Filomat 38:24 (2024), 8689–8700
https://doi.org/10.2298/FIL2424689G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Split continuity of functions between topological spaces

Argha Ghosh[a]

[a]*Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India*

Abstract. In this article, we introduce the notion of split continuity of functions between topological spaces. Also, we give various characterizations of such functions and establish some basic properties. We observe that a function is continuous if and only if it is split continuous and has a closed graph. Furthermore, we study the set of points of split continuity of a quasi-continuous function, and we show that the set of all points of split continuity of a quasi-continuous function from a Baire space X into Y contains a dense G_δ subset of X, where Y is Hausdorff and has some additional properties.

1. Introduction

In 2019, Beer et al. [5] introduced the notion of split continuity of functions between metric spaces to provide answers to some fundamental questions such as what property a function between metric spaces X and Y must have so that when it is followed by an arbitrary continuous real-valued function, the composition is either upper semicontinuous or lower semicontinuous at each point of X. They proved that a function f between two metric spaces X and Y is split continuous at $p \in X$ if and only if whenever g is a real-valued continuous function on Y, then $g \circ f$ is either upper semicontinuous or lower semicontinuous at p. Here are some important facts that they proved about split continuous functions:

(i) Let (X, d) and (Y, ρ) be metric spaces. Then the function $f : X \to Y$ is split continuous on X if and only if there exists a function $g : X \to Y$ such that $\Gamma(x) = \{f(x), g(x)\}$ is a globally upper semicontinuous multifunction on X.

(ii) Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be a function that is split continuous at p. Then f is either upper semicontinuous or lower semicontinuous at p.

(iii) Let (X, d) and (Y, ρ) be metric spaces and the function $f : X \to Y$ is split continuous on X. Then the its oscillation function $\omega(f; \cdot)$ is split continuous on X.

Further, in [9], Gupta and Aggarwal studied split continuity in comparison to other notions of weak continuity. Following are some interesting facts they proved:

(i) A metric space (X, d) is complete if and only if every real-valued split continuous function is Cauchy-subregular on X.

(ii) A metric space (X, d) is discrete if and only if every real-valued subcontinuous function is split continuous on X.

2020 *Mathematics Subject Classification*. Primary 54E40; Secondary 54C08, 54E18, 91A44.
Keywords. Split continuity, Quasi-continuity, Topological Properties, Topological games.
Received: 02 October 2023; Revised: 06 February 2024; Accepted: 07 May 2024
Communicated by Santi Spadaro
Email address: buagbu@yahoo.co.in, argha.ghosh@manipal.edu (Argha Ghosh)

(iii) Let (X, d) and (Y, ρ) be metric spaces. Then $f : X \to Y$ is split continuous at x if and only if for every open cover $\mathcal{V}$ of Y, there exist $V_1, V_2 \in \mathcal{V}$ and $\delta > 0$ such that $f(B(x, \delta)) \subseteq V_1 \cup V_2$.

Since metric spaces are a special type of topological spaces, in this article, we go further in this direction and formulate a definition of split continuity of functions between topological spaces. We give two characterizations of split continuous functions and study various properties of such functions. Many of these properties are direct generalizations of properties studied in metric space contexts, although some of the proofs are not direct because of the absence of the notion of distance in general topological spaces, and the information about functions between topological spaces is not entirely encoded by sequences, which makes the study interesting.

In Section 4, we go one step further than the previous studies on split continuity to investigate the set of points of split continuity of quasi-continuous functions defined on Baire spaces. According to the article [11], Volterra first used the notion of quasi-continuity to study separately continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ (see also [4]). And later in [13], Kempisty formulated this notion for functions from $\mathbb{R}^n$ to $\mathbb{R}$. Further in [19], Neubrunn extended this definition to mappings between topological spaces. In 2001, based on the notion of topological games, Kenderov et al. [14] studied the set of points of continuity of quasi-continuous functions between topological spaces. Further in 2009, Holá et al. [21] studied the set of points of continuity of quasi-continuous functions with values in generalized metric spaces. Later in 2022, Holá et al. [12] studied the set of points of subcontinuity of quasi-continuous functions between topological spaces. Further studies in this direction can be found in [11, 16, 17] and references therein. The following are some interesting results, which we will prove in that section:

(i) If X is a Baire space, Y is a Hausdorff space with the property $(\mathcal{B}_2)$ and $f : X \to Y$ is a quasi-continuous mapping, then f is split continuous at points of a dense G_δ subset of X (Theorem 4.8).
(ii) If X is a Baire space, Y is a Hausdorff space such that $G_2(Y)$ is Ω-favorable and $f : X \to Y$ is a quasi-continuous mapping, then f is split continuous on a dense subset of X (Theorem 4.13).
(iii) If X be α-favorable, Y is a Hausdorff space such that $G_2(Y)$ is Σ-unfavorable and $f : X \to Y$ is a quasi-continuous mapping, then the set of split continuity points of f intersects some residual subset of every non-empty open subset of X (Theorem 4.15).

2. Preliminaries

In this section, we include all the notations, definitions and results that help readers to understand the subsequent sections thoroughly. We write $\mathbb{R}$ and $\mathbb{N}$ to denote the set of all real numbers and the set of all positive integers, respectively. Throughout the article, by a neighbourhood of a point a in a topological space X, we mean an open subset of X containing a.

Definition 2.1 ([5]). *Let (X, d) and (Y, ρ) be metric spaces. Let $f : X \to Y$ be a function and $p \in X$. Then f is said to be split continuous at p if there exists $y \in Y$ such that*

(i) *for each $\varepsilon > 0$, $f(B_d(p, \varepsilon)) \cap B_\rho(y, \varepsilon) \neq \emptyset$;*
(ii) *for each $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_d(p, \delta)) \subseteq B_\rho(f(p), \varepsilon) \cup B_\rho(y, \varepsilon)$.*

We say that f is split continuous on X if f is split continuous at each point of X.

Note 2.2. *If f is continuous at $p \in X$, then the y in the above definition is equal to $f(p)$. Otherwise, there exists a unique $y \in Y$ that satisfies the above definition, and $y \neq f(p)$. In the latter case, f is said to be strictly split continuous at p.*

Result 2.3 ([5]). *Let (X, d) and (Y, ρ) be metric spaces. Then a function $f : X \to Y$ is strictly split continuous at p if and only if there is $y(\neq f(p)) \in Y$ such that (a) there exists a sequence (x_n) that converges to p such that $(f(x_n))$ converges to y, and (b) each sequence which is convergent to p has a subsequence along which f is convergent to either y or $f(p)$.*

Definition 2.4 ([8]). *Let X and Y be topological spaces. Then a function $f : X \to Y$ is said to be subcontinuous at $p \in X$ if for every net $\langle x_\lambda : \lambda \in \Lambda \rangle$ in X converging to p, there is a convergent subnet of $\langle f(x_\lambda) : \lambda \in \Lambda \rangle$. We say that f is subcontinuous on X if f is subcontinuous at each point of X.*

Result 2.5 ([8]). *Let (X, d) and (Y, ρ) be metric spaces. Then a function $f : X \to Y$ is continuous if and only if it is subcontinuous and has closed graph.*

Definition 2.6 ([15]). *Let X and Y be topological spaces. Then a function $f : X \to Y$ is said to be locally compact at $x \in X$ if there exists a compact subset K of Y such that x is an interior point of $f^{-1}(K)$.*

In [18], Neubrunn gave the definition of quasi-continuity for general topological spaces, and later he reformulated the definition in [19]. In our study the notion of quasi-continuity plays an important role. Thus following [19], we recall the definition of quasi-continuity for topological spaces.

Definition 2.7 ([19]). *Let X and Y be topological spaces. Let $f : X \to Y$ be a function and $p \in X$. Then f is said to be quasi-continuous at p if for every pair of neighbourhoods U of p and V of $f(p)$, there exists a non-empty open set $G \subseteq U$ such that $f(G) \subseteq V$. We say that f is quasi-continuous on X if f is quasi-continuous at each point of X.*

In [14], Kenderov et al. gave the definition of quasi-continuity for general topological spaces as follows:

Definition 2.8 ([14]). *Let X and Y be topological spaces. Let $f : X \to Y$ be a function and $p \in X$. Then f is said to be quasi-continuous at p if for every neighbourhood V of $f(p)$, there exists a non-empty open set U such that $p \in \overline{U}$ and $f(U) \subseteq V$. We say that f is quasi-continuous on X if f is quasi-continuous at each point of X.*

Now we prove that both definitions are equivalent. Let X and Y be topological spaces, and $f : X \to Y$ be a function. Let f be a quasi-continuous function in the sense of Definition 2.8. We prove that f is quasi-continuous in the sense of Definition 2.7. Let $p \in X$. Let U be a neighbourhood of p and V be a neighbourhood of $f(p)$. Then there exists an open set $W \subseteq X$ such that $p \in \overline{W}$ and $f(W) \subseteq V$. Then $G = W \cap U$ is the non-empty open subset of U for which $f(G) \subseteq V$.

Conversely, suppose that f is a quasi-continuous function in the sense of Definition 2.7. We prove that f is quasi-continuous in the sense of Definition 2.8. Let $p \in X$ and V be a neighbourhood of $f(p)$. Set

$$L = \{G \subseteq X : G \text{ is open and } f(G) \subseteq V\}.$$

Let $W = \bigcup_{G \in L} G$. Then W is the largest open subset of X such that $f(W) \subseteq V$. We claim that $p \in \overline{W}$. Suppose to the contrary that $p \notin \overline{W}$. Then $X - \overline{W}$ is an open subset of X containing p. Then there exists an open subset $H \subseteq X - \overline{W}$ such that $f(H) \subseteq V$. Thus $H \in L$, which is a contradiction. Hence $p \in \overline{W}$ and $f(W) \subseteq V$.

Obviously, any continuous mapping is quasi-continuous, but a quasi-continuous mapping may not be continuous.

The notion of a sequence of covers endowed with various properties plays an important role in our study. We give a list of such properties.

Let X be a topological space and $\mathcal{G}$ be a collection of subsets of X. For each $x \in X$, we define

$$\mathrm{st}(x, \mathcal{G}) = \bigcup \{G \in \mathcal{G} : x \in G\}.$$

Definition 2.9 ([2, 7]). *Let X be a topological space and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of X.*

(i) *If for each $x \in X$, the set $\{\mathrm{st}(x, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a base at x, then $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is called a development on X and the space X is called developable. A regular developable space is called a Moore space.*

(ii) *If for every sequence $(G_n)_{n\in\mathbb{N}}$ such that $G_n \in \mathcal{G}_n$ for every $n \in \mathbb{N}$ and for every $x \in \bigcap_n G_n$, the sequence $\{\bigcap_{i=1}^n G_i : n \in \mathbb{N}\}$ is a base at x, we say that $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a weak development on X and that the space X is weakly developable.*

(iii) *If for every $x \in X$, $\{x\} = \bigcap \mathrm{st}(x, \mathcal{G}_n)$, then we say that $(\mathcal{G}_n)$ is a G_δ-development and that the space X is G_δ-developable.*

Remark 2.10. *In the literature, a G_δ-development is also called a G_δ-diagonal sequence and a G_δ-developable space is correspondingly called a space with a G_δ-diagonal (see [2, Page 24]).*

The notion of p-spaces was originally introduced by Arhangel'skiĭ [3] as follows:

Definition 2.11 ([3]). *A completely regular space X is said to be a p-space if there exists a sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of collections of open subsets of the Stone-Čech compactification $\beta(X)$ of X such that each $\mathcal{G}_n$ covers X, and for every $x \in X$, we have $\bigcap_n \mathrm{st}(x, \mathcal{G}_n) \subseteq X$.*

In [6, Theorem 1.3], Burke gave the following internal characterization of p-spaces: A completely regular space X is a p-space if and only if it satisfies the property $(\mathcal{B})$. Recall from [1], the definition of the property $(\mathcal{B})$.

Definition 2.12 ([1]). *A space X is said to have the property $(\mathcal{B})$ if there exists a sequence of open covers $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of X such that whenever $x \in X$ and $G_n \in \mathcal{G}_n$ are such that $x \in G_n$ for each n, then*

(i) $\bigcap_{n=1}^{\infty} \overline{G_n}$ *is compact;*
(ii) *every neighbourhood of* $\bigcap_{n=1}^{\infty} \overline{G_n}$ *contains some* $\bigcap_{n=1}^{k} \overline{G_n}$.

3. Split continuity of functions between topological spaces

In this section, we introduce the definition of split continuity for topological spaces and show that several of the properties of split continuous functions that hold in metric spaces also hold in the case of a split continuous function from a topological space to a Hausdorff space. Although these results parallel the results obtained in [5, 9], the proofs given here are significantly different.

Definition 3.1. *Let X and Y be topological spaces. Let $f : X \to Y$ be a function and $p \in X$. Then f is said to be split continuous at p if there exists $y \in Y$ such that:*

(i) *for every pair of neighbourhoods U_p of p and V_y of y, $f(U_p) \cap V_y \neq \emptyset$;*
(ii) *for every pair of neighbourhoods $V_{f(p)}$ of $f(p)$ and V_y of y, there exists a neighbourhood U_p of p such that $f(U_p) \subseteq V_{f(p)} \cup V_y$.*

In the above definition, $y \in Y$ is not unique as we can observe in the following example:

Example 3.2. *Consider the real line $\mathbb{R}_{cf}$ with the co-finite topology. Then the identity function on $\mathbb{R}_{cf}$ is split continuous everywhere and y can be any real number.*

Now suppose the co-domain space Y is Hausdorff. Let there exist $y_1, y_2 \in Y$ satisfying the above conditions, such that neither of them is equal to $f(p)$. We show that $y_1 = y_2$. Suppose to the contrary that $y_1 \neq y_2$. Since Y is Hausdorff, there exist pair-wise disjoint neighbourhoods V_{y_1}, V_{y_2}, and $V_{f(p)}$ of y_1, y_2, and $f(p)$, respectively. Then there exists a neighbourhood U_p of p such that $f(U_p) \subseteq V_{f(p)} \cup V_{y_1}$. Consequently, $f(U_p) \cap V_{y_2} = \emptyset$, which is a contradiction.

Therefore, from now on, we will always consider the co-domain space of a function as a Hausdorff space, unless otherwise stated.

Theorem 3.3. *Let $f : X \to Y$ be a function. Then the following conditions are equivalent:*

(i) *f is strictly split continuous at $p \in X$.*
(ii) *There exists $y \in Y$ and $y \neq f(p)$ such that (a) there exists a net $\langle x_\lambda : \lambda \in \Lambda \rangle$ that converges to p such that $\langle f(x_\lambda) : \lambda \in \Lambda \rangle$ converges to y, and (b) each net which is convergent to p has a subnet along which f converges to either y or $f(p)$.*

Proof. (i) $\implies$ (ii) Since for all neighbourhoods U of p and V of y_p, $f(U) \cap V \neq \emptyset$, we can pick $x_{(U,V)} \in U$ such that $f(x_{(U,V)}) \in V$. Put

$$\Lambda = \Big\{(U, V) : U \text{ is a neighbourhood of } p \text{ and } V \text{ is a neighbourhood of } y_p\Big\}.$$

Define the following direction on Λ: $(U_1, V_1) \geq (U_2, V_2)$ if and only if $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$. Clearly, the net $\langle x_{(U,V)} : (U, V) \in \Lambda\rangle$ converges to p and the net $\langle f(x_{(U,V)}) : (U, V) \in \Lambda\rangle$ converges to y_p. Therefore the condition (a) of (ii) holds for $y = y_p$.

Let $\langle x_\lambda : \lambda \in \Lambda\rangle$ converges to p. We will show that either $f(p)$ or y_p is a cluster point of the net $\langle f(x_\lambda) : \lambda \in \Lambda\rangle$. If not, there exist neighbourhoods $V_{f(p)}$ and V_{y_p} of $f(p)$ and y_p respectively such that $f(x_\lambda) \notin V_{f(p)}$ for all $\lambda \geq \lambda_1$ and $f(x_\lambda) \notin V_{y_p}$ for all $\lambda \geq \lambda_2$ for some $\lambda_1, \lambda_2 \in \Lambda$. Pick $\lambda_0 \in \Lambda$ such that $\lambda_0 \geq \lambda_1$ and $\lambda_0 \geq \lambda_2$. Then for all $\lambda \geq \lambda_0$, we have $f(x_\lambda) \notin V_{f(p)}$ and $f(x_\lambda) \notin V_{y_p}$. Thus for all $\lambda \geq \lambda_0$, we have $f(x_\lambda) \notin V_{f(p)} \cup V_{y_p}$, which is a contradiction because f is split continuous at p and $\langle x_\lambda : \lambda \in \Lambda\rangle$ converges to p. Hence either $f(p)$ or y_p is a cluster point of the net $\langle f(x_\lambda) : \lambda \in \Lambda\rangle$. Thus $\langle x_\lambda : \lambda \in \Lambda\rangle$ has a subnet along which f converges to either y or $f(p)$.

(ii) $\implies$ (i) Let V be a neighbourhood of y and U be a neighbourhood of p. Then there exist $\lambda_1, \lambda_2 \in \Lambda$ such that for all $\lambda \geq \lambda_1$, we have $f(x_\lambda) \in V$ and for all $\lambda \geq \lambda_2$, we have $x_\lambda \in U$. Pick $\lambda_0 \in \Lambda$ such that $\lambda_0 \geq \lambda_1$ and $\lambda_0 \geq \lambda_2$. Then clearly $f(x_\lambda) \in f(U) \cap V$ for all $\lambda \geq \lambda_0$. Therefore, the first condition of the definition of split continuity is satisfied.

Now suppose the second condition of the definition does not hold with respect to this y. Then there exist neighbourhoods $V_{f(p)}$ and V_y of $f(p)$ and y respectively, such that for each neighbourhood U of p, $f(U) \nsubseteq V_{f(p)} \cup V_y$. Choose $x_U \in U$ such that $f(x_U) \notin V_{f(p)} \cup V_y$. Let $\Lambda' = \{U : U \text{ is a neighbourhood of } p\}$. Define the following direction on Λ': $U_1 \geq U_2$ if and only if $U_1 \subseteq U_2$. Clearly, the net $\langle x_{\lambda'} : \lambda' \in \Lambda'\rangle$ converges to p, but neither y nor $f(p)$ is a cluster point of $\langle f(x_{\lambda'}) : \lambda' \in \Lambda'\rangle$, a contradiction to the condition (b) of (ii). □

Now we provide a condition by which we can verify the split continuity of f without ever finding the y in the definition of split continuity. A metric space version of the same can be found in [9].

Theorem 3.4. *Let $f : X \to Y$ be a function and $p \in X$. Then the following conditions are equivalent:*

(i) *f is split continuous at $p \in X$.*
(ii) *For every open cover $\mathcal{V}$ of Y, there exist $V_1, V_2 \in \mathcal{V}$ and a neighbourhood U of p such that $f(U) \subseteq V_1 \cup V_2$.*

Proof. (i) $\implies$ (ii) Let $\mathcal{V}$ be an open cover of Y and f be split continuous at p. Then there exist $V_1, V_2 \in \mathcal{V}$ such that $f(p) \in V_1$ and $y_p \in V_2$. Then by the definition of split continuity there exists a neighbourhood U of p such that $f(U) \subseteq V_1 \cup V_2$.

(ii) $\implies$ (i) The given condition implies that every net converging to p has a subnet along which f is convergent (see [20, Theorem 2.1]). Let S_p be the set of all $y \in Y$ such that there exists a net in X converging to p along which f converges to y. Clearly, $f(p) \in S_p$. It is sufficient to show that S_p contains no more than two points. Suppose to the contrary that S_p contains three distinct points $y_1 = f(p)$, y_2, and y_3. Since Y is a Hausdorff space, there exist neighbourhoods V_{y_j} of y_j whose closures do not contain y_i if $i \neq j$, where $i, j = 1, 2, 3$. In addition, we can pick a neighbourhood V_z of each $z \in Y \setminus \{y_1, y_2, y_3\}$ such that $\overline{V_z}$ does not contain any of y_1, y_2, and y_3. Then $\mathcal{V} = \{V_z : z \in Y \setminus \{y_1, y_2, y_3\}\} \cup \{V_{y_j} : j = 1, 2, 3\}$ is an open cover of Y. Then by the given condition there exist $V_1, V_2 \in \mathcal{V}$ and a neighbourhood U of p such that $f(U) \subseteq V_1 \cup V_2$. Suppose $y_1 = f(p) \in V_1$. Then by the construction of the open cover $V_1 = V_{y_1}$. Since $y_2 \in S_p$, there exists a net $\langle x_{\lambda'} : \lambda' \in \Lambda'\rangle$ converging to p such that $\langle f(x_{\lambda'}) : \lambda' \in \Lambda'\rangle$ converges to y_2. Therefore, $y_2 \in \overline{(V_1 \cup V_2)} = \overline{V_1} \cup \overline{V_2}$. Since $\overline{V_1} = \overline{V_{y_1}}$ does not contain y_2, we have $y_2 \in \overline{V_2}$. Thus $V_2 = V_{y_2}$. Similarly, $y_3 \in \overline{V_1} \cup \overline{V_2}$, which is a contradiction. Consequently, S_p contains at most two elements. □

Corollary 3.5. *Let Y be a locally compact Hausdorff space. If $f : X \to Y$ is split continuous at p, then it is locally compact at p.*

Proof. Since Y is locally compact, for each $y \in Y$ there exists a neighbourhood V_y of y and a compact subset C_y of Y such that $V_y \subseteq C_y$. Now $\mathcal{V} = \{V_y : y \in Y\}$ is an open cover of Y. Since f is split continuous at p, there exist $V_{y_1}, V_{y_2} \in \mathcal{V}$ and a neighbourhood U of p such that $f(U) \subseteq V_{y_1} \cup V_{y_2} \subseteq C_{y_1} \cup C_{y_2}$. Since $C_{y_1} \cup C_{y_2}$ is compact, f is locally compact at p. □

Recall from Theorem 3.3 that if f is a strictly split continuous function from X to a Hausdorff space Y, then for each $x \in X$, there exists a unique $y_x \neq f(x)$ satisfying the definition of split continuity. Following the definition given in [5], we define the star function $f^*(x)$ of a split continuous function $f(x)$. If $f : X \to Y$ is a split continuous function on X, define $f^* : X \to Y$ by $f^*(x) = f(x)$, if f is continuous at $x \in X$, and $f(x) = y_x$ if f is strictly split continuous at $x \in X$.

Example 3.6. *Consider the non-metrizable space $\mathbb{R}_l$, the set of all real numbers with the lower limit topology. Define a function $f : \mathbb{R}_l \to \mathbb{R}_l$ as follows:*

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x = 0 \\ -1, & \text{if } x > 0. \end{cases}$$

The function f is continuous everywhere except at $x = 0$, where it is strictly split continuous. Then the star function of f is

$$f^*(x) = \begin{cases} 0, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0. \end{cases}$$

Observe that f^* is continuous everywhere.

Proposition 3.7. *Let $f : X \to Y$ be strictly split continuous at $p \in X$. If there exists $y \in Y$ such that $y \neq f(p)$ and for every pair of neighbourhoods U_p of p and V_y of y, $f(U_p) \cap V_y \neq \emptyset$, then $f^*(p) = y$.*

Proof. Suppose to the contrary that $f^*(p) \neq y$. Since Y is Hausdorff, there exist pair-wise disjoint neighbourhoods $V_y, V_{f^*(p)}$, and $V_{f(p)}$ of $y, f^*(p)$, and $f(p)$, respectively. Again, since f is strictly split continuous at p, there exists a neighbourhood U of p such that $f(U) \subseteq V_{f^*(p)} \cup V_{f(p)}$. Consequently, $f(U) \cap V_y = \emptyset$, which is a contradiction. Hence $f^*(p) = y$. □

Theorem 3.8. *Let X be a topological space and Y be a regular Hausdorff space. If $f : X \to Y$ is a split continuous function on X such that f is continuous at p, i.e., $f^*(p) = f(p)$, then $f^* : X \to Y$ is continuous at p.*

Proof. Let V be a neighbourhood of $f^*(p) = f(p)$. Since Y is regular, there exists an open set W such that $f^*(p) \in W \subseteq \overline{W} \subseteq V$. Again, since f is continuous at p, there exists a neighbourhood U of p such that $f(U) \subseteq W$. As f is split continuous, for each $x \in U$, there exists a net $\langle x_\gamma : \gamma \in \Gamma \rangle$ converging to x in U such that $\langle f(x_\gamma) : \gamma \in \Gamma \rangle$ converges to $f^*(x)$. So $f^*(x) \in \overline{W}$. Thus $f^*(U) \subseteq \overline{W} \subseteq V$. Hence f^* is continuous at p. □

Theorem 3.9. *Let X be a topological space and Y be a regular Hausdorff space. If $f : X \to Y$ is split continuous on X, then $f^* : X \to Y$ is split continuous on X.*

Proof. Let $p \in X$. If f^* is continuous at p, then there is nothing to prove. Let f^* be not continuous at p. Let $V_{f(p)}$ and $V_{f^*(p)}$ be neighbourhoods of $f(p)$ and $f^*(p)$, respectively. Since Y is regular, there exist open sets $W_{f(p)}$ and $W_{f^*(p)}$ such that

$$f(p) \in W_{f(p)} \subseteq \overline{W_{f(p)}} \subseteq V_{f(p)},$$

and

$$f^*(p) \in W_{f^*(p)} \subseteq \overline{W_{f^*(p)}} \subseteq V_{f^*(p)}.$$

Now, since f is split continuous at p, there exists a neighbourhood U of p such that $f(U) \subseteq W_{f(p)} \cup W_{f^*(p)}$. Again, since f is split continuous, for each $x \in U$, there exists a net $\langle x_\gamma : \gamma \in \Gamma \rangle$ converging to x in U such that $\langle f(x_\gamma) : \gamma \in \Gamma\rangle$ converges to $f^*(x)$. So either $f^*(x) \in \overline{W_{f(p)}}$ or $f^*(x) \in \overline{W_{f^*(p)}}$. Hence $f^*(U) \subseteq V_{f(p)} \cup V_{f^*(p)}$. Therefore, f^* satisfies the condition (ii) of the definition of split continuity at p with respect to $y = f(p)$. Now we prove that f^* satisfies the condition (i) of the definition of split continuity at p with respect to $y = f(p)$. Since f^* is not continuous at p, there exists a neighbourhood $V_{f^*(p)}$ of $f^*(p)$ such that for each neighbourhood U of p, we have $f^*(U) \nsubseteq V_{f^*(P)}$. Let $V_{f(p)}$ be a neighbourhood of $f(p)$. Then there exists a neighbourhood U' of p such that $f^*(U') \subseteq V_{f(p)} \cup V_{f^*(p)}$. Let U be a neighbourhood of p. Then $f^*(U \cap U') \subseteq V_{f(p)} \cup V_{f^*(p)}$. Since $f^*(U \cap U') \nsubseteq V_{f^*(p)}$, we have $f^*(U \cap U') \cap V_{f(p)} \neq \emptyset$. Hence $f^*(U) \cap V_{f(p)} \neq \emptyset$, which is what we had to prove. □

If Y is a non-regular Hausdorff space, then the above theorems (Theorem 3.8 and Theorem 3.9) may not be true. To show this, we consider the following example:

Example 3.10. *Let $X = \mathbb{R}$ with usual topology. Put $Y = \mathbb{R}$, where the topology of Y consists of the usual topology and, moreover, of all the sets $G \setminus K$, where G is open in the usual topology and $K = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Define $f : X \to Y$ by*

$$f(x) = \begin{cases} x, & \text{if } x \notin K \\ 0, & \text{if } x \in K. \end{cases}$$

Clearly, f is continuous everywhere except at $\frac{1}{n}$ for each $n \in \mathbb{N}$. However, f is split continuous at $\frac{1}{n}$ for each $n \in \mathbb{N}$. In fact, $f^*(x) = x$ for each $x \in X$. Thus f^* is neither continuous nor split continuous at 0*.*

Proposition 3.11. *Let X be a topological space and Y be a Hausdorff space. Let $f : X \to Y$ be split continuous on X. Then the following conditions are equivalent:*

(i) *f^* is strictly split continuous at $p \in X$ with respect to $y = f(p)$.*
(ii) *f^* is not continuous at p, and each net which is convergent to p has a subnet along which f^* converges to either $f^*(p)$ or $f(p)$.*

Proof. The proof of (i) $\implies$ (ii) is obvious. We only have to prove (ii) $\implies$ (i). Since f^* is not continuous at p, there exists a neighbourhood $V_{f^*(p)}$ of $f^*(p)$ such that for each neighbourhood U of p, there exists $x_U \in U$ such that $f^*(x_U) \notin V_{f^*(p)}$. Let $\mathcal{N}(p)$ be the collection of all neighbourhoods of p. Define a direction $\geq$ on $\mathcal{N}(p)$ by reverse inclusion. Then $\langle x_U : U \in \mathcal{N}(p)\rangle$ is a net in X converging to p. Now by assumption, $\langle f^*(x_U) : U \in \mathcal{N}(p)\rangle$ has a subnet which is convergent to $f^*(p)$ or $f(p)$. However, no subnet of $\langle f^*(x_U) : U \in \mathcal{N}(p)\rangle$ can converge to $f^*(p)$. Thus $\langle f^*(x_U) : U \in \mathcal{N}(p)\rangle$ has a subnet which is convergent to $f(p)$. Therefore, there exists a net $\langle x_\lambda : \lambda \in \Lambda\rangle$ converging to p such that $\langle f^*(x_\lambda) : \lambda \in \Lambda\rangle$ converges to $f(p)$. Hence by Theorem 3.3, f^* is strictly split continuous at p with respect to $y = f(p)$. □

The following proposition can be proved along the same lines as [5, Proposition 3.6.].

Proposition 3.12. *If $f : X \to Y$ is split continuous on X, then* $\mathrm{gr}(f) \cup \mathrm{gr}(f^*) = \overline{\mathrm{gr}\, f}$*.*

The following proposition is a direct consequence of [8, Theorem 3.4] and the fact that any split continuous function is subcontinuous.

Proposition 3.13. *A function $f : X \to Y$ is continuous if and only if it is split continuous and has a closed graph.*

In [5], a relationship between split continuity and multifunctions whose values consist of no more than two points has been established in the context of metric spaces. Similarly, one can investigate the relation between split continuity and multifunctions whose values consist of no more than two points in a more general context, namely in topological spaces. In this article, we forego our investigation in the aforementioned direction and instead we examine the set of points of split-continuity of quasi-continuous functions. We conclude this section with the following open problems:

Open Problem 3.14. *Let X be a topological space and Y be a regular Hausdorff space. Let $f : X \to Y$ be split continuous on X. Define $f_1 = f^*$, $f_2 = (f^*)^*$, and in this way, recursively define f_n for any positive integer n. Is there a positive integer k such that f_k is continuous on X? If not, does the sequence of functions $(f_n)_n$ converge to some continuous function in the topology of pointwise convergence?*

4. Points of split continuity of quasi-continuous functions

We have divided the whole section into two subsections, Subsection A and Subsection B. In Subsection A, we consider the quasi-continuous functions with values in a topological space with the property $(\mathcal{B}_2)$ and in Subsection B, we consider the quasi-continuous functions with values in a topological space whose topology guarantees the existence of a winning strategy for one of the players in a special two-player fragmenting game.

4.1. Subsection A

Definition 4.1. *A topological space X is said to satisfy the property $(\mathcal{B}_i)$, where i is a positive integer, if X has a sequence of open covers $(\mathcal{G}_n)_{n\in\mathbb{N}}$ such that whenever $x \in X$ and $G_n \in \mathcal{G}_n$ are such that $x \in G_n$ for each n, then*

(i) *$\bigcap_{n=1}^{\infty} \overline{G_n}$ contains at most i number of elements;*
(ii) *every neighbourhood of $\bigcap_{n=1}^{\infty} \overline{G_n}$ contains some $\bigcap_{n=1}^{k} \overline{G_n}$.*

Remark 4.2. *Clearly, if a topological space X satisfies the property $(\mathcal{B}_i)$, then it also satisfies the property $(\mathcal{B}_j)$ for all $j > i$, and the property $(\mathcal{B})$. If X is a regular and weakly developable space, then X has the property $(\mathcal{B}_i)$ for all i.*

In addition, we have the following fact discussed in [1, 2]: A completely regular space is weakly developable if and only if it is G_δ-developable and has the property $(\mathcal{B})$. In fact, one can easily prove that a completely regular space is weakly developable if and only if it is G_δ-developable and has the property $(\mathcal{B}_i)$.

The following example shows that there is a weakly developable non-Hausdorff space having the property $(\mathcal{B}_2)$ but not the property $(\mathcal{B}_1)$.

Example 4.3. *We recall the weakly developable non-Hausdorff topological space $X = \mathbb{N} \cup \{\infty_1, \infty_2\}$, where $\infty_1, \infty_2 \notin \mathbb{N}$, and $\infty_1 \neq \infty_2$, and X is topologized as follows: $\{m\}$ is an open discrete subset in X for every $m \in \mathbb{N}$, and the point ∞_i has the neighbourhood base $\{V_n^i : n \in \mathbb{N}\}$, where $V_n^i = \{\infty_i\} \cup \{k \in \mathbb{N} : k \geq n\}$, for every $n \in \mathbb{N}$, and $i = 1, 2$ (see [1, Example 3.3]). The space has the property $(\mathcal{B}_2)$ because of the sequence of open covers $(\mathcal{G}_n)$ of X, where $\mathcal{G}_n = \{\{m\} : m \in \mathbb{N}, m < n\} \cup \{V_n^i : i = 1, 2\}$ for every $n \in \mathbb{N}$. We will now prove that X does not have the property $(\mathcal{B}_1)$. Let $(\mathcal{G}'_n)$ be a sequence of open covers of X. Then for every n, there exists $G' \in \mathcal{G}'_n$ such that $\infty_1 \in G'$. Set $G' = G'_n$. Then $\infty_1 \in G'_n$ for all n. Then for each n, there exists $k_n \in \mathbb{N}$ such that $V_{k_n}^1 \subseteq G'_n$. Clearly, $\infty_2 \in \overline{V_{k_n}^1} \subseteq \overline{G'_n}$. Hence $\bigcap_n \overline{G'_n}$ contains more than one element. Consequently, X does not have the property $(\mathcal{B}_1)$.*

The following example shows that there is a weakly developable Hausdorff space without the property $(\mathcal{B}_1)$.

Example 4.4. *Consider the topological space $X = \mathbb{R} \times \left(\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}\right)$ from [1, Example 3.4]. The space is topologized as follows: $\mathbb{R} \times \{\frac{1}{n}\}$ is an open set in X for every $n \in \mathbb{N}$, and every $\mathbf{x} = (x, 0) \in \mathbb{R} \times \{0\}$ has the neighbourhood base $\{N(\mathbf{x}, \varepsilon) : \varepsilon > 0\}$, where $N(\mathbf{x}, \varepsilon) = (\{x\} \times [0, \varepsilon[) \cup (]x - 2 - \varepsilon, x - 2 + \varepsilon[\setminus\{x + 2\} \times]0, \varepsilon[) \cap X$. It is proved in [1, Example 3.4] that this space is weakly-developable and Hausdorff. It is easy to see that this space does not have the property $(\mathcal{B}_1)$.*

Proposition 4.5. *Let A be a non-empty subset of a Hausdorff space Y. If Y has the property $(\mathcal{B}_i)$, then A has the property $(\mathcal{B}_i)$ in Y with the induced topology.*

Proof. Let $(\mathcal{G}_n)$ be a sequence of open covers of Y which guarantees the property $(\mathcal{B}_i)$. Set $\mathcal{G}'_n = \{G \cap A : G \in \mathcal{G}_n\}$ for every $n \in \mathbb{N}$. Then $(\mathcal{G}'_n)$ will guarantee the property $(\mathcal{B}_i)$ for A in Y with the induced topology. □

Proposition 4.6. *Let G and H be two disjoint closed subsets of a Hausdorff space Y. If G and H have the property $(\mathcal{B}_i)$ in Y with the induced topology, then also $G \cup H$ has the property $(\mathcal{B}_i)$ in Y with the induced topology.*

Proof. Let $(\mathcal{G}_n)$ be a sequence of open covers of G and $(\mathcal{H}_n)$ be a sequence of open covers of H with respect to which G and H have the property $(\mathcal{B}_i)$, respectively. Since G and H are disjoint closed subsets of Y, $\mathcal{I}_n = \mathcal{G}_n \cup \mathcal{H}_n$ is an open cover of $G \cup H$, for every n. It is easy to establish that with respect to the sequence of open covers $(\mathcal{I}_n)$, $G \cup H$ has the property $(\mathcal{B}_i)$. □

The following proposition is a generalization of [1, Theorem 4.1].

Proposition 4.7. *Let X be a topological space having the property $(\mathcal{B}_i)$. Then there exists a sequence of open covers $(\mathcal{G}_n)$ such that for every $x \in X$ and for every sequence (G_n) of open sets where, for each n, $x \in G_n \in \mathcal{G}_n$ and if (x_n) is a sequence, where $x_n \in \bigcap_{k \leq n} G_n$, then the set of all cluster points of (x_n) is a non-empty set with no more than i number of elements.*

Theorem 4.8. *Let X be a Baire space and Y be a Hausdorff space with the property $(\mathcal{B}_2)$. If $f : X \to Y$ is a quasi-continuous mapping, then the set of points of split continuity of f is a dense G_δ subset in X.*

Proof. Let $(\mathcal{G}_n)$ be a sequence of open covers of Y which guarantees the property $(\mathcal{B}_2)$. For each $n \in \mathbb{N}$, define D_n to be the set of all $x \in X$ for which there exist neighbourhoods U of x, and $G_n \in \mathcal{G}_n$ of $f(x)$ such that $f(U) \subseteq G_n$. By construction, D_n is open in X. We prove that D_n is dense in X for each $n \in \mathbb{N}$. Let U be an open set in X and $n \in \mathbb{N}$. Pick $x \in U$. Choose $G_n \in \mathcal{G}_n$ so that $f(x) \in G_n$. Since f is quasi-continuous at x, there exists a non-empty open subset U' of U such that $f(U') \subseteq G_n$. Clearly, $U' \subseteq U \cap D_n$. Hence D_n is dense in X. Let $D = \bigcap_n D_n$. Since X is Baire, D is a dense G_δ subset of X. Now we show that f is split continuous at each point of D. Let $x \in D$ and $\mathcal{V}$ be an open cover of Y. Since $x \in D_n$ for each $n \in \mathbb{N}$, there are neighbourhoods U_n of x and G_n of $f(x)$ such that $f(U_n) \subseteq G_n$. Clearly, $f(x) \in G_n$ for each n. Since Y has the property $(\mathcal{B}_2)$, $\bigcap_n \overline{G_n}$ can have at most two points. Then there exist $V_1, V_2 \in \mathcal{V}$ such that $\bigcap_n \overline{G_n} \subseteq V_1 \cup V_2$. Furthermore, $\bigcap_{n \leq k} \overline{G_n} \subseteq V_1 \cup V_2$ for some $k \in \mathbb{N}$. Set $U = \bigcap_{n \leq k} U_n$. Then $x \in U$ and $f(U) \subseteq V_1 \cup V_2$. Hence f is split continuous at x. Consequently, f is split continuous on D. □

Corollary 4.9. *Let X be a Baire space and Y be a Hausdorff space with the property $(\mathcal{B}_2)$. If $f : X \to Y$ is a quasi-continuous mapping, then the set of points of split continuity of f is of the second category.*

Proof. It follows from the proof of Theorem 4.8 that the set of points of split continuity of f is an intersection of countably many open dense sets in X. Since the intersection of countably many open dense sets in a Baire space is a set of the second category, the set of points of split continuity of f is of the second category. □

Example 4.10. *Consider the space $Y = [0, 1)$ with lower limit topology and $X = (0, 1)$ with standard subspace topology of $\mathbb{R}$. Consider the inclusion mapping $f(x) = x$. Clearly, f is quasi-continuous on X, but it is not split-continuous at any point of X. Since X is a Baire space, $Y = [0, 1)$ does not have the property $(\mathcal{B}_2)$ by the Theorem 4.8.*

Theorem 4.11. *Let X be a Baire space and Y be a regular Hausdorff space with the property $(\mathcal{B}_2)$ locally. If $f : X \to Y$ is a quasi-continuous mapping, then the set of points of split continuity of f is of the second category.*

Proof. Let $L = \{x \in X : f$ is not split continuous at $x\}$. Since X is a Baire space, it sufficient to show that L is of first category in X. Put $\mathcal{G} = \{V : V$ is open in X and $V \cap L$ is of first Baire category in $V\}$. We prove that $\mathcal{G}$ is a π-base in X. Let U be a non-empty open subset of X and let $x \in U$. Let V be a neighbourhood of $f(x)$. Since Y is regular space having the property $(\mathcal{B}_2)$ locally, we can pick a neighbourhood W of $f(x)$ such that $\overline{W}$ has the property $(\mathcal{B}_2)$ in Y with the induced topology and $f(x) \in \overline{W} \subseteq V$. Since f is quasi-continuous at x, there exists a nonempty open subset U_1 of U such that $f(U_1) \subseteq W$. Consider the restricted mapping $f|U_1 : U_1 \to \overline{W}$. Clearly, $f|U_1$ satisfies all the assumptions of Theorem 4.8. Thus there exists a dense G_δ subset S of U_1 such that f is split continuous on S. Then $U_1 \in \mathcal{G}$ and $U_1 \subseteq U$. By [11, Lemma 3.1], the set L is of first category in X. □

4.2. Subsection B

Let X be a topological space. The Banach–Mazur game $BM(X)$ can be defined in the following way: Two players α and β alternately choose non-empty open subsets of X to form a sequence $W_0 \supseteq W_1 \supseteq W_2 \supseteq \cdots$, and the player α wins if and only if $\bigcap_n W_n \neq \emptyset$. It has been proved that X is a Baire space if and only if player β has no winning strategy in the game $BM(X)$ (see [22]). Later, Kenderov et al. [14] introduced the idea of the fragmenting game $G(X)$ in the space X as follows: The game involves two players Σ and Ω.

The players alternately select non-empty subsets of X. The player Ω starts the game by selecting the whole space X. Then Σ responds by picking a subset A_1 of X and Ω answers by picking a relatively open subset B_1 of A_1. In this way, at the n^{th} stage of the game, Σ picks any relatively open subset A_n of B_{n-1} in response to the last move B_{n-1} of Ω, and Ω responds by picking a relatively open subset B_n of A_n. In this manner, the players generate a sequence of non-empty sets $A_1 \supseteq B_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq B_n \supseteq \cdots$, which is said to be a play, and is denoted by $p = ((A_n, B_n))_{n\geq 1}$. The player Ω is said to have won the game if the set $\bigcap_n A_n$ contains at most one point. Otherwise, the player Σ is said to have won the game.

A finite sequence $((A_k, B_k))_{k=1}^{i}$ of pairs of nonempty open sets consisting of the first several moves of the play $((A_n, B_n))_{n\geq 1}$ will be called a partial play. A strategy σ of the player Σ is a rule that species each move of the player Σ in every possible situation. More precisely, a strategy $\sigma = (\sigma_n)_n$ for the player Σ is a sequence of set-valued mappings such that $A_1 = \sigma_1(X)$ and for each $n \geq 2$, we have

$$\emptyset \neq \sigma_n(B_1, B_2, \ldots, B_{n-1}) \subseteq B_{n-1}$$

and $\sigma_n(B_1, B_2, \ldots, B_{n-1})$ is a relatively open subset of B_{n-1}. By a σ-play, we mean a sequence of non-empty subsets $(B_n)_n$ of X such that the domain of σ_n contains the element $(B_1, B_2, \ldots, B_{n-1})$, for all $n \geq 2$. A strategy σ is called a winning strategy for the player Σ if Σ wins each σ-play. A winning strategy ω for the player Ω can be defined similarly. The game $G(X)$ or the space X is called Ω-favorable (Σ-favorable), if there is a wining strategy for the player Ω (Σ). The game $G(X)$ or the space X is called Σ-unfavorable (Ω-unfavorable), if there does not exist winning strategy for the player Σ (Ω). Note that $G(X)$ is Σ-unfavorable does not imply that $G(X)$ is Ω-favorable.

In the same paper [14], by altering the rule of winning in the game $G(X)$, the games $G'(X)$ and $DG(X)$ were introduced. A space X is said to be game determined if $DG(X)$ is Ω-favorable. Similarly, by altering only the winning criteria keeping everything else same in the game $G(X)$, we define a new game $G_2(X)$ as follows: The player Ω is said to have won the game if the set $A = \bigcap_n \overline{A_n}$ is either empty or contains at most two points such that for every neighbourhood U of A there exists A_n contained in U. Otherwise, the player Σ is said to have won the game. We will say that the game $G_2(X)$ or the space X is Ω-favorable (Σ-favorable), if there is a wining strategy for the player Ω (Σ). Clearly, if the game $G'(X)$ is Ω-favorable, then $G_2(X)$ is Ω-favorable. Therefore, for every metric space X, $G_2(X)$ is Ω-favorable. Also, if $G_2(X)$ is Ω-favorable, then the game $DG(X)$ is Ω-favorable as well.

In [12], Holá et al. proved the following result:

Result 4.12 ([12]). *Let X be a Baire space, Y be a game determined Hausdorff space and $f : X \to Y$ be a quasi-continuous mapping. Then the set of points of subcontinuity of f is a dense subset in X.*

We will now prove a result analogous to the above result.

Theorem 4.13. *Let X be a Baire space, and Y be a Hausdorff space such that $G_2(Y)$ is Ω-favorable. If $f : X \to Y$ is a quasi-continuous mapping, then f is split continuous on a dense subset of X.*

Proof. Let U be an arbitrary open subset of X. It is sufficient to show that f is split continuous at some points of U. Let ω be a winning strategy for Ω in the game $G_2(Y)$. We will construct a strategy b for the player β in the Banach-Mazur game $BM(X)$. Set $b(\emptyset) = U$. Let V_1 be a non-empty open subset of U. Consider the set $A_1 = f(V_1)$. Let $B_1 = \omega(A_1)$. Since B_1 is relatively open in A_1 and f is quasi-continuous, there exists an open subset $U_1 \subseteq V_1$ such that $f(U_1) \subseteq B_1$. Set $b(V_1) = U_1$.

Proceeding inductively we can construct the strategy b for player β in such a way that any b-play $(U, V_1, U_1, V_2, \ldots V_n, U_n, \ldots)$ is accompanied by some ω-play $(A_1, B_1, A_2, B_2, \ldots, A_n, B_n, \ldots)$ such that for every $n \geq 1$, $f(V_n) = A_n$ and $f(U_n) \subseteq B_n$. Since X is a Baire space, the strategy b for β in $BM(X)$ is not a winning one. Thus there exists a play $(V_1, V_2, \ldots, V_n, \ldots)$ of α against this strategy. Thus $\bigcap_n U_n = \bigcap_n V_n \neq \emptyset$. Let $x \in \bigcap_n V_n \subseteq U$. We prove that f is split continuous at x. Clearly, $f(x) \in \bigcap_n A_n$. Since ω is a winning strategy for Ω in the game $G_2(Y)$, the set $A = \bigcap_n \overline{A_n}$ contains at most two points, and for every open subset G of Y containing A, there is some $k \geq 1$ such that $A_k \subseteq G$.

Now let $\mathcal{W}$ be an open cover of Y. Since A has at most two elements, there exist $W_1, W_2 \in \mathcal{W}$ such that $A \subseteq W_1 \cup W_2$. Then there exists some $k \geq 1$ such that $A_k \subseteq W_1 \cup W_2$. Thus $f(V_k) \subseteq W_1 \cup W_2$. Since V_k is a neighbourhood of x, f is split continuous at x. □

Corollary 4.14. *For the space $Y = (0, 1)$ with the lower limit topology, $G_2(Y)$ is not Ω-favorable.*

Proof. Consider $X = (0, 1)$ with the standard topology. Then X is a Baire space. Consider the identity function I from X to Y. Clearly, I is a quasi-continuous mapping on X. Now we show that I is not split continuous at any point $x \in (0, 1)$. Clearly, I is not continuous at x. Let $y \in (0, 1)$ be such that $y \neq x$. Set $r = |y - x|$. Then $I\left(x - \frac{r}{4}, x + \frac{r}{4}\right) \cap \left[y, \frac{r}{4}\right) = \left(x - \frac{r}{4}, x + \frac{r}{4}\right) \cap \left[y, \frac{r}{4}\right) = \emptyset$. Thus I can not be split continuous at x. Hence by Theorem 4.13, we conclude that $G_2(Y)$ is not Ω-favorable. □

A space X is called α-favorable if there exists a winning strategy for the player α in the game $BM(X)$. The following theorem is analogous to (i) $\implies$ (iv) of Theorem 2 in [14], and we employ a similar technique in the proof.

Theorem 4.15. *Let X be α-favorable, and Y be a Hausdorff space such that $G_2(Y)$ is Σ-unfavorable. If $f : X \to Y$ is a quasi-continuous mapping, then the set of split continuity points of f is of the second category in every non-empty open subset of X. Moreover, the set is dense in X.*

Proof. Let H be a first category subset of X and t be a winning strategy for player α such that $\bigcap_n W_n \neq \emptyset$ and $H \cap (\bigcap_n W_n) = \emptyset$ whenever $(V_n, W_n)_n$ is a t-play. Let V_0 be a non-empty open subset of X. We will show that f is split continuous at some point of $V_0 \setminus H$. We first construct a strategy σ for the player Σ in the game $G_2(Y)$. Let V_0 be the first move of the player β in the game $BM(X)$ and let $W_1 = t(V_0)$. Consider the set $A_1 = f(W_1)$. Assume that A_1 is the first move of the player Σ under the strategy σ. Let $B_1 = \sigma(A_1)$ be the response of the player Ω. Since B_1 is a relatively open subset of A_1 and f is quasi-continuous, there exists a non-empty open subset V_1 of W_1 such that $f(V_1) \subseteq B_1$. Suppose that V_1 is the next move of the player β in the game $BM(X)$. Set $W_2 = t(V_0, W_1, V_1)$.

Proceeding inductively, we can construct the strategy σ for player Σ in such a way that any σ-play $(A_1, B_1, A_2, B_2, \ldots, A_n, B_n, \ldots)$ is accompanied by some t-play $(V_0, W_1, V_1, W_2, V_2, \ldots, W_n, V_n, \ldots)$ such that for all every $n \geq 1$, $f(W_n) = A_n$ and $f(V_n) \subseteq B_n$.

Since X is α-favorable, $\bigcap_n W_n \neq \emptyset$. Consequently $\bigcap_n A_n = \bigcap_n f(W_n) \supseteq f(\bigcap_n W_n) \neq \emptyset$. Since $G_2(Y)$ is Σ-unfavorable, there is some σ-play $(A_n, B_n)_n$ which is won by Ω. Hence the non-empty set $A = \bigcap_n \overline{A_n}$ has at most two points such that for every neighbourhood U of A there exists A_n contained in U. Let $x \in \bigcap_n W_n$ and $\mathcal{V}$ be an open cover of Y. Then there exist $V_1, V_2 \in \mathcal{V}$ such that $A \subseteq V_1 \cup V_2$. Therefore, $A_k \subseteq V_1 \cup V_2$ for some positive integer k. Thus $f(W_k) \subseteq V_1 \cup V_2$. Since W_k is a neighbourhood of x, we conclude that f is split continuous at $x \in \bigcap_n W_n \subseteq V_0 \setminus H$. □

Acknowledgement: The author is grateful to the referees for carefully reading the paper and making valuable suggestions and comments. The author thanks of the referees for recognizing that the generalization of Theorem 3.8 to non-regular spaces included in the previous version of the paper was erroneous and for providing Example 3.10 as a counterexample. The author also acknowledges Dr. Vinay Madhusudanan of MIT MAHE for assisting him in dispelling some doubts.

References

[1] B. Alleche, *Weakly developable and weakly k-developable spaces, and the Vietoris topology* , Top. Appl. **111(1-2)** (2001), 3–19.
[2] B. Alleche, A. V. Arhangel'skiĭ, J. Calbrix, *Weak developments and metrization*, Top. Appl. **100(1)** (2000), 23–38.
[3] A. V. Arkhangel'skiĭ, *A class of spaces which contains all metric and all locally compact spaces*, Matematicheskiĭ Sbornik, **109(1)** (1965), 55–88.
[4] R. Baire, *Sur la représentation des fonctions discontinues*, Acta Math. **30(1)** (1906), 1–48.
[5] G. Beer, C. Bloomfield, G. Robinson, *Split continuity*, J. Math. Anal. Appl. **474(2)** (2019), 833–851.
[6] D. Burke, *On p-spaces and wΔ-spaces*, Pacific Journal of Mathematics, **35(2)** (1970), 285–296.
[7] J. Calbrix, B. Alleche, *Multifunctions and Čech-complete spaces*, Topology Atlas (Proceeding of the 8th Prague Topological Symposium), (1997), 30–36.
[8] R. V. Fuller, *Relations among continuous and various noncontinuous functions*, Pacific J. Math. **25** (1968), 495—509.
[9] L. Gupta, M. Aggarwal, *Split continuity: A different perspective*, Top. Appl. **317** (2022), 108189.
[10] L. Holá, B. Novotnỳ, *Subcontinuity*, Math. Slovaca, **62(2)** (2012), 345–362.

[11] L. Holá, A. K. Mirmostafaee, *On continuity of set-valued mappings* , Top. Appl. **320** (2022), 108200.
[12] L. Holá, A. K. Mirmostafaee, *Joint continuity of separately continuous mappings*, Top. Appl. **307** (2022), 107881.
[13] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math. **1(19)** (1932), 184–197.
[14] P. Kenderov, S. I. Kortezov, W. B. Moors, *Continuity points of quasi-continuous mappings*, Top. Appl. **109(3)** (2001), 321–346.
[15] R. A. Mimna, E. J. Wingler, *Locally bounded functions*, Real Analysis Exchange, **23(1)** (1997-8), 251—258.
[16] A. K. Mirmostafaee, *Continuity of separately continuous mappings*, Math. Slovaca, **64** (2014), 1019–1026.
[17] A. K. Mirmostafaee, *Points of upper and lower semicontinuity of multivalued functions*, Ukrainian Math. J. **69(9)** (2017), 1224–1231.
[18] T. Neubrunn, *A generalized continuity and product spaces*, Math. Slovaca **26** (1976), 97–99.
[19] T. Neubrunn, *Quasi-continuity*, Real Anal. Exchange, **14(2)** (1988), 259–306.
[20] B. Novotnỳ, *On subcontinuity* , Real Anal. Exchange, **31(2)** (2005/2006), 535–546.
[21] L. Holá, Z, Piotrowski, *Set of continuity points of functions with values in generalized metric spaces*, Tatra Mt. Math. Publ. **42** (2009), 149–160.
[22] R. Telgársky, *Topological games: on the 50th anniversary of the Banach-Mazur game*, Rocky Mt. J. Math. **17(2)** (1987), 227-276.