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Uniform structure with iterated function system, step skew product and their uniform entropy

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Abstract. Let $F : \sum_m^+ \times X \to \sum_m^+ \times X$ by $(\omega, x) \mapsto (\sigma_m(\omega), f_{\omega_0}(x))$, be skew product of IFS $\mathcal{F} = \{f_i : X \to X | i = 0, 1, \dots, m-1\}$ on uniform space $(X, \mathcal{U}_X)$. In this paper, we prove that the equivalence of the chain transitive, topologically transitive and topological shadowing property between IFS $\mathcal{F}$ and step skew product F. Moreover, we give two version of entropy, uniform entropy and uniform covering entropy, for IFS $\mathcal{F}$ on uniform space $(X, \mathcal{U}_X)$, and prove that the basic properties of them. Finally, we show that $h_u(F) = \log m + h_u(\mathcal{F})$, where h_u is uniform entropy.

1. Introduction

In topological dynamics, which studies the behavior of continuous transformation on topologica spaces, topological entropy and topological transitivity are fundamental tools for understanding the behavior of a dynamical system.
Topological entropy measures the complexity of a dynamical system, while topological transitivity describes the behavior of the system. There are spaces that every topologically transitive map on them, have necessary positive topological entropy. For instant by [16, Theorem 9.1], every transitive map on interval $[0, 1]$ has topological entropy atleast $\frac{\log 2}{2}$, also, if an interval map f has positive topological entropy, then it is transitive on a closed invariant set which has no isolated point. Topological entropy can be defined in different ways, dependeing on the type of system and tools used to analyze it. Adler, Konhelm and McAndrew [1] used the idea of open covers of a compact topological space with continuous mapping . Another definition uses the idea of metrics which are functions that measure the distance between points in the space [5]. However, for general topological spaces such distance-or size-related concepts cannot be defined unless there exists some kind of structure in addition to what the topology itself provides. This can be solved by considering a completely regular, and not necessarily metrizable, topological space which equipped with a structure, called a uniformity, enabling one to control the distance between points in the space and generalize notion of topological entropy on uniform space which is called iniform entropy. Uniform entropy is defined in

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term of uniform cover of a space. Bowen [5] defined uniform entropy of a uniformly continuous self map of a metric space, that coincides with the topological entropy when the metric space is compact. This approach was later extended by Hood [11] and Hofer [12] for uniform spaces. The relation between uniform entropy and topological entropy is discussed in [7]. Although, there is a relation between positive topological entropy and topological transitivity of systems on interval $[0, 1]$, but transitivity is a global characteristics. A map having two invariant sets A, B with nonempty interiors cannot be transitive. But the positivity of the entropy on A implies that the system does have positive topological entropy. Topological transitivity is strongly of chain transitive, this means that in transitive systems every two point of space can be chained by pseudo-orbit. If the system does have shadowing property, then the notions of chain transitivity and topological transitivity are coincide. Note that the shadowing property is a fundamental concept in the theory of dynamical systems. It describes how well an approximate trajectory can be shadowed by an exact trajectory. In [17], Author gave a characterization of zero topological entropy maps which have the shadowing property.

A family $\{f_\sigma : \sigma \in \Gamma\} \subseteq C(X)$ such that Λ is a finite set, is said to be an Iterated Function System (IFS for short). This notion introduced by Hutchinson to study hyperbolic IFS. Many authors generalized notion of classical dynamical system to IFS, see notions of transitivity, mixing, ergodicity in [10, 15], various types shadowing properties in [4, 9, 20], some criteria for transitivity and accessibility in [18, 19]

Compositions of uniformly continuous maps $f_0, \ldots, f_k$ on X can be studied in a single framework given by a skew product system

$$\mathrm{F}(\omega, x) = (\sigma_m(\omega), f_{\omega_0}(x)), \quad \forall(\omega = \omega_0\omega_1\omega_2\cdots, x) \in \Sigma_m^+ \times X.$$

Bahabadi [4] obtained the equivalence of the shadowing property between IFS F and step skew product F. Also, in [23], authors showed that this also holds for chain transitivity and chain mixing.
In this paper we study notions of shadowing property, topological entropy, chain transitive and chain mixing for an IFS on a uniform space. Indeed, We define step skew product and equip it with a uniform structure and then we prove the inherited properties of this mapping with that uniform structure, including the topologically chain transitive, topologically chain mixing, topologically shadowing property and transitive. Moreover, we extend Hood's definition of uniform entropy from a single self-mapping to a finite number of self-mappings(IFS).

2. Preliminaries

For non-empty set X, take $\Delta_X = \{(x, x) : x \in X\}$, which is called the diagonalof $X \times X$. For $\mathscr{A} \subset X \times X$ consider $\mathscr{A}^{-1} = \{(b, a) : (a, b) \in \mathscr{A}\}$. We say $\mathscr{A}$ is *symmetric*, if $\mathscr{A} = \mathscr{A}^{-1}$. For any $\mathscr{A}, \mathscr{B} \subset X \times X$, *composite* $\mathscr{A}$ and $\mathscr{B}$ is

$$\mathscr{A} \circ \mathscr{B} = \{(a, b) : \exists\, (a, x) \in \mathscr{B}, (x, b) \in \mathscr{A}\}. \tag{1}$$

Definition 2.1. *[14, p. 176] A uniform structure on X is a non-empty collection $\mathcal{U}_X$ of subsets of $X \times X$ satisfying the following:*
(U1) If $\mathscr{A} \in \mathcal{U}_X$, then $\mathscr{A}^{-1} \in \mathcal{U}_X$ and $\Delta_X \subset \mathscr{A}$;
(U2) If $\mathscr{A} \in \mathcal{U}_X$ and $\mathscr{A} \subset \mathscr{B} \subset X \times X$, then $\mathscr{B} \in \mathcal{U}_X$;
(U3) If $\mathscr{A}, \mathscr{B} \in \mathcal{U}_X$, then $\mathscr{A} \cap \mathscr{B} \in \mathcal{U}_X$;
(U4) For any $\mathscr{A} \in \mathcal{U}_X$, there exists a $\mathscr{B} \in \mathcal{U}_X$ such that $\mathscr{B}\ o\ \mathscr{B} \subset \mathscr{A}$.

The pair $(X, \mathcal{U}_X)$ is called a *uniform space* and the members of $\mathcal{U}_X$ are called *entourages*. The largest uniformity on X, *discrete uniformity*, is the family of all those subsets $X \times X$ which contains the diagonal, and the smallest uniformity on X, *trivial uniformity*, is the set $X \times X$.

Definition 2.2. *[14, p. 180] The map $f : (X, \mathcal{U}_X) \to (Y, \mathcal{U}_Y)$ is uniformly continuous if $(f \times f)^{-1}(\mathscr{U}) \in \mathcal{U}_X$ for all $\mathscr{U} \in \mathcal{U}_X$.*

Every uniformity $\mathcal{U}_X$ on X induces a topology $\mathcal{T}_{\mathcal{U}}$ as follows: A subset $T \subseteq X$ belongs to $\mathcal{T}_{\mathcal{U}}$ if and only if for every $x \in T$ there is $\mathscr{U} \in \mathcal{U}$ such that $\mathscr{U}[x] \subseteq T$ where $\mathscr{U}[x] = \{y \in X : (x, y) \in \mathscr{U}\}$.

Remark 2.3. *Different two-uniformity may create equal topology, and there are topologies of space that never obtained by uniformity except those that are completely regular [14, Corollary 6.17].*

Definition 2.4. *If* $\{(X_i, \mathcal{U}_i)\}_{i\in\mathbb{N}}$ *is the family of uniform spaces then the product uniformity for* $\prod X_i$ *is the smallest uniformity such that projection into each coordinate space is uniformly continuous.*
The family of all sets of the form $V=\{(x, y); x=(a_1, a_2, \cdots), y=(b_1, b_2, \cdots):(a_i, b_i)$ *all elements of a* $\mathscr{V} \in \mathcal{U}_i\}$ *is a subbase for the product uniformity.*

In the next two examples, we define uniform structures on two finite sets and then obtain the subbasis of the product uniformity on them. The reason that the finite structures are discussed is that in the Section 2.1, theorems will be presented in this field.

Example 2.5. *Let* $X = \{a, b, c, d\}$ *be a set , then one of the fifteen(The reason for this number is in the Subsection 2.2) uniform structures on X is* $\mathcal{U}_X = \{\mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3, \cdots, \mathscr{U}_{256}\}$ *where*

$$\mathscr{U}_1 = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\},$$
$$\mathscr{U}_2 = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c), (a, c)\},$$
$$\mathscr{U}_3 = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c), (c, a)\},$$
$$\vdots$$

the remaining members are added until finally, $\mathscr{U}_{256} = X \times X$.
Let $g : (X, \mathcal{U}_X) \to (X, \mathcal{U}_X)$ *that* $g(a) := c, g(b) := d, g(c) := a, g(d) := b$ *and* $h : (X, \mathcal{U}_X) \to (X, \mathcal{U}_X)$ *that* $h(a) := b, h(b) := a, h(c) := d, h(d) := c$ *is uniformly continuous but* $f : (X, \mathcal{U}_X) \to (X, \mathcal{U}_X)$ *that* $f(a) := b, f(b) := c, f(c) := d, f(d) := a$ *because* $(f \times f)^{-1}(\mathscr{U}_1) = \{(d, d), (a, a), (b, b), (c, c), (d, a), (a, d), (b, c), (c, b)\}$ *is not element of* $\mathcal{U}_X$ *hence f is not uniformly continuous.*
For uniform space $(X, \mathcal{U}_X)$ *we have* $\mathscr{U}_1[a] = \{a, b\}, \mathscr{U}_1[b] = \{a, b\}, \mathscr{U}_1[c] = \{c, d\}, \mathscr{U}_1[d] = \{c, d\}$ *and* $\mathscr{U}_2[a] = \{a, b, c\}$,- $\mathscr{U}_2[b] = \{a, b\}, \mathscr{U}_2[c] = \{c, d\}, \mathscr{U}_2[d] = \{c, d\}$ *and so on. Therefore* $\mathcal{T}_{\mathcal{U}} = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$.

Example 2.6. *Let* $Y = \{1, 2\}$ *be a set. Then one of the two uniform structures on X is* $\mathcal{U}_Y = \{\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \mathscr{V}_4\}$ *where*

$$\mathscr{V}_1 = \{(1, 1), (2, 2)\}, \qquad \mathscr{V}_2 = \{(1, 1), (2, 2), (1, 2)\},$$
$$\mathscr{V}_3 = \{(1, 1), (2, 2), (2, 1)\}, \quad \mathscr{V}_4 = \{(1, 1), (2, 2), (1, 2), (2, 1)\}.$$

For uniform space $(Y, \mathcal{U}_Y)$ *if we define any map* $f : (Y, \mathcal{U}_Y) \to (Y, \mathcal{U}_Y)$, *then f is always uniformly continuous. For uniform space* $(Y, \mathcal{U}_Y)$ *we have* $\mathscr{V}_1[1] = \{1\}, \mathscr{V}_1[2] = \{2\}$ *and* $\mathscr{V}_2[1] = \{1, 2\}, \mathscr{V}_2[2] = \{2\}$ *and so on. Therefore* $\mathcal{T}_{\mathcal{U}} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Example 2.7. *If* $\mathcal{U}_X, \mathcal{U}_Y$ *are uniform structures defined in the Examples 2.5 and 2.6, then according to the Definition 2.4 the subbase of* $\mathcal{U}_X \times \mathcal{U}_Y$ *is*
$\mathfrak{B} = \{V_{\mathscr{U}_1,\mathscr{V}_1}, V_{\mathscr{U}_2,\mathscr{V}_1}, V_{\mathscr{U}_3,\mathscr{V}_1}, \ldots, V_{\mathscr{U}_1,\mathscr{V}_2}, V_{\mathscr{U}_2,\mathscr{V}_2}, V_{\mathscr{U}_3,\mathscr{V}_2}, \ldots, V_{\mathscr{U}_{256},\mathscr{V}_4}\}$ *where*
$V_{\mathscr{U}_1,\mathscr{V}_1} = \{((a, 1), (a, 1)), ((b, 1), (b, 1)), ((c, 1), (c, 1)), ..., ((a, 2), (a, 2)), \ldots\}$,
$V_{\mathscr{U}_2,\mathscr{V}_1} = \{((a, 1), (a, 1)), ((b, 1), (b, 1)), ((c, 1), (c, 1)), ..., ((a, 1), (c, 1)), \ldots\}$,
and so on.

In the following, we give definition of IFS which introduced by Hutchinson in [13].

Definition 2.8. *Let* $(X, \mathcal{U}_X)$ *be a nontrivial uniform space,* $\Gamma=\{0, 1, ..., m-1\}$. *If* $\mathcal{F}=\{f_\sigma:X \to X; \sigma \in \Gamma\}$ *that any* $f_\sigma:X \to X$ *be a uniformly continuous map, then system* $(X, \mathcal{U}_X, \mathcal{F})$ *is an iterated function system on uniform space which we write as* $\mathcal{IFS}_u$ *in short, of course, until the initial space does not change.*

We denote $\mathbb{Z}^+ = \{0, 1, 2, ...\}$, $\Sigma_m^+ = \Gamma^{\mathbb{Z}^+}$(*Symbolic space*) and $\omega = \omega_0\omega_1\omega_2\cdots \in \Sigma_m^+$(*Word*) and define

$$f_\omega^0 := id_X, \qquad f_\omega^n = f^n_{\omega_0\omega_1\omega_2\cdots} := f_{\omega_{n-1}} \circ f_\omega^{n-1} \quad (n \in \mathbb{N}).$$

Let $\mathscr{A} \in \mathcal{U}_X$. We say that there is an $\mathscr{A}$-chain from a to b in X if there exists a finite sequence $\{a_k\}_{k=0}^{m}$ such that $a_0=a \ldots,\ a_m=b$ and $(f_{\sigma_k}(a_k), a_{k+1}) \in \mathscr{A}$ for every $k \in \Gamma$ and $\sigma_k \in \Gamma$. A sequence $\{x_k\}_{k\geq 0}$ is an ($\mathscr{A}$, ω)*-pseudo-orbit* of $\mathcal{F}$ if $\left(f_{\omega_k}(x_k), x_{k+1}\right) \in \mathscr{A}$. An ($\mathscr{A}$, ω)-pseudo-orbit to be $\mathscr{B}$*-traced* by a point $y \in X$ if $(f_{\omega}^{k}(y), x_k) \in \mathscr{B}$.

Definition 2.9. *The* $\mathcal{IFS}_u$ *:*

- *is topologically chain transitive, if for any* $\mathscr{A} \in \mathcal{U}_X$ *and for any* $a, b \in X$*, there exists an* $\mathscr{A}$*-chain* $\{a_k\}_{k=0}^{m}$ *from* a *to* b*.*
- *is topologically chain mixing , if for any* $\mathscr{A} \in \mathcal{U}_X$ *and for any* $a, b \in X$ *there exists* $N \in \mathbb{N}$ *such that for any* $n \geq N$*, there is an* $\mathscr{A}$*-chain* $\{a_k\}_{k=0}^{n}$ *from* a *to* b *.*
- *is topologically transitive, if for any pair of nonempty open sets* $O_1, O_2 \subseteq X$*, there exists an* $\omega \in \Sigma_m^+$ *and* $n \in \mathbb{N}$ *such that* $f_{\omega}^{n}(O_1) \cap O_2 \neq \varnothing$*.*
- *has topological shadowing property if for any* $\mathscr{B} \in \mathcal{U}_X$*, there exists an* $\mathscr{A} \in \mathcal{U}_X$*,* $\omega \in \Sigma_m^+$ *such that every (*$\mathscr{A}$*,* ω*)-pseudo-orbit to be* $\mathscr{B}$*-traced by some point* $y \in X$*.*

In the Example 2.5, we difined two uniformity continuous maps. In next example we examine their topologically chain transitive.

Example 2.10. *System* $(X, \mathcal{U}_X, h)$ *where* $h{:}(X, \mathcal{U}_X) \to (X, \mathcal{U}_X)$ *such that* $h(a) := b, h(b) := a, h(c) := d, h(d) := c$*, is not topologically chain transitive because from a to c there is not* $\mathscr{U}_1$*-chain but system* $(X, \mathcal{U}_X, \mathcal{F})$ *that* $\mathcal{F} = \{g, h\}$ *as an* $\mathcal{IFS}_u$ *is topologically chain transitive*

Remark 2.11. *Any metric space X with the metric d can be endowed by a uniform structure. The family* $\mathcal{U}_d = \{\mathscr{U} \subset X \times X : \exists r > 0; U_r \subset \mathscr{U}\}$ *that* $U_r = \{(x, y) : d(x, y) < r \text{ for } r > 0\}$ *is satisfied all condition of uniformity. Different two-metric may create equal uniformity[14, p. 184].*

Example 2.12. *Denote by* Σ_2^+*, the set of all possible sequences of 0's and 1's. A word or "code" in this space is therefore an infinite sequence of the form* $s = (s_0 s_1 s_2 \cdots)$*. let* $s = (s_0 s_1 s_2 \cdots)$ *and* $t = (t_0 t_1 t_2 \cdots)$ *be points in* Σ_2^+*. A distance function or metric on* Σ_2^+ *is:*

$$d(s,t) = \begin{cases} 0 & s = t \\ \frac{1}{\min\{i \in \mathbb{Z}^+ : s_i \neq t_i\} + 1} & s \neq t. \end{cases}$$

According to the Remark 2.11 we can define $\mathcal{U}_{\Sigma_2^+} = \left\{\mathscr{U} \subset \Sigma_2^+ \times \Sigma_2^+; \exists r > 0 : U_r \subset \mathscr{U}\right\}$ *as an uniformity on the* Σ_2^+*. Of course, this definition can be generalized to any* Σ_m^+*.*

We difine the *shift map* $\sigma_2 : \Sigma_2^+ \to \Sigma_2^+$ by $\sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 s_3 \cdots)$.

Remark 2.13. *The shift map* $\sigma_m : \Sigma_m^+ \to \Sigma_m^+$ *is uniformly continuous.*

Indeed if $\mathscr{U} \in \mathcal{U}_{\Sigma_m^+}$ so there exist $r > 0$ that $U_r \subset \mathscr{U}$, then $\sigma_m^{-1}(\mathscr{U})$ Adds a member to the beginning of the sequence for each point, so the distance between the two points either stays the same or decreases.Thus $U_r \subset \sigma_m^{-1}(\mathscr{U}) \in \mathcal{U}_{\Sigma_m^+}$.

2.1. The view of set theory

We know a *relation* R on a set X is a subset of $X \times X$ and use the notation aRb to denote that $(a, b) \in R$. Therefore an uniform structure on X is a collection of relationes on X. A relation on X is called *reflexive* if every element of X is related to itself, so by U1 of the Definition 2.1 all relationes of an uniform structure are reflexive. A relation on X is called *symmetric* if bRa whenever aRb and it is called *transitive* if aRb, bRc, then aRc.

A relation on a set X is called an *equivalence relation* if it is reflexive, symmetric and transitive. Two elements

a and b that are related by an equivalence relation are called equivalent and denote by $a \sim b$. The set of all elements that are related to an element a of X is called the *equivalence class* of a and denote by $[a]$. Let R, S be two relation on X. The *composite* of two relations is the same as the Definition 1. The powers R^n, $n = 1, 2, 3, ...$ are difined recursively by

$$R^1 = R \quad and \quad R^{n+1} = R^n oR.$$

We note that if R is reflexive, then $R \subseteq RoR$.

Remark 2.14. *The relation R on a set X is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, ...$ [21, Theorem 1.6.1].*

2.2. Finite uniform structure

When the number of entourages of a uniform structure is finite, we obtain results that lead to counting the number of possible structures on a finite set.

Theorem 2.15. *If $\mathcal{U}_X$ is a finite uniform structure on X, then there existes $\mathscr{A} \in \mathcal{U}_X$ such that $\mathscr{A} \subseteq \mathscr{B}$ for all $\mathscr{B} \in \mathcal{U}_X$ and $\mathscr{A}$ is an equivalence relation.*

Proof. We have $\mathscr{V} = \mathscr{U} \cap \mathscr{U}^{-1} \in \mathcal{U}_X$, so $\mathscr{V}$ having two properties symmetric and reflexive. If we subscribe all of them, we will reach a relation $\mathscr{A}$ that by repeating property U4, must be $\mathscr{A}^n \subset \mathscr{A} (n = 1, 2, ...)$. According to the Remark 2.14, relation $\mathscr{A}$ is also transitive and therefore it is an equivalence relation. □

From the above theorem, the following results are immediately obtained one after the other:

- The number of uniform structures on a finite set X is equal to the number of equivalence relations on the same set.
- Definition of uniform structure on finite set X : Family of all relations $\mathscr{U}$ on X that $\mathscr{A} \subseteq \mathscr{U}$ where $\mathscr{A}$ is an equivalence relation.
  A *partition* of a set X is a collection of disjoint nonempty subsets of X that have X their union. We know that union of the equivalence classes of X is all of X and these equivalence classes are either equal or disjoint. So an equivalence relation on a set X, partitions the set X and conversely. Therefore, the number of equivalence relations is equal to the number of partitions of a set. *Bell number* [3, Theorem 5.6] is the total number of partitions of a set with n members,

  $$B(n) = \Sigma_{k=0}^{n-1} \binom{n-1}{k} B(k) \quad \text{for all } n \geq 1, \qquad B(0) = 1.$$

- Number of uniform structure on a set with n members equals of $B(n)$.
- Let f be a map(relation) on the finite space $(X, \mathcal{U}_X)$. f is uniformly continuous if and only if it has one of the following conditions :
  1. All members to be related only with one member.(Fixed map).
  2. The members of each equivalence class to be related with the members of the same class.
  3. The members of each equivalence class to be related only to the members of one another class, so that no class remains unrelated.
  4. A combination of type 2 and 3

  And this map is a topologically chain transitive when defined in state 1 or 3.

Look at the Example 2.10 once again from this point of view.

3. Step skew product

Consider product uniformity $\mathcal{U} = \mathcal{U}_{\Sigma_m^+} \times \mathcal{U}_X$ on $\Sigma_m^+ \times X$ where $\mathcal{U}_{\Sigma_m^+}$ is the uniform structure defined using the metric of this space (Ex mple 2.12) and $\mathcal{U}_X$ is uniform structure on X.

The *step skew product* $\mathrm{F} : \Sigma_m^+ \times X \to \Sigma_m^+ \times X$ corresponding to $\mathcal{IFS}_u$ defined by

$$\mathrm{F}(\omega, x) = (\sigma_m(\omega), f_{\omega_0}(x)), \quad \forall(\omega = \omega_0\omega_1\omega_2 \cdots, x) \in \Sigma_m^+ \times X.$$

Remark 3.1. *According to the Proposition 2.13 and uniform continuity f, The step skew product F corresponding to the $\mathcal{IFS}_u$ is uniformly continuous.*

In this section, we study the equivalence of the chain transitive, topologically transitive and topological shadowing property between $\mathcal{IFS}_u$ and step skew product F.

The following theorem generalizes [23, Theorem 2.1] from metric spaces to uniform spaces.

Theorem 3.2. *The step skew product* F *corresponding to the $\mathcal{IFS}_u$ is topologically chain transitive if and only if $\mathcal{IFS}_u$ is topologically chain transitive.*

Proof. First, we assume that $\mathcal{IFS}_u$ is topologically chain transitive. Fix $\mathscr{A} \in \mathcal{U}$ and $\left(\omega^1, x_1\right), \left(\omega^2, x_2\right) \in \Sigma_m^+ \times X$. According to the Remark 2.11 and product uniformity, there exists $r > 0$, $\mathscr{A}' \in \mathcal{U}_X$ such that $U_r \times \mathscr{A}' \subseteq \mathscr{A}$ where $U_r = \{(s,t) \in \Sigma_m^+ \times \Sigma_m^+ : d(s,t) < r\}$. Let $k \in \mathbb{N}$ that $\frac{1}{k} < \frac{r}{2}$. Because $\mathcal{IFS}_u$ is topologically chain transitive so there is $\mathscr{A}'$-chain $\{z_i\}_{i=0}^m$ from $f_{\omega^1}^{k+1}(x_1)$ to x_2. This means that there is $\xi = \xi_0\xi_1 \ldots \xi_{m-1}$ such that $(f_{\xi_i}(z_i),\ z_{i+1}) \in \mathscr{A}'$. Consider $\omega = \omega_0^1\omega_1^1\omega_2^1 \ldots \omega_k^1\xi_0\xi_1 \ldots \xi_{m-1}\omega^2$ and

$$\varphi_i = \begin{cases} (\omega^1, x_1) & i = 0 \\ \left(\sigma_m^i(\omega),\ f_\omega^i(x_1)\right) & 1 \le i \le k+1 \\ \left(\sigma_m^i(\omega),\ z_{i-k-1}\right) & k+2 \le i \le k+m. \end{cases}$$

We claim $\{\varphi_i\}_{i=0}^m$ is the same $\mathscr{A}$-chain desired from $\left(\omega^1, x_1\right)$ to $\left(\omega^2, x_2\right)$.
For $i = 0$ we have

$$(\mathrm{F}(\varphi_0),\ \varphi_1) = \left(\mathrm{F}(\omega^1, x_1), \left(\sigma_m(\omega), f_\omega^1(x_1)\right)\right) = \left(\left(\sigma_m(\omega^1), f_{\omega_0^1}(x_1)\right),\ \left(\sigma_m(\omega), f_{\omega_0^1}(x_1)\right)\right) \in \mathscr{A}.$$

For $1 \le i \le k+1$ we have $\mathrm{F}(\varphi_i) = \varphi_{i+1}$.
For $1 \ge k+2$,

$$(\mathrm{F}(\varphi_i),\ \varphi_{i+1}) = \left(\mathrm{F}(\sigma_m^i(\omega),\ z_{i-k-1}),\ \left(\sigma_m^{i+1}(\omega),\ z_{i-k}\right)\right) = \left(\left(\sigma_m^{i+1}(\omega)), f_{\xi_{i-k-1}}(z_{i-k-1})\right),\ \left(\sigma_m^{i+1}(\omega),\ z_{i-k}\right)\right) \in \mathscr{A}.$$

Conversely, for any two arbitrary points of the space X and a constant entourage of $\mathcal{U}_X$, we can easily reach the members of at least one entourage of $\mathcal{U}$ by product uniformity an arbitrary U_r and that constant entourage. Then from existence of topologically chain transitive in this space, we get the desired chain from the second components of this chain. □

The following theorem generalizes [23, Theorem 2.3], from metric spaces to uniform spaces.

Theorem 3.3. *The step skew product* F *corresponding to the $\mathcal{IFS}_u$ is topologically chain mixing if and only if $\mathcal{IFS}_u$ is topologically chain mixing.*

Proof. According to the process of proving Theorem 3.2 and the relationship between the length of chains in the two spaces X and $\Sigma_m^+ \times X$, equivalence of topologically chain mixing is easily obtained. □

Definition 3.4. *A map $f : X \to X$ is semi-open if for any non-empty open subset U of X , $f(U)$ has non-empty interior.*

The following theorem generalizes [23, Theorem 2.4] from metric spaces to uniform spaces.

Theorem 3.5. *Assuming that $f_\sigma:X \to X$ ($\sigma \in \Gamma$) be a semi-open, the step skew product* F *corresponding to the $\mathcal{IFS}_u$ is transitive if and only if $\mathcal{IFS}_u$ is transitive.*

Proof. First, we assume that $\mathcal{IFS}_u$ is transitive. Fix U, $V \in \Sigma_m^+ \times X$. If $(\omega, x) \in U$, $(\xi, y) \in V$, then there exists $k \in \mathbb{N}$ such that $B(\omega, 1/k) \times U_x \subset U$ and $B(\xi, 1/k) \times V_y \subset V$. As f_σ are semi-open, then $f_\omega^{k+1}(U_x)$ has non-empty interior so, there is $U_{x'} \in X$ such that $f_\omega^{k+1}(U_x) \in U_{x'}$. According to the assumption, there is $\upsilon = \upsilon_0\upsilon_1\upsilon_2\dots\upsilon_n$ such that $f_\upsilon^{n+1}(U_{x'}) \cap V_y \neq \emptyset$. We define $\omega' = \omega_0\omega_1\dots\omega_k\upsilon_0\upsilon_1\dots\upsilon_n\xi_0\xi_1\dots$ then $(\omega', x') \in B(\omega, 1/k) \times U_x \subset U$ and we have for it, $\mathrm{F}^{n+k+2}(\omega', x') = \left(\sigma^{n+k+2}(\omega'), f_{\omega'}^{n+k+2}(x')\right) \in B(\xi, 1/k) \times V_y \subset V$, therefore F is transitive.
Conversely, if F is transitive Obviously $\mathcal{IFS}_u$ is transitive. □

The following theorem generalizes [4, Theorem 1.3] from metric spaces to uniform spaces.

Theorem 3.6. *The step skew product* F *corresponding to the $\mathcal{IFS}_u$ has topological shadowing property if and only if $\mathcal{IFS}_u$ has topological shadowing property.*

Proof. First, we assume that $\mathcal{IFS}_u$ has topological shadowing property. Fix $\mathscr{B} \in \mathcal{U}$. There exists $\mathscr{U} \in \mathcal{U}_{\Sigma_m^+}$, $\mathscr{B}' \in \mathcal{U}_X$ such that $\mathscr{U} \times \mathscr{B}' \subseteq \mathscr{B}$. According to the assumption, there exists $\mathscr{A}' \in \mathcal{U}_X$, $\omega \in \Sigma_m^+$ such that $\{x_k\}_{k\geq 0}$ is an $(\mathscr{A}', \omega)$-pseudo-orbit that $\mathscr{B}'$-traced by some point $y \in X$. This means that $\left(f_{\omega_k}(x_k), x_{k+1}\right) \in \mathscr{A}'$ and $\left(f_\omega^k(y), x_k\right) \in \mathscr{B}'$. There exists $\mathscr{A} \in \mathcal{U}$ that $\mathscr{U} \times \mathscr{A}' \subseteq \mathscr{A}$. We define $\{\varphi_i\}_{i\geq 0} = \left\{(\sigma^i(\omega), x_i)\right\}_{i\geq 0}$. The sequence $\{\varphi_i\}_{i\geq 0}$ is an $\mathscr{A}$-pseudo-orbit, because

$$\left(F(\varphi_i), \varphi_{i+1}\right) = \left(F\left(\sigma^i(\omega), x_i\right), \left(\sigma^{i+1}(\omega), x_{i+1}\right)\right) = \left(\left(\sigma^{i+1}(\omega), f_{\omega_i}(x_i)\right), \left(\sigma^{i+1}(\omega), x_{i+1}\right)\right) \in \mathscr{A}.$$

The sequence $\{\varphi_i\}_{i\geq 0}$ is $\mathscr{B}$-traced by $(\omega, y) \in \Sigma_m^+ \times X$, because

$$\left(F^n(\omega, y), \varphi_n\right) = \left(\left(\sigma^n(\omega), f_\omega^n(y)\right), \left(\sigma^n(\omega), x_n\right)\right) \in \mathscr{B}.$$

And this process is also established for every other $\mathscr{A}$-pseudo-orbit.
Conversely, if F has topological shadowing property Obviously $\mathcal{IFS}_u$ has topological shadowing property. □

4. Uniform entropy

Topological entropy was extended by Bowen [5] to uniformly continuous self-maps of a metric space. Later on, Hood [12] adapted Bowen's definition to uniformly continuous self-maps on a uniform space. In this subsection, we extend that definition from a single self-mapping to a finite number of self-mappings(IFS).
Before the original definitions, we introduce some notations. Let F_m^+ be the set of all finite words of Σ_m^+. For any $\omega, \omega' \in F_m^+$, let $\omega\omega'$ be the concatenation of ω and ω'. We write $\omega' \leq \omega$ if there exists a word $\omega'' \in F_m^+$ such that $\omega = \omega'\omega''$. For $\omega \in F_m^+, \sigma = \sigma_0\sigma_1\sigma_2\cdots \in \Sigma_m^+, a, b \in \mathbb{Z}^+$ and $a \leq b$, we write $\sigma|_{[a,b]} = \omega$ if $\omega = \sigma_a\sigma_{a+1}\cdots\sigma_{b-1}\sigma_b$. Now, define the uniform entropy of an iterated function system on uniform space by using separated sets and spanning sets.

4.1. Uniform entropy on the $\mathcal{IFS}_u$

Let $\mathcal{IFS}_u$ be iterated function system on uniform space $(X, \mathcal{U}_X)$, $\mathcal{K}(X) = \{K : K$ is a compact subset of X$\}$, $\mathscr{A} \in \mathcal{U}_X$, $\omega \in F_m^+$ and $K \in \mathcal{K}(X)$.

- a subset $\mathrm{F} \subseteq X$ is said to *$(\omega, \mathscr{A}, \mathcal{F})$-span K*, if for every $x \in K$ there is $y \in \mathrm{F}$ such that $(f_{\omega'}(x), f_{\omega'}(y)) \in \mathscr{A}$ for each $\omega' < \omega$,
  $r_\omega(\mathscr{A}, K, \mathcal{F}) = min\{|\mathrm{F}| : \mathrm{F}$ *is* $(\omega, \mathscr{A}, \mathcal{F})$-spans $K\}$;

- a subset $E \subseteq X$ is said to be an *$(\omega, \mathscr{A}, \mathcal{F})$-separated* set, if for each pair of distinct points $x, y \in E$ there exists ω' such that $\omega' < \omega$ and $(f_{\omega'}(x), f_{\omega'}(y)) \notin \mathscr{A}$,
  $s_\omega(\mathscr{A}, K, \mathcal{F}) = max\{|E| : E \subseteq K \text{ and } E \text{ is } (\omega, \mathscr{A}, \mathcal{F})\text{-separated set}\}$

Since K is compact, the numbers $r_\omega(\mathscr{A}, K, \mathcal{F})$ and $s_\omega(\mathscr{A}, K, \mathcal{F})$ are finite. It follows directly from the definition, that if $\mathscr{A} \subset \mathscr{B}$, then $r_\omega(\mathscr{A}, K, \mathcal{F}) \geq r_\omega(\mathscr{B}, K, \mathcal{F})$ and $s_\omega(\mathscr{A}, K, \mathcal{F}) \geq s_\omega(\mathscr{B}, K, \mathcal{F})$.
For brevity from here σ stands either for r or for s. So we define

$$\sigma_n(\mathscr{A}, K, \mathcal{F}) = \frac{1}{m^n} \sum_{|\omega|=n} \sigma_\omega(\mathscr{A}, K, \mathcal{F}). \tag{2}$$

Now define

$$\sigma(\mathscr{A}, K, \mathcal{F}) = \limsup_{n\to\infty} \frac{\log \sigma_n(\mathscr{A}, K, \mathcal{F})}{n}, \tag{3}$$

and

$$h_\sigma(K, \mathcal{F}) = \sup\{\sigma(\mathscr{A}, K, \mathcal{F}) : \mathscr{A} \in \mathcal{U}_X\}. \tag{4}$$

And finally, we define

$$h_r(\mathcal{F}) = \sup\{h_r(K, \mathcal{F}) : K \in \mathcal{K}(X)\}, \tag{5}$$

$$h_s(\mathcal{F}) = \sup\{h_s(K, \mathcal{F}) : K \in \mathcal{K}(X)\}. \tag{6}$$

The next lemma allows us to present each of the two relations as a definition of uniform entropy.

Lemma 4.1. *Let $\mathcal{IFS}_u$ be iterated function system on uniform space $(X, \mathcal{U}_X)$, $\omega \in F_m^+$ and $K \in \mathcal{K}(X)$.*

1. *If $\mathscr{A}, \mathscr{B} \in \mathcal{U}_X$ such that $\mathscr{B} \circ \mathscr{B} \subset \mathscr{A}$, then*
$$r_\omega(\mathscr{A}, K, \mathcal{F}) \leq s_\omega(\mathscr{A}, K, \mathcal{F}) \leq r_\omega(\mathscr{B}, K, \mathcal{F}).$$
2. $h_r(\mathcal{F}) = h_s(\mathcal{F})$.

Proof. (1)A maximal $(\omega, \mathscr{A}, \mathcal{F})$-separated subset of K is an $(\omega, \mathscr{A}, \mathcal{F})$-spans K. Hence $r_\omega(\mathscr{A}, K, \mathcal{F}) \leq s_\omega(\mathscr{A}, K, \mathcal{F})$. Suppose E is an $(\omega, \mathscr{A}, \mathcal{F})$-separated subset of K and F is an $(\omega, \mathscr{B}, \mathcal{F})$-spans K. Define ϕ :E→F that for each $x \in E$, there exists $\phi(x) \in F$ such that $(f_{\omega'}(x), f_{\omega'}(\phi(x))) \in \mathscr{B}$ for all $\omega' < \omega$. If $\phi(x) = \phi(y)$, then $(f_{\omega'}(x), f_{\omega'}(\phi(x))) \in \mathscr{B}$, $(f_{\omega'}(y), f_{\omega'}(\phi(x))) \in \mathscr{B}$. Because $\mathscr{B}$ is symmetric so $(f_{\omega'}(x), f_{\omega'}(y)) \in \mathscr{B} \circ \mathscr{B}^{-1} = \mathscr{B} \circ \mathscr{B} \subset \mathscr{A}$. Since E is an $(\omega, \mathscr{A}, \mathcal{F})$-separated subset of K and $x, y \in E$, it follows that ϕ is injective and therefore the cardinality of E is not greater than of F. Hence $s_\omega(\mathscr{A}, K, \mathcal{F}) \leq r_\omega(\mathscr{B}, K, \mathcal{F})$.
(2) It immediately results from (1). □

Definition 4.2. *Let $\mathcal{IFS}_u$ be iterated function system on uniform space $(X, \mathcal{U}_X)$. We define uniform entropy $h_u(\mathcal{F})$ of $\mathcal{F}$ with respect to $\mathcal{U}_X$: $h_u(\mathcal{F}) = h_r(\mathcal{F}) = h_s(\mathcal{F})$.*

4.2. Uniform covering entropy on the $\mathcal{IFS}_u$

Topological entropy by open covers of the phase space was first defined by Adler, Konheim and McAndrew [1]. In [8] for uniform spaces presented an approach to the uniform entropy similar to the definition of topological entropy and proved that this definition and the previous one are equivalent and in [22] topological entropy of free semi-group action was defined. In this subsection,we extend the uniform entropy to the iterated function system($\mathcal{IFS}_u$).

For structure $\mathcal{U}_X$ of X, *uniform covers* $\mathcal{C}_\mathcal{U} = \{\mathcal{C}(\mathscr{U}) : \mathscr{U} \in \mathcal{U}_X\}$ defined as $\mathcal{C}(\mathscr{U}) = \{\mathscr{U}[x] : x \in X\}$ for $\mathscr{U} \in \mathcal{U}_X$. For $\omega = \omega_0\omega_1 \cdots \omega_{k-1} \in F_m^+$, denote $f_\omega^{-1} = f_{\omega_0}^{-1} o f_{\omega_1}^{-1} o \cdots o f_{\omega_{k-1}}^{-1}$. In the following, for $\mathscr{A} \in \mathcal{U}$, we denote the corresponding uniform cover by $\mathcal{A} = \mathcal{C}(\mathscr{A})$. If K is a compact subset of X, then define the number $N(K, \mathcal{A}) = min\{|\mathcal{A}_K| : \mathcal{A}_K \subset \mathcal{A} \text{ and } K \subset \bigcup \mathcal{A}_K\}$ and take $\mathcal{A} \bigvee \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$

$$f_\omega^{-1}(\mathcal{A}) = \{f_\omega^{-1}(A) : A \in \mathcal{A}\} \quad \text{and} \quad \mathcal{A}_\omega(\mathcal{F}) = \bigvee_{\omega' \leq \omega} f_{\omega'}^{-1}(\mathcal{A}),$$

Definition 4.3. *Let $\mathcal{IFS}_u$ be iterated function system on uniform space $(X, \mathcal{U}_X)$, $\mathcal{K}(X) = \{K : K$ is a compact subset of $X\}$, $\omega \in F_m^+$, $K \in \mathcal{K}(X)$.*

$$c_\omega(\mathscr{A}, K, \mathcal{F}) = N\Big(K, \mathcal{A}_\omega(\mathcal{F})\Big), \tag{7}$$

$$c_n(\mathscr{A}, K, \mathcal{F}) = \frac{1}{m^n} \sum_{|\omega|=n} c_\omega(\mathscr{A}, K, \mathcal{F}), \tag{8}$$

$$c(\mathscr{A}, K, \mathcal{F}) = \limsup_{n\to\infty} \frac{\log c_n(\mathscr{A}, K, \mathcal{F})}{n} \tag{9}$$

and

$$h_{uc}(K, \mathcal{F}) = \sup\{c(\mathscr{A}, K, \mathcal{F}) : \mathscr{A} \in \mathcal{U}_X\}.$$

And finally, we define Uniform covering entropy:

$$h_{uc}(\mathcal{F}) = \sup\{h_{uc}(K, \mathcal{F}) : K \in \mathcal{K}(X)\}.$$

The next lemma compares the two definitions of uniform entropy and proves that they are the same.

Lemma 4.4. *Let $\mathcal{IFS}_u$ be iterated function system on uniform space $(X, \mathcal{U}_X)$, $\omega \in F_m^+$ and $K \in \mathcal{K}(X)$.*

1. *If $\mathscr{A} \in \mathcal{U}_X$, then $c_\omega(\mathscr{A}, K, \mathcal{F}) \leq r_\omega(\mathscr{A}, K, \mathcal{F})$.*
2. *If $\mathscr{A}, \mathscr{B} \in \mathcal{U}_X$ such that $\mathscr{B} \circ \mathscr{B} \subset \mathscr{A}$, then $s_\omega(\mathscr{A}, K, \mathcal{F}) \leq c_\omega(\mathscr{B}, K, \mathcal{F})$.*

Proof. (1)Suppose F $=\{x_i\}_{i=1}^s$ be a subset of K of minimal cardinality which $(\omega, \mathscr{A}, \mathcal{F})$-spans K. By definition, given $k \in K$, there exists $x_i \in$ F such that $(f_{\omega'}(k), f_{\omega'}(x_i)) \in \mathscr{A}$ for all $\omega' < \omega$. that's mean,

$$k \in f_{\omega'}^{-1}\Big(\mathscr{A}(f_{\omega'}(x_i))\Big) \; for \; all \; \omega' < \omega. \tag{10}$$

Hence the family $\{f_{\omega'}^{-1}\Big(\mathscr{A}(f_{\omega'}(x_i))\Big)\}_{i=1}^s$ covers K. Now, for each $i = 1, 2, ..., s$, consider

$$B_i = \bigcap_{\omega' \leq \omega} f_{\omega'}^{-1}\Big(\mathscr{A}(f_{\omega'}(x_i))\Big).$$

Notice that $x_i \in B_i (i = 1, 2, ..., s)$, so that (10) tells us that the family $\{B_i\}_{i=1}^s$ is a subcover of $\mathcal{A}_\omega(\mathcal{F}) \bigcap K$ of cardinality s. Thus, by the definition of $c_\omega(\mathscr{A}, K, \mathcal{F})$, we have $c_\omega(\mathscr{A}, K, \mathcal{F}) \leq s$.
(2)Suppose E be a subset of K of maximal cardinality which is $(\omega, \mathscr{A}, \mathcal{F})$-separated set. Note that every member of the cover $\mathcal{A}_\omega(\mathcal{F})$ can contain at most one point of E, and hence $s_\omega(\mathscr{A}, K, \mathcal{F}) \leq c_\omega(\mathscr{B}, K, \mathcal{F})$. □

Theorem 4.5. *If $\mathcal{IFS}_u$ is iterated function system on uniform space $(X, \mathcal{U}_X)$, then $h_{uc}(\mathcal{F}) = h_u(\mathcal{F})$.*

Proof. According to Lemma 4.1 and Lemma 4.4 the result is obtained. □

In the following, which is an extension of topics [22, Subsection 2.2.2], we need to extend the definition of uniform entropy from a space to a set. Let Y be a nonempty subset of the space X. If $\mathcal{A}$ is a uniform cover of X, we denote by $\mathcal{A}|_Y$ the uniform cover $\{A \cap Y : A \in \mathcal{A}\}$ of the set Y and denote the any $K \in \mathcal{K}(Y)$ by $K^{\subseteq Y}$, where $\mathcal{K}(Y) = \{K \in \mathcal{K}(X) : K \subseteq Y\}$. Then, define the uniform entropy on the set Y as

$$h_{uc}(\mathcal{F}; Y) = \sup\{h_{uc}(K^{\subseteq Y}, \mathcal{F}; Y) : K^{\subseteq Y} \in \mathcal{K}(Y)\},$$

where

$$h_{uc}(K^{\subseteq Y}, \mathcal{F}; Y) = \sup\{c(\mathscr{A}, K^{\subseteq Y}, \mathcal{F}; Y) : \mathscr{A} \in \mathcal{U}_X\},$$

$$c(\mathscr{A}, K^{\subseteq Y}, \mathcal{F}; Y) = \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{m^n} \sum_{|\omega|=n} N\Big(K^{\subseteq Y}, \mathcal{A}_\omega(\mathcal{F})|_Y\Big).$$

Obviously, $h_{uc}(\mathcal{F}; Y) = h_{uc}(\mathcal{F})$ if $Y = X$.

Remark 4.6. *To prove the following theorem, we need to consider the following equality [2, Lemma 4.1.9]. If $k \in N$ and $b_{n,i}$ be positive numbers, where $1 \le i \le k, n = 0, 1, 2, ...,$ Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{k} b_{n,i} = \max_{1\le i\le k} \limsup_{n\to\infty} \frac{1}{n} \log b_{n,i}. \tag{11}$$

Theorem 4.7. *If $\mathcal{IFS}_u$ is iterated function system on uniform space $(X, \mathcal{U}_X)$, $X = \bigcup_{i=1}^{k} K_i$, then*

$$h_{uc}(\mathcal{F}) = \max_{1\le i\le k} h_u(\mathcal{F}; K_i).$$

Proof. Since $h_{uc}(\mathcal{F}) = h_{uc}(\mathcal{F}; X) \ge h_{uc}(\mathcal{F}; K_i)$ so $h_{uc}(\mathcal{F}) \ge \max_{1\le i\le k} h_{uc}(\mathcal{F}; K_i)$. We take a uniform cover $\mathcal{A}$ of X, for any $\mathscr{A} \in \mathcal{U}_X$, $\omega \in F_m^+$, $K \in \mathcal{K}(X)$. Let $\{\mathcal{B}_i\}_{i=1}^k$ be subovers chosen from the covers $\{\mathcal{A}_\omega(\mathcal{F})|_{K_i}\}_{i=1}^k$ respectively. So each element of $\mathcal{B} = \bigcup_{i=1}^{k} \mathcal{B}_i$ is contained in some element of $\mathcal{A}_\omega(\mathcal{F})$ and $\mathcal{B}$ is also a uniform cover of X. Thus

$$N\Big(K, \mathcal{A}_\omega(\mathcal{F})\Big) \le \sum_{i=1}^{k} N\Big(K^{\subseteq K_i}, \mathcal{A}_\omega(\mathcal{F})|_{K_i}\Big).$$

This implies

$$\begin{aligned}
c(\mathscr{A}, K, \mathcal{F}) &= \limsup_{n\to\infty} \frac{1}{n} log \frac{1}{m^n} \sum_{|\omega|=n} N\Big(K, \mathcal{A}_\omega(\mathcal{F})\Big) \\
&\le \limsup_{n\to\infty} \frac{1}{n} log \frac{1}{m^n} \sum_{|\omega|=n} \sum_{i=1}^{k} N\Big(K^{\subseteq K_i}, \mathcal{A}_\omega(\mathcal{F})|_{K_i}\Big) \\
&= \max_{1\le i\le k} \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{m^n} \sum_{|\omega|=n} N\Big(K^{\subseteq K_i}, \mathcal{A}_\omega(\mathcal{F})|_{K_i}\Big) \\
&= \max_{1\le i\le k} c(\mathscr{A}, K_i, \mathcal{F}) \le \max_{1\le i\le k} c(K_i, \mathcal{F}).
\end{aligned}$$

Note that the Equation (11) was used above. □

A basic statement of topological entropy is the *power rule*. In the next theorem, an extension of [22, Theorem 2.10] for $\mathcal{IFS}_u$ is presented.

Theorem 4.8. *If $\mathcal{IFS}_u$ is iterated function system on uniform space $(X, \mathcal{U}_X)$ and $\mathcal{F}^k = \{g_1 \circ g_2 \circ ... \circ g_k : g_1, g_2, ..., g_k \in \mathcal{F}\}$, then*

$$h_u(\mathcal{F}^k) \le k.h_u(\mathcal{F}).$$

Proof. Consider $K \in \mathcal{K}(X)$, $\mathscr{A} \in \mathcal{U}$, $\omega = \omega_0\omega_1 \cdots \omega_{nk-1} \in F_m^+$ that $f_\omega = f_{\omega_0} o f_{\omega_1} o \cdots o f_{\omega_{nk-1}}$. Denote $g_{u_i} = f_{\omega_{ik}} o \cdots o f_{\omega_{(i+1)k-1}} \in \mathcal{F}^k$, for $i = 0,\ 1,\ ...,\ n-1$ and $u = u_0u_1 \cdots u_{n-1}$. Let $F \subseteq X$ be an $(\omega, \mathscr{A}, \mathcal{F})$-span K with minimal cardinality $r_\omega(\mathscr{A},\ K,\ \mathcal{F})$. According to the definition, for any $x \in X$, there exists a $y \in F$ such that $(f_{\omega'}(x), f_{\omega'}(y)) \in \mathscr{A}$ for each $\omega' < \omega$. In particular, for any $g_{u_0}, g_{u_1}, \cdots g_{u_{n-1}}$, we have $(g_{u'}(x),\ g_{u'}(y)) \in \mathscr{A}$ for each $u' < u$. Therefore, F is also a $(u,\ \mathscr{A},\ \mathcal{F}^k)$-span K and if $r_u(\mathscr{A},\ K,\ \mathcal{F}^k)$ be the minimal cardinality of all $(u,\ \mathscr{A},\ \mathcal{F}^k)$-spans K, then

$$r_u(\mathscr{A},\ K,\ \mathcal{F}^k) \le r_\omega(\mathscr{A},\ K,\ \mathcal{F}).$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{m^{nk}} \sum_{|u|=n} r_u(\mathscr{A},\ K, \mathcal{F}^k) \le \limsup_{n\to\infty} \frac{k}{nk} \log \frac{1}{m^{nk}} \sum_{|\omega|=nk} r_u(\mathscr{A},\ K,\ \mathcal{F}),$$

and according to the definition the proof is complete. □

4.3. Uniform entropy of step skew product

We remind the step skew product $\mathrm{F} : \Sigma_m^+ \times X \to \Sigma_m^+ \times X$ corresponding to iterated function system $\mathcal{IFS}_u$ defined by

$$\mathrm{F}(\omega, x) = (\sigma_m(\omega), f_{\omega_0}(x)), \quad \forall(\omega = \omega_0\omega_1\omega_2 \cdots, x) \in \Sigma_m^+ \times X.$$

In the next theorem, an extension of [6, Main Theorem] for $\mathcal{IFS}_u$ is presented, which actually expresses the relationship between the uniform entropies of the $\mathcal{IFS}_u$ and the step skew product F corresponding to the $\mathcal{IFS}_u$.

Theorem 4.9. *If* F *is the step skew product of iterated function system* $\mathcal{IFS}_u$ *on uniform space* $(X, \mathcal{U}_X)$*, then*

$$h_u(\mathrm{F}) = \log m + h_u(\mathcal{F}).$$

We knowe that $\mathcal{K}(X)$ is all of compact subsets X and Σ_m^+ is a compact space so according to the compactness of the product of compacts, denote $\mathcal{K}(\Sigma_m^+ \times X)$ be all of compact subsets $\Sigma_m^+ \times X$. As we said, $\mathcal{U}_{\Sigma_m^+}$, $\mathcal{U}_X$ and $\mathcal{U}$ are uniform structures on Σ_m^+, X and $\Sigma_m^+ \times X$, respectively. For each $\mathscr{A} \in \mathcal{U}$, we can consider $\mathscr{U} \in \mathcal{U}_{\Sigma_m^+}$ and $\mathscr{A}' \in \mathcal{U}_X$, which is $\mathscr{U} \times \mathscr{A}' \subseteq \mathscr{A}$ that $\mathscr{A}'$ can be called the corresponding entourage of $\mathscr{A}$. For proof, we need a lemma:

Lemma 4.10. *Let* F *be the step skew product of an* $\mathcal{IFS}_u$ *on uniform space* $(X, \mathcal{U}_X)$*. For any* $\omega \in F_m^+$, $\mathscr{A} \in \mathcal{U}$ *and* $\mathscr{A}' \in \mathcal{U}_X$ *that* $\mathscr{A}'$ *is corresponding entourage of* A, $K' \in \mathcal{K}(X)$, $K \in \mathcal{K}(\Sigma_m^+ \times X)$,

1. $s_n(\mathscr{A}, K, F) \geq \Sigma_{|\omega|=n} s_\omega(\mathscr{A}', K', \mathcal{F})$
2. $r_n(\mathscr{A}, K, F) \leq \Sigma_{|\omega|=n} r_\omega(\mathscr{A}', K', \mathcal{F})$

Proof. (1) Let $N = m^n$. There are N distinct words of length n in F_m^+. Denote these words by $\omega_1, \omega_2, ..., \omega_N$. For any $i = 1, ..., N$, consider $\omega(i) \in \Sigma_m^+$ be an arbitrary sequence such that $\omega(i)|_{[0,n-1]} = \omega_i$. There exists $\mathscr{U} \in \mathcal{U}_{\Sigma_m^+}$ that sequence $\omega(i)$, $i = 1, ..., N$ form a $(\mathscr{U}, \Sigma_m^+, \sigma_m^+)$-separating subset of Σ_m^+. Let $s_i = s_{\omega_i}(\mathscr{A}', K', \mathcal{F})$. Therefore, the points $x_1^i, ..., x_{s_i}^i$, form a $s_{\omega_i}(\mathscr{A}', K', \mathcal{F})$-separating subset of X. Then the points

$$\left(\omega(i),\ x_j^i\right) \in \Sigma_m^+ \times X, \quad i = 1, ..., N, \quad j = 1, ..., N_i,$$

form a $(\mathscr{A}, K, \mathrm{F})$-separating subset of $\Sigma_m^+ \times X$ that the number of its elements is exactly $\Sigma_{|\omega|=n} s_\omega(\mathscr{A}', K', \mathcal{F})$. According to the definition 2 is proved.
(2). The proof is similar to the proof of the first part, only in the end, according to the Definition $r_n(\mathscr{A}, K, \mathrm{F})$, which is the cardinal minimum of separating subsets, the inequality is proved. □

Proof. (Theorem 4.9). From Equation (2) and inequality (1) of Lemma 4.10 we have

$$s_n(\mathscr{A}, K, \mathrm{F}) \geq m^n . s_n(\mathscr{A}', K', \mathcal{F}),$$

and using the Equations (3), (4) and (6) results

$$h_s(\mathrm{F}) \geq \log m + h_s(\mathcal{F}).$$

Similarly Using inequality (2) of Lemma 4.10,

$$h_r(\mathrm{F}) \leq \log m + h_r(\mathcal{F}).$$

By Definition 4.2, the proof is complete. □

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