}{2} + \sum_{q=1}^{\infty} (a_q \cos(qx) + b_q \sin(qx)),\ \forall\, q \geq 1, \tag{1}$$

where a_o, a_q, b_q are the Fourier coefficients and '$\sim$' is the notation for "asymptotically equal to", that is, if $M(x) \sim N(x)$, then $\lim\limits_{x\to\infty} \frac{M(x)}{N(x)} = 1$.

Approximation of $f \in H_p^{(\omega)}$ by its Fourier series is called trigonometric Fourier approximation of f and the degree of approximation $E_n(f)$ is given by

$$E_n(f) = \underset{n}{Min} \|f(x) - T_n(f;x)\|_p^{(\omega)},$$

where $T_n(f;x)$ is the $n-$th degree Fourier series associated with the function f [16].

For any given sequence $\{a_k(x)\}$, the Borel's exponential means $B_r(a;x)$ is defined by

$$B_r(a;x) = e^{-r} \sum_{k=0}^{\infty} \frac{r^k}{k!} a_k(x),\ \text{for } r > 0. \tag{2}$$

If $\lim\limits_{r\to\infty} B_r(a;x) = s$, then the series (2) is summable by the Borel method to s.
Regularity of the above summation method is demonstrated by Hardy [[5],p. 182].

We use the following notations:

$$\phi_x(t) = 2^{-1}[f(x+t) + f(x-t) - 2f(x)], \tag{3}$$

$$F(t) = \phi_{x+y}(t) - \phi_x(t), \tag{4}$$

$$K(r,t) = e^{-r(1-\cos t)} \sin\left(r \sin t + \frac{1}{2} t\right), \tag{5}$$

$$H(r,t) = e^{-r(1-\cos t)} - e^{-r(1-\cos(t+\frac{\pi}{4}))}. \tag{6}$$

3. Some known Results

For Borel means Chandra [1] established the following theorem

Theorem 3.1. *Let $0 \leq p < \alpha < 1$ and let $f \in H_\alpha$. Then*

$$\|B(r,f) - f\|_p = O(r^{p-\alpha} \log r).$$

Next, Padhy et al. [14] estimated the degree of approximation of Fourier series of functions by Borel means in the Besov $B_r^\beta(L_p)$ space and the result is as follows

Theorem 3.2. *Let $0 \leq \beta < \alpha < 2$. If $f \in B_r^\beta(L_p)$, then*

$$\|B(r,f) - f\|_p = O\left(\frac{1}{r^\alpha}\right) + O\left(\frac{1}{r^{\alpha-\beta-p}}\right) + O\left(\frac{1}{r^{\alpha-\beta}}\right).$$

4. Main Result

In the present paper, we have estimated the degree of approximation of functions in the $H_p^{(\omega)}$ space by using Borel's exponential means of Fourier series.

The following lemmas are used to prove our main result.

Lemma 4.1. *Let $\omega(t)$, $\nu(t)$ be integral moduli of continuity such that $\frac{\omega(t)}{\nu(t)}$ is non-decreasing. If $f \in H_p^{(\omega)}$, $p \geq 1$, then*

$$\|F(t) - F(t+h)\|_p = O(1)\left\{\frac{\omega(t)}{\nu(t)}\nu(y) + \frac{\omega(t+h)}{\nu(t+h)}\nu(y)\right\}.$$

Proof. We have

$$\begin{aligned}\|F(t) - F(t+h)\|_p &\leq \|\phi_{x+y}(t) - \phi_x(t)\|_p + \|\phi_{x+y}(t+h) - \phi_x(t+h)\|_p \\ &= \frac{1}{2}\|\{f(x+y+t) + f(x+y-t) - 2f(x+y)\} - \{f(x+t) + f(x-t) - 2f(x)\}\|_p \\ &+ \frac{1}{2}\|\{f(x+y+t+h) + f(x+y-t-h) - 2f(x+y)\} - \{f(x+t+h) + f(x-t-h) - 2f(x)\}\|_p \\ &\leq \frac{1}{2}\Big[\|f(x+y+t) - f(x+y)\|_p + \|f(x+y-t) - f(x+y)\|_p \\ &+ \|f(x+t) - f(x)\|_p + \|f(x-t) - f(x)\|_p\Big] \\ &+ \frac{1}{2}\Big[\|f(x+y+t+h) - f(x+y)\|_p + \|f(x+y-t-h) - f(x+y)\|_p \\ &+ \|f(x+t+h) - f(x)\|_p + \|f(x-t-h) - f(x)\|_p\Big].\end{aligned}$$

Using the definition of integral modulus of continuity and Lemma 5 [2], we obtain

$$\begin{aligned}\|F(t) - F(t+h)\|_p &= O\left(\frac{\omega(t)}{\nu(t)}\nu(y)\right) + O\left(\frac{\omega(t+h)}{\nu(t+h)}\nu(y)\right) \\ &= O(1)\left\{\frac{\omega(t)}{\nu(t)}\nu(y) + \frac{\omega(t+h)}{\nu(t+h)}\nu(y)\right\}.\end{aligned}$$

□

Lemma 4.2. *Let $0 < t \leq \pi$. Then $|K(r,t)| = O(t^{-1})$.*

Proof.

$$\begin{aligned}|K(r,t)| &= \left|e^{-r(1-\cos t)}\sin\left(r\sin t + \frac{1}{2}t\right)\right| \\ &= \left|e^{-r(1-\cos t)}\right|.\left|\sin\left(r\sin t + \frac{1}{2}t\right)\right| \\ &\leq e^{-2r(t/\pi)^2}.1 = O(t^{-1}).\end{aligned}$$

□

Lemma 4.3. $H(r,t) = O\left(re^{-rt^2/5}\right)$

Proof. Using the fact

$$1 - \cos t \geq 2\frac{t^2}{\pi^2} = O\left(\frac{t^2}{5}\right),\ \forall t \in [0,\pi].$$

Implies that

$$e^{-r(1-\cos t)} = O(e^{-rt^2/5}) \tag{7}$$

Now,

$$\begin{aligned} H(r,t) &= \exp\{-r(1-\cos t)\} - \exp\{-r(1-\cos\left(t+\frac{\pi}{r}\right))\} \\ &= \exp\{-r(1-\cos t)\} - \exp\{-r(1-\cos t) + r\cos\left(t+\frac{\pi}{r}\right) - r\cos t\} \\ &= \exp\{-r(1-\cos t)\}[1-\exp\{r\cos\left(t+\frac{\pi}{r}\right) - r\cos t\}]. \end{aligned}$$

By equation 7, we have

$$\begin{aligned} H(r,t) &= O(e^{-rt^2/5})[1-\exp\{r\cos\left(t+\frac{\pi}{r}\right) - r\cos t\}] \\ &= O(re^{-rt^2/5}). \end{aligned}$$

□

Theorem 4.4. *Let $s_n(x)$ be the $n-$th partial sum of the Fourier series associated with the function $f \in H_p^{(\omega)}$. Then the degree of approximation of f by using Borel's exponential means of its Fourier series is*

$$\|\tau_r\|_p^{(\nu)} = \|B_r(x) - f(x)\|_p^{(\nu)} = O(1)\int_{\pi/r}^{\pi} \frac{\omega(t)}{t\nu(t)}\frac{1}{r}dt,$$

where $\omega(t)$, $\nu(t)$ are integral moduli of continuity such that $\frac{\omega(t)}{\nu(t)}$ is non-decreasing.

Proof. It is known that [16]

$$s_n(x) - f(x) = \frac{2}{\pi}\int_1^{\pi} \phi_x(t)\frac{\sin\left(n+\frac{1}{2}\right)}{2\sin\frac{t}{2}}dt. \tag{8}$$

Denoting the Borel's exponential means of the sequence $\{s_n(x)\}$ by $B_r(f;x)$, we have

$$\tau_r(x) = B_r(f;x) - f(x) = e^{-r}\sum_{n=0}^{\infty} s_n(f;x)\frac{r^n}{n!} - f(x).$$

By using (8), we get

$$\begin{aligned} B_r(f;x) - f(x) &= \frac{2}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{2\sin(t/2)} e^{-r(1-\cos t)}\sin\left(r\sin t + \frac{1}{2}t\right)dt \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{\phi_x(t)}{2\sin(t/2)}K(r,t)dt, \end{aligned}$$

where $K(r,t) = e^{-r(1-\cos t)}\sin\left(r\sin t + \frac{1}{2}t\right)$.

Now, $\tau_r(x) - \tau_r(x+y) = \frac{2}{\pi}\int_0^{\pi} \frac{\phi_{x+y}(t) - \phi_x(t)}{2\sin(t/2)}K(r,t)dt.$

By using generalized Minkowski inequality, we obtain

$$\begin{aligned}\|\tau_r(x)-\tau_r(x+y)\|_p &\leq \frac{2}{\pi}\int_0^{\pi}\frac{\|\phi_{x+y}(t)-\phi_x(t)\|_p}{|2\sin(t/2)|}|K(r,t)|dt\\ &= \frac{2}{\pi}\left(\int_0^{\pi/r}+\int_{\pi/r}^{\delta}+\int_{\delta}^{\pi}\right)\frac{\|F(t)\|_p}{2\sin(t/2)}|K(r,t)|dt\\ &= I_1+I_2+I_3. \end{aligned} \tag{9}$$

Now,

$$\begin{aligned} I_1 &= \frac{2}{\pi}\int_0^{\pi/r}\frac{\|F(t)\|_p}{2\sin(t/2)}|K(r,t)|dt\\ &= O(1)\nu(y)\int_0^{\pi/r}\frac{\omega(t)}{\nu(t)}\frac{|K(r,t)|}{|2\sin(t/2)|}dt = O(1)\nu(y)\int_0^{\pi/r}\frac{\omega(t)}{t\nu(t)}t^{-1}dt\\ &= O(1)\nu(y)\int_0^{\pi/r}\frac{\omega(t)}{t^2\nu(t)}dt. \end{aligned} \tag{10}$$

Using Lemma 4.2, we get

$$\begin{aligned} I_3 &= \frac{2}{\pi}\int_{\delta}^{\pi}\frac{\|F(t)\|_p}{2\sin(t/2)}|K(r,t)|dt\\ &= O(1)\nu(y)\int_{\delta}^{\pi}\frac{\omega(t)}{t\nu(t)}.\frac{1}{t}dt\\ &= O(1)\nu(y)\int_{\delta}^{\pi}\frac{\omega(t)}{t^2\nu(t)}. \end{aligned} \tag{11}$$

Using the fact

$$\sin\left(r\sin t+\frac{1}{2}t\right) = \{\sin(r\sin t)-\sin rt\}\cos(t/2)+\sin rt\cos(t/2)+\cos(r\sin t)\sin(t/2),$$

we have

$$\begin{aligned} I_2 &\leq \frac{2}{\pi}\int_{\pi/r}^{\delta}\frac{\|F(t)\|_p}{2\sin(t/2)}e^{-r(1-\cos t)}|\cos(t/2)|.|\sin(r\sin t)-\sin rt|dt\\ &+\frac{2}{\pi}\int_{\pi/r}^{\delta}\|F(t)\|_p\frac{|\cot(t/2)|}{2}e^{-r(1-\cos t)}|\sin rt|\\ &+\frac{1}{\pi}\int_{\pi/r}^{\delta}\|F(t)\|_p|\cos(r\sin t)|e^{-r(1-\cos t)}\\ &= I_{21}+I_{22}+I_{23}. \end{aligned} \tag{12}$$

Now,

$$\begin{aligned} I_{21} &= O(1)\nu(y)\int_{\pi/r}^{\delta}\frac{\omega(t)}{\nu(t)}rt^3.t^{-1}e^{-rt^2/5}dt\\ &= O(1)\nu(y)\int_{\pi/r}^{\delta}\frac{\omega(t)}{\nu(t)}rt^2e^{-rt^2/5}dt. \end{aligned} \tag{13}$$

Similarly,

$$I_{23} = O(1)\nu(y)\int_{\pi/r}^{\delta}\frac{\omega(t)}{\nu(t)}e^{-rt^2/5}dt. \tag{14}$$

We write

$$
\begin{aligned}
I_{22} &= \frac{2}{\pi}\int_{\pi/r}^{\delta} \|F(t)\|_p \frac{|t^{-1} + \cot(t/2) - t^{-1}|}{2} e^{-r(1-\cos t)} |\sin rt| \\
&\le \frac{2}{\pi}\int_{\pi/r}^{\delta} \|F(t)\|_p t^{-1} e^{-r(1-\cos t)} |\sin rt| dt \\
&+ \frac{2}{\pi}\int_{\pi/r}^{\delta} \|F(t)\|_p \left|\frac{\cot(t/2)}{2} - \frac{1}{t}\right| e^{-r(1-\cos t)} |\sin rt| dt \\
&= L_1 + L_2. \qquad (15)
\end{aligned}
$$

Now,

$$
\begin{aligned}
L_2 &= O(1)\nu(y)\int_{\pi/r}^{\delta} \frac{\omega(t)}{\nu(t)} e^{-rt^2/5} \left|\frac{\cot(t/2)}{2} - \frac{1}{t}\right| dt \\
&= O(1)\nu(y)\int_{\pi/r}^{\delta} \frac{\omega(t)}{\nu(t)} t e^{-rt^2/5} dt. \qquad (16)
\end{aligned}
$$

Putting $\frac{\pi}{r} = h$ in L_1, we get

$$
\begin{aligned}
L_1 &= \frac{2}{\pi}\int_{h}^{\delta} \left\{ \frac{\|F(t)\|_p}{t} e^{-r(1-\cos t)} - \frac{\|F(t+h)\|_p}{t+h} e^{-r(1-\cos t)} \right\} |\sin rt| dt \\
&+ \frac{2}{\pi}\int_{h}^{\delta} \|F(t+h)\|_p \left\{ \frac{1}{t} - \frac{1}{t+h} \right\} e^{-r(1-\cos t)} |\sin rt| dt \\
&+ \frac{2}{\pi}\int_{h}^{\delta} \frac{\|F(t+h)\|_p}{t+h} H(r,t) |\sin rt| dt \\
&+ \frac{2}{\pi}\int_{\delta-h}^{\delta} \frac{\|F(t+h)\|_p}{t+h} e^{-r(1-\cos(t+h))} |\sin rt| dt \qquad (17) \\
&- \frac{2}{\pi}\int_{0}^{h} \frac{\|F(t+h)\|_p}{t+h} e^{-r(1-\cos(t+h))} |\sin rt| dt \\
&= R_1 + R_2 + R_3 + R_4 - R_5. \qquad (18)
\end{aligned}
$$

We write

$$
\begin{aligned}
R_1 &= \left\{ \int_{\pi/r}^{\frac{\log r}{\sqrt{r}}} + \int_{\frac{\log r}{\sqrt{r}}}^{\delta} \right\} \frac{\|F(t) - F(t+\frac{\pi}{r})\|_p}{t} e^{-r(1-\cos t)} |\sin rt| dt \\
&= \theta_1 + \theta_2. \qquad (19)
\end{aligned}
$$

Using Lemma 4.1, it is easy to estimate

$$
\theta_1 = O(1)\nu(y)\int_{\pi/r}^{\frac{\log r}{\sqrt{r}}} \left\{ \frac{\omega(t)}{\nu(t)} + \frac{\omega(t+\frac{\pi}{r})}{\nu(t+\frac{\pi}{r})} \right\} \frac{e^{-rt^2/5}}{t} dt, \qquad (20)
$$

$$
\theta_2 = O(1)\nu(y)\int_{\frac{\log r}{\sqrt{r}}}^{\delta} \left\{ \frac{\omega(t)}{\nu(t)} + \frac{\omega(t+\frac{\pi}{r})}{\nu(t+\frac{\pi}{r})} \right\} \frac{e^{-rt^2/5}}{t} dt. \qquad (21)
$$

Hence, (19) becomes

$$
R_1 = O(1)\nu(y)\int_{\frac{\log r}{\sqrt{r}}}^{\delta} \left\{ \frac{\omega(t)}{\nu(t)} + \frac{\omega(t+\frac{\pi}{r})}{\nu(t+\frac{\pi}{r})} \right\} \frac{e^{-rt^2/5}}{t} dt. \qquad (22)
$$

By Lemma 4.1 and using the fact $\frac{\omega(t)}{\nu(t)}$ is non-decreasing, we have

$$\begin{aligned} R_2 &= O(1)\nu(y)\int_h^\delta \left\{\frac{\omega(t)}{\nu(t)} + \frac{\omega(t+h)}{\nu(t+h)}\right\} e^{-r(1-\cos(t+h))}|\sin rt|dt \\ &= O(1)\nu(y)\int_h^\delta \frac{\omega(t+h)}{\nu(t+h)}\frac{1}{t(t+h)}dt. \end{aligned} \tag{23}$$

Again by using the same arguments we obtain

$$R_4 = O(1)\nu(y)\int_{\delta-h}^\delta \frac{\omega(t+h)}{(t+h).\nu(t+h)}r^{-1}dt, \tag{24}$$

and

$$R_5 = O(1)\nu(y)\int_0^h \frac{\omega(t+h)}{(t+h).\nu(t+h)}r^{-1}dt. \tag{25}$$

By splitting R_3 like L_1 and using Lemma 4.3, we obtain the following estimation

$$R_3 = O(1)\nu(y)\int_h^\delta \frac{\omega(t+h)}{\nu(t+h)}e^{-rt^2/5}dt. \tag{26}$$

By collecting the estimations, we have

$$I_2 = O(1)\nu(y)\int_{\pi/r}^\delta \frac{\omega(t)}{t\nu(t)}r^{-1}dt. \tag{27}$$

From (10), (27) and (11), we get

$$\|\tau_r(x) - \tau_r(x+y)\|_p = O(1)\nu(y)\int_{\pi/r}^\pi \frac{\omega(t)}{rt\nu(t)}dt.$$

That implies

$$\begin{aligned} A(f,\nu) &= \sup_{y\neq 0}\frac{\|\tau_r(x+y) - \tau_r(x)\|_p}{\nu(y)} \\ &= O(1)\int_{\pi/r}^\pi \frac{\omega(t)}{rt\nu(t)}dt. \end{aligned} \tag{28}$$

By proceeding as above, we obtain

$$\|\tau_r\|_p = O\left(\int_{\pi/r}^\pi \frac{\omega(t)}{\nu(t)}(rt)^{-1}dt\right). \tag{29}$$

Therefore,

$$\begin{aligned} \|\tau_r\|_p^{(\nu)} &= \|\tau_r\|_p + A(f,\nu) \\ &= O\left(\int_{\pi/r}^\pi \frac{\omega(t)}{t\nu(t)}.\frac{1}{r}dt\right). \end{aligned}$$

This completes the proof. □

Remark 4.5. *If we do the above estimation by using "ess* sup *norm" instead of "$p-$ norm", we get the degree of approximation for $H^{(\omega)}$ class of functions.*

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References

[1] P. Chandra, *Degree of Approximation of Functions in the Hölder Metric by Borel's Means*, J. Math. Anal. Appl., **149** (1990), 236–248.
[2] G. Das, A. Nath, B. K. Ray, *A new trigonometric method of summation and its application to the degree of approximation*, Ind. Acad. Sci., **112**(2) (2002), 299–319.
[3] G. Das, L. Nayak, B. K. Ray, *An Estimate of the Rate of Convergence of the Fourier Series in the Generalized Hölder Metric by Delayed Arithmetic Mean*, Int. journal of Anal., (2014).
[4] G. Das, A. K. Ojha, B. K. Ray, *Degree of Approximation of Functions Associated with Hardy-Littlewood Series in the Hölder Metric by Borel Means*, J. Math. Anal. Appl., **219** (1998), 279–293.
[5] G.H. Hardy, *Divergent Series*, Clarendon, Oxford, 1949.
[6] B. B. Jena, S. K. Paikray, M. Mursaleen, *On the degree of approximation of Fourier series based on a certain class of product deferred summability means*, J. Inequal. Appl., **18** (2023).
[7] Xh. Z. Krasniqi, *On the degree of approximation by Fourier series of functions from the Banach space H_p^{ω}, $p \geq 1$, in Generalized Hölder metric*, International Mathematical Forum, **6**(13) (2011), 613–625.
[8] Xh. Z. Krasniqi, Lenski, B. Szal,*Seminormed approximation by deferred matrix means of integrable functions in $H_p^{(\omega)}$ space*, Results in Mathematics, **77** (2022).
[9] S. Lal, Shiree, *Approximation of functions by Matrix-Euler summability means of Fourier Series in Genaeralized Hölder Metric*, Bull. Math. Anal. Appl., **4**(3) (2012), 12–22.
[10] H.K. Nigam , S. Rani, *Approximation of conjugate of a function in Genaeralized Hölder class*, Journal of Mathematical and Computational Science, **10**(5) (2020), 1851–1866.
[11] H.K. Nigam, Md. Hadish, *Best approximation of functions in generalized Hölder class*, J. Inequal. Appl., **276** (2018).
[12] H. K. Nigam, M. Mursaleen, M. K. Sah, *Degree of convergence of function in generalized Zygmund space*, Mathematical Foundations of Computing, **6** (2023), 484–499.
[13] H. K. Nigam, M. Mursaleen, S. Rani, *Approximation of functions in generalized Zygmund class by double Hausdorff matrix*, Adv. Differential Equations, **317** (2020).
[14] B.P. Padhy,A. Mishra, S. Nanda, *Approximation of Fourier series of functions in Besov space by Borel means*, J. Indian Math. Soc., **89**(3) (2023), 373–385.
[15] S. S. Volosivets, *Modified modulus of smoothness and approximation in weighted Lorentz spaces by Borel snd Euler means*, Probl. Anal. Issues Anal., **10**(1) (2021), 87–100.
[16] A. Zygmund, *Trigonometric Series*, Vol.1, Cambridge Univ. Press, New York, 1968.