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# New regular perturbation for a sequential random differential problem of Airy type

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**Abstract.** In this paper, we study a new problem of random differential equations of Airy type by means of the stochastic mean square theory. A new perturbation problem is introduced and some existence and uniqueness results for "stochastic process" solutions" are established. At the end, an example is discussed in details.

## 1. Introduction

Fractional differential equations are important for studying mathematical models in many fields. For more details, see [2, 8, 11, 14] and the references therein.
Recently, the researchers have bifurcated to investigate another important class of fractional differential problems which are the random equations [7, 9, 10, 18]. To motivate this new way, we begin by citing the Airy differential equation which has a great interest in developing the applied mathematics since it appears when we deal with solving partial differential equations for mathematical physics. The Airy equation has the following form, see [15]:

$$Z'' - tZ = 0, \quad t \in \mathbb{R}.$$

For the above equation, in [4], Cortez et al. have constructed power series solutions of random Airy type differential equations containing uncertainty through the coefficients as well as the initial conditions over the whole real line. The studied problem is the following

$$v'' - Atv = 0, \quad t \in \mathbb{R},$$

under two random conditions.
To present to the reader other works that have motivated the present paper, we recall the work ([3]), where the random problem has been investigated:

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$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0^+}Y(t) - Bt^{\beta}Y(t) = 0, \quad t > 0 \\ \quad n-1 < \alpha \leq n, \quad \beta > 0, \\ Y^{(j)}(0) = A_j, \quad j = 0, 1, \cdots, n-1, \end{cases}$$

such that ${}^{c}\mathcal{D}^{\alpha}_{0^+}Y(t)$ is the mean square random Caputo fractional derivative of order $\alpha$ of the stochastic process $Y(t)$.

Also, we find that the paper of R. Spigler [13] is important to cite since the author has been concerned with the following perturbed fractional differential problem:

$$\begin{cases} \mathcal{D}^{\alpha}_{t}U + b\mathcal{D}^{\beta}_{t}U + \varepsilon cU = 0 \\ \quad a, b, c \in \mathbb{R}^*, \varepsilon > 0. \end{cases}$$

Very recently, H. Bedani et al. [1] have investigated the following problem:

$$\begin{cases} \mathcal{D}^{\alpha_1}_{0^+}\left(\mathcal{D}^{\alpha_2}_{0^+}Y(t)\right) = aAf(t, Y(t)) + bBg(t, \mathcal{D}^{\gamma}_{0^+}Y(t), \mathcal{I}^{\rho}_{0^+}Y(t)), t \in J = [0, T] \\ \qquad 0 < \alpha_1, \alpha_2 \leq 1, \alpha_1 + \alpha_2 > 1, \alpha_2 > \gamma > 0, \text{ and } \rho > 0 \\ \qquad A, B \in L^2(\Omega), a, b \in \mathbb{R} \\ \qquad Y(0) = Y^*, \text{ and } Y(T) = \sum\limits_{i=\overline{1,n}} \lambda_i Y(\zeta_i),\ 0 < \zeta_i < T, \end{cases} \tag{1}$$

where $\mathcal{D}^{\alpha_1}_{0^+}, \mathcal{D}^{\alpha_2}_{0^+}$, and $\mathcal{D}^{\gamma}_{0^+}$ are the mean square Caputo derivatives of orders $\alpha_1, \alpha_2, \gamma$, and $\mathcal{I}^{\rho}_{0^+}$ is the stochastic mean square integral of order $\rho$, where $0 < \alpha_1, \alpha_2 \leq 1, \alpha_1 + \alpha_2 > 1, \alpha_2 > \gamma > 0, \rho > 0, f : J \times L^2(\Omega) \to L^2(\Omega)$.
In the present work, we study the existence of solutions as well as the existence of unique solutions in the sense of stochastic mean square calculus for the following regularly perturbed problem of Airy type:

$$\begin{cases} \mathcal{D}^{\alpha}_{0^+}\left(\mathcal{D}^{\beta}_{0^+}Y(t)\right) = aAf(t, Y(t), \mathcal{D}^{\gamma}_{0^+}Y(t)) + bBg(t, Y(t), \mathcal{I}^{\rho}_{0^+}Y(t)) + \varepsilon Y(t), t \in J = [0, \lambda] \\ \qquad \varepsilon > 0, Y(0) = Y^*, \\ \qquad \text{and } Y(\lambda) = \sum\limits_{i=\overline{1,n}} \lambda_i Y(\zeta_i),\ 0 < \zeta_i < \lambda, \end{cases} \tag{2}$$

where $\mathcal{D}^{\beta}_{0^+}, \mathcal{D}^{\alpha}_{0^+}$, and $\mathcal{D}^{\gamma}_{0^+}$ are the mean square derivative of Caputo of orders $\beta, \alpha, \gamma$, and $\mathcal{I}^{\rho}_{0^+}$ The stochastic mean square non integer order of orders $\rho$ and $0 < \alpha, \beta \leq 1, \alpha + \beta > 1, \beta > \gamma > 0, \rho > 0$, $A, B$ and $Y^*$ are three bounded random variables, $a, b \in \mathbb{R}$, and $f, g : J \times L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega)$ are a given functions satisfying some assumptions that will be specified later.
We find that the above introduced problem is interesting in the sense that it can allow us to obtain some recent fractional random works as special cases. It allows us also to obtain the classical Airy random differential equation.

## 2. Stochastic Mean Square Calculus

In this section, we introduce some notations and definitions of mean square fractional calculus and present preliminary results needed in our proofs later, for details, see [5, 6].

**Definition 2.1.** *Let $\alpha > 0$ and $Y \in C(J, L^2(\Omega))$. The stochastic mean square fractional integral $\mathcal{I}^{\alpha}_{0^+}Y(t)$ is defined by*

$$\mathcal{I}^{\alpha}_{0^+}Y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Y(s)ds.$$

**Theorem 2.2.** *Let $\alpha > 0$ and $\beta > 0$. If $Y \in C(J, L^2(\Omega))$, then $\mathcal{I}^{\alpha}_{0^+}Y(t)$ exists in mean square sense as a second-order mean square continuous second-order process $\mathcal{I}^{\alpha}_{0^+}Y(t)$ $C(J, L^2(\Omega))$ with the following properties*

•) $\mathcal{I}^{\alpha}_{0^+} : C(J, L^2(\Omega)) \to C(J, L^2(\Omega))$,

•) $\mathcal{I}^{\alpha}_{0^+}\mathcal{I}^{\beta}_{0^+}Y(t) = \mathcal{I}^{\beta}_{0^+}\mathcal{I}^{\alpha}_{0^+}Y(t) = \mathcal{I}^{\alpha+\beta}_{0^+}Y(t)$.

**Definition 2.3.** *The Caputo fractional derivative of order $\alpha \in (0, 1]$ of the stochastic process $Y$, denoted by $\mathcal{D}^{\alpha}_{0^+}Y(t)$ is defined by*

$$\mathcal{D}^{\alpha}_{0^+}Y(t) = \mathcal{I}^{1-\alpha}_{0^+}\frac{d}{dt}Y(t).$$

*where, $\frac{d}{dt}Y(t)$ denotes the mean square differentiation of $Y(t)$.*

**Theorem 2.4.** *Let $\alpha > 0$. If $Y$ is mean square differentiable with mean square integrable second-order derivative, then*

•) $\lim\limits_{\alpha\to 1}\mathcal{D}^{\alpha}_{0^+}Y(t) = \frac{d}{dt}Y(t)$,

•) $\lim\limits_{\alpha\to 0}\mathcal{D}^{\alpha}_{0^+}Y(t) = Y(t) - Y(0)$,

•) $\mathcal{I}^{\alpha}_{0^+}\mathcal{D}^{\alpha}_{0^+}Y(t) = Y(t) - Y(0)$,

•) $\mathcal{D}^{\alpha}_{0^+}\mathcal{I}^{\alpha}_{0^+}Y(t) = Y(t)$.

**Lemma 2.5.** *Let $Y \in C(J, L_2(\Omega))$ and suppose that $Y^*$ is a bounded random variable. If $G$ is a continuous function on $J$, then the unique solution-stochastic process of the following non-local random problem*

$$\begin{cases} \mathcal{D}^{\alpha}_{0^+}\left(\mathcal{D}^{\beta}_{0^+}Y(t)\right) = G(t), t \in J \\ 0 < \alpha, \beta \leq 1, \ \text{and} \ \alpha + \beta > 1 \\ Y(0) = Y^* \\ Y(\lambda) = \sum\limits_{i=\overline{1,n}} \lambda_i Y(\zeta_i), \ 0 < \zeta_i < \lambda \end{cases} \tag{3}$$

*is given by*

$$\begin{aligned} Y(t) \quad = \quad & \frac{1}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1}G(s)ds + \frac{(Y(\lambda) - Y^*)}{\lambda^{\beta}}t^{\beta} \\ & -\frac{t^{\beta}}{\lambda^{\beta}\Gamma(\beta+\alpha)}\int_0^{\lambda}(\lambda-s)^{\beta+\alpha-1}G(s)ds + Y^*. \end{aligned} \tag{4}$$

*Proof.* Let $0 < \alpha, \beta < 1$. Then, thanks to point 2 of Theorem 2.2 and point 3 of Theorem 2.4, we can write

$$\begin{aligned} \mathcal{I}^{\alpha}_{0^+}\mathcal{D}^{\alpha}_{0^+}\left(\mathcal{D}^{\beta}_{0^+}Y(t)\right) \quad &= \quad \mathcal{I}^{\alpha}_{0^+}G(t) + C_1 \Longrightarrow \mathcal{D}^{\beta}_{0^+}Y(t) = \mathcal{I}^{\alpha}_{0^+}G(t) + C_1 \\ &\Longrightarrow \quad \mathcal{I}^{\beta}_{0^+}\mathcal{D}^{\beta}_{0^+}Y(t) = \mathcal{I}^{\beta}_{0^+}\mathcal{I}^{\alpha}_{0^+}G(t) + \mathcal{I}^{\beta}_{0^+}C_1 + C_2, \end{aligned}$$

so,

$$Y(t) = \mathcal{I}^{\beta+\alpha}_{0^+}G(t) + \mathcal{I}^{\beta}_{0^+}C_1 + C_2,$$

where, $C_1, C_2$ are random variables.
By (3), we get

$$C_2 = Y^*, \ \text{and} \ C_1 = \frac{\Gamma(\beta+1)}{\lambda^{\beta}}\left(Y(\lambda) - Y^* - \mathcal{I}^{\beta+\alpha}_{0^+}G(\lambda)\right).$$

This completes the proof. □

## 3. Main Results

Let $C^\gamma = C^\gamma(J, L_2(\Omega)) = \left\{Y : Y, \mathcal{D}^\gamma_{0^+} Y \in C(J, L_2(\Omega))\right\}$ be the Banach space of all mean square continuous second order stochastic processes with the norm

$$\|Y\|_{C^\gamma} = \max\left\{\|Y\|_C, \left\|\mathcal{D}^\gamma_{0^+} Y\right\|_C\right\},$$

where,

$$\|Y\|_C = \sup_{t\in J} \|Y(t)\|_2, \text{ and } \left\|\mathcal{D}^\gamma_{0^+} Y\right\|_C = \sup_{t\in J} \left\|\mathcal{D}^\gamma_{0^+} Y(t)\right\|_2.$$

### 3.1. Criteria For Uniqueness Solution-Stochastic Process

Now, we need to consider the assumptions:

- $\mathbb{A}_1$) $f, g$ are continuous functions.
- $\mathbb{A}_2$) $\exists(\delta_1, \delta_2) \in (\mathbb{R}^*_+)^2$ such that

$$\left\|f(t, Y(t), X(t)) - f(t, U(t), V(t))\right\|_2 \leq \delta_1 \left(\|Y(t) - U(t)\|_2 + \|X(t) - V(t)\|_2\right),$$

and

$$\left\|g(t, Y(t), X(t)) - g(t, U(t), V(t))\right\|_2 \leq \delta_2 \left(\|Y(t) - U(t)\|_2 + \|X(t) - V(t)\|_2\right),$$

for any $X, Y, U, V : J \to L^2(\Omega)$.

- $\mathbb{A}_3$) There are continuous functions $\varphi_i : J \to \mathbb{R}^+ (i = \overline{1,4})$ such that

$$\left\|f(t, Y(t), U(t))\right\|_2 \leq \varphi_1(t) \|Y(t)\|_2 + \varphi_2(t) \|U(t)\|_2,$$

and

$$\left\|g(t, Y, U)\right\|_2 \leq \varphi_3(t) \|Y\|_2 + \varphi_4(t) \|U\|_2,$$

such that

$$\varphi_i^* = \sup_{t\in J} \left|\varphi_i(t)\right|, \quad (i = \overline{1,4}).$$

Now, we define the following constants:

$$\begin{aligned}
\mathcal{V}_1 &= \varphi_1^* |a| \|A\|_2 + \varphi_3^* |b| \|B\|_2 + \frac{\varphi_4^* \lambda^\rho |b| \|B\|_2}{\Gamma(\rho+1)} + \varepsilon \\
\mathcal{V}_2 &= \varphi_2^* |a| \|A\|_2 \\
\mathcal{V}_3 &= 2|a| \|A\|_2 \delta_1 + |b| \|B\|_2 \delta_2 + \frac{|b| \|B\|_2 \lambda^\rho}{\Gamma(\rho+1)} \delta_2 + \varepsilon \\
\Lambda_1 &= \mathcal{V}_1 \frac{2\lambda^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + n \sup_{i=\overline{1,n}} |\lambda_i| \\
\Lambda_2 &= \frac{2\lambda^{\beta+\alpha}\mathcal{V}_2}{\Gamma(\beta+\alpha+1)} \\
\Lambda_3 &= \frac{\lambda^{\beta+\alpha-\gamma}\mathcal{V}_1}{\Gamma(\beta+\alpha-\gamma+1)} + \frac{\Gamma(\beta+1)\lambda^{\beta+\alpha-\gamma}\mathcal{V}_1}{\Gamma(\beta-\gamma+1)\Gamma(\beta+\alpha+1)} \\
&\quad + \frac{n\Gamma(\beta+1)\lambda^{-\gamma}}{\Gamma(\beta-\gamma+1)} \sup_{i=\overline{1,n}} |\lambda_i| \\
\Lambda_4 &= \frac{\lambda^{\beta+\alpha-\gamma}\mathcal{V}_2}{\Gamma(\beta+\alpha-\gamma+1)} + \frac{\Gamma(\beta+1)\lambda^{\beta+\alpha-\gamma}\mathcal{V}_2}{\Gamma(\beta-\gamma+1)\Gamma(\beta+\alpha+1)}
\end{aligned}$$

**Theorem 3.1.** *Assume that the assumptions ($\mathbb{A}_2$) and ($\mathbb{A}_3$) are satisfied and $Y^*$ is a bounded random variable. If*

$$\max\{\Upsilon_1, \Upsilon_2\} < 1,$$

*where,*

$$\Upsilon_1 = \frac{2\lambda^{\beta+\alpha}\mathcal{V}_3}{\Gamma(\beta+\alpha+1)} + n\sup_{i=\overline{1,n}}|\lambda_i|$$

*and*

$$\Upsilon_2 = \frac{\lambda^{\beta+\alpha-\gamma}\mathcal{V}_3}{\Gamma(\beta+\alpha-\gamma+1)} + \frac{\Gamma(\beta+1)\lambda^{\beta+\alpha-\gamma}\mathcal{V}_3}{\Gamma(\beta-\gamma+1)\Gamma(\beta+\alpha+1)} + \frac{n\Gamma(\beta+1)\lambda^{-\gamma}}{\Gamma(\beta-\gamma+1)}\sup_{i=\overline{1,n}}|\lambda_i|$$

*then the problem (2) has a unique solution-stochastic process on J.*

*Proof.* **A:** Let us define the operator $\mathbb{L} : C^\gamma \to C^\gamma$ by

$$\begin{aligned}(\mathbb{L}Y)(t) &= \frac{1}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1} G_Y(s)ds + \frac{(Y(\lambda)-Y^*)}{\lambda^\beta}t^\beta \\ &\quad -\frac{t^\beta}{\lambda^\beta\Gamma(\beta+\alpha)}\int_0^\lambda (\lambda-s)^{\beta+\alpha-1} G_Y(s)ds + Y^*.\end{aligned}$$

Then, we have

$$\begin{aligned}\left(\mathcal{D}^\gamma_{0^+}\mathbb{L}Y\right)(t) &= \frac{1}{\Gamma(\beta+\alpha-\gamma)}\int_0^t (t-s)^{\beta+\alpha-\gamma-1} G_Y(s)ds \\ &\quad -\frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)\Gamma(\beta+\alpha)}\frac{t^{\beta-\gamma}}{\lambda^\beta}\int_0^\lambda (\lambda-s)^{\beta+\alpha-1} G_Y(s)ds \\ &\quad +\frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}\frac{t^{\beta-\gamma}}{\lambda^\beta}(Y(\lambda)-Y^*),\end{aligned}$$

where,

$$G_Y(t) = aAf(t, Y(t), \mathcal{D}^\gamma_{0^+}Y(t)) + bBg(t, Y(t), \mathcal{I}^\rho_{0^+}Y(t)) + \varepsilon Y(t).$$

We consider the set $\mathcal{U}_r = \{Y \in C^\gamma : \|Y\|_{C^\gamma} \leq r\}$, so that

$$\max\left\{2\|Y^*\|_2, \frac{\Gamma(\beta+1)\lambda^{-\gamma}\|Y^*\|_2}{\Gamma(\beta-\gamma+1)}, \Lambda_1, \Lambda_3, \Lambda_2, \Lambda_4\right\} \leq \frac{r}{3}. \tag{5}$$

We prove that $\mathbb{L}Y \in \mathcal{U}_r$, for any $Y \in \mathcal{U}_r$.
let $Y \in \mathcal{U}_r$, so we have

$$\begin{aligned}\|(\mathbb{L}Y)(t)\|_2 &= \left\|\frac{1}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1} G_Y(s)ds + \frac{(Y(\lambda)-Y^*)}{\lambda^\beta}t^\beta \right. \\ &\quad \left. -\frac{t^\beta}{\lambda^\beta\Gamma(\beta+\alpha)}\int_0^\lambda (\lambda-s)^{\beta+\alpha-1} G_Y(s)ds + Y^*\right\|_2 \\ &\leq \frac{2\|G_Y(t)\|_2 T^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + 2\|Y^*\|_2 + \|Y(\lambda)\|_2.\end{aligned} \tag{6}$$

According to ($\mathbb{A}_3$), we have

$$\begin{aligned}
&\|G_Y(t)\|_2 \qquad (7)\\
\leq\ & |a|\,\|A\|_2 \left\| f(t, Y(t), \mathcal{D}^{\gamma}_{0^+} Y(t)) \right\|_2 + |b|\,\|B\|_2 \left\| g(t, Y(t), \mathcal{I}^{\rho}_{0^+} Y(t)) \right\|_2 \\
&+\varepsilon \|Y(t)\|_2 \\
\leq\ & |a|\,\|A\|_2 \left( \varphi_1^* \|Y\|_C + \varphi_2^* \left\| \mathcal{D}^{\gamma}_{0^+} Y \right\|_C \right) + |b|\,\|A\|_2 \left( \varphi_3^* \|Y\|_C + \varphi_4^* \left\| \mathcal{I}^{\rho}_{0^+} Y \right\|_C \right) \\
&+\varepsilon \|Y\|_C \\
\leq\ & \left( \varphi_1^* |a|\,\|A\|_2 + \varphi_3^* |b|\,\|A\|_2 + \frac{\varphi_4^* \lambda^{\rho} |b|\,\|B\|_2}{\Gamma(\rho+1)} + \varepsilon \right) \|Y\|_C \\
&+\varphi_2^* |a|\,\|A\|_2 \left\| \mathcal{D}^{\gamma}_{0^+} Y \right\|_C \\
\leq\ & \mathcal{V}_1 \|Y\|_C + \mathcal{V}_2 \left\| \mathcal{D}^{\gamma}_{0^+} Y \right\|_C .
\end{aligned}$$

Substituting (7) in (6), we get

$$\begin{aligned}
\|(\mathbb{L}Y)(t)\|_2 \leq\ & \left( \mathcal{V}_1 \frac{2\lambda^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + n \sup_{i=\overline{1,n}} |\lambda_i| \right) \|Y\|_C \qquad (8)\\
&+ \frac{2\lambda^{\beta+\alpha} \mathcal{V}_2}{\Gamma(\beta+\alpha+1)} \left\| \mathcal{D}^{\gamma}_{0^+} Y \right\|_C + 2\|Y^*\|_2 \\
\leq\ & \Lambda_1 \|Y\|_C + \Lambda_2 \left\| \mathcal{D}^{\gamma}_{0^+} Y \right\|_C + 2\|Y^*\|_2 .
\end{aligned}$$

Also, one can observe that

$$\begin{aligned}
\left\| \left( \mathcal{D}^{\gamma}_{0^+} \mathbb{L}Y \right)(t) \right\|_2 \leq\ & \left( \frac{\lambda^{\beta+\alpha-\gamma}}{\Gamma(\beta+\alpha-\gamma+1)} + \frac{\Gamma(\beta+1)\,\lambda^{\beta+\alpha-\gamma}}{\Gamma(\beta-\gamma+1)\,\Gamma(\beta+\alpha+1)} \right) \|G_Y(t)\|_2 \qquad (9)\\
&+ \frac{\Gamma(\beta+1)\,\lambda^{-\gamma}}{\Gamma(\beta-\gamma+1)} \left( n \sup_{i=\overline{1,n}} |\lambda_i|\,\|Y\|_C + \|Y^*\|_2 \right) \\
\leq\ & \left\{ \frac{\lambda^{\beta+\alpha-\gamma} \mathcal{V}_1}{\Gamma(\beta+\alpha-\gamma+1)} + \frac{\Gamma(\beta+1)\,\lambda^{\beta+\alpha-\gamma} \mathcal{V}_1}{\Gamma(\beta-\gamma+1)\,\Gamma(\beta+\alpha+1)} \right. \\
&\left. + \frac{n\Gamma(\beta+1)\,\lambda^{-\gamma}}{\Gamma(\beta-\gamma+1)} \sup_{i=\overline{1,n}} |\lambda_i| \right\} \|Y\|_C + \frac{\Gamma(\beta+1)\,\lambda^{-\gamma} \|Y^*\|_2}{\Gamma(\beta-\gamma+1)} \\
&+ \left\{ \frac{\lambda^{\beta+\alpha-\gamma} \mathcal{V}_2}{\Gamma(\beta+\alpha-\gamma+1)} + \frac{\Gamma(\beta+1)\,\lambda^{\beta+\alpha-\gamma} \mathcal{V}_2}{\Gamma(\beta-\gamma+1)\,\Gamma(\beta+\alpha+1)} \right\} \left\| \mathcal{D}^{\gamma}_{0^+} Y \right\|_C \\
\leq\ & \Lambda_3 \|Y\|_C + \Lambda_4 \left\| \mathcal{D}^{\gamma}_{0^+} Y \right\|_C + \frac{\Gamma(\beta+1)\,v^{-\gamma} \|Y^*\|_2}{\Gamma(\beta-\gamma+1)}
\end{aligned}$$

By (8) and (9), we state that

$$\begin{aligned}
&\|\mathbb{L}Y\|_{C^{\gamma}} \qquad (10)\\
=\ & \max \left\{ \sup_{t\in J} \|(\mathbb{L}Y)(t)\|_2 , \sup_{t\in J} \left\| \left( \mathcal{D}^{\gamma}_{0^+} \mathbb{L}Y \right)(t) \right\|_2 \right\} \\
\leq\ & \max \left\{ 2, \frac{\Gamma(\beta+1)\,\lambda^{-\gamma}}{\Gamma(\beta-\gamma+1)} \right\} \|Y^*\|_2 + \max\{\Lambda_1, \Lambda_3\} \|Y\|_C + \max\{\Lambda_2, \Lambda_4\} \left\| \mathcal{D}^{\gamma}_{0^+} Y \right\|_C \\
\leq\ & r.
\end{aligned}$$

That is to say that $\mathbb{L}Y \in \mathcal{U}_r$.

**B:** We prove that the operator $\mathbb{L}$ is a contraction. For any $Y, Z \in \mathcal{U}_r$, we have the following estimate

$$\begin{aligned} & \left\| (\mathbb{L}Y)(t) - (\mathbb{L}U)(t) \right\|_2 \\ \leq & \left\| \frac{1}{\Gamma(\beta+\alpha)} \int_0^t (t-s)^{\beta+\alpha-1} \left[ G_Y(s) - G_U(s) \right] ds \right. \\ & - \frac{t^\beta}{\lambda^\beta} \frac{1}{\Gamma(\beta+\alpha)} \int_0^\lambda (\lambda-s)^{\beta+\alpha-1} \left[ G_Y(s) - G_U(s) \right] ds \\ & \left. - \frac{t^\beta}{\lambda^\beta} \sum_{i=\overline{1,n}} \lambda_i \left[ Y - U \right](\zeta_i) \right\|_2 . \end{aligned} \tag{11}$$

Thanks to ($\mathbb{A}_2$), we can write

$$\begin{aligned} & \left\| G_Y(s) - G_U(s) \right\|_2 \\ \leq & |a| \, \|A\|_2 \, \delta_1 \left( \|Y(t) - U(t)\|_2 + \left\| \mathcal{D}_{0^+}^{\gamma} \left( Y(t) - U(t) \right) \right\|_2 \right) \\ & + |b| \, \|B\|_2 \, \delta_2 \left( \|Y(t) - U(t)\|_2 + \left\| \mathcal{I}_{0^+}^{\rho} \left( Y(t) - U(t) \right) \right\|_2 \right) \\ & + \varepsilon \, \|Y(t) - U(t)\|_2 \\ \leq & \left( 2 |a| \, \|A\|_2 \, \delta_1 + |b| \, \|B\|_2 \, \delta_2 + \frac{|b| \, \|B\|_2 \, \lambda^\rho}{\Gamma(\rho+1)} \delta_2 + \varepsilon \right) \|Y - U\|_{C^\gamma} \\ \leq & \mathcal{V}_3 \, \|Y - U\|_{C^\gamma} . \end{aligned} \tag{12}$$

Substituting (12) in (11), we get

$$\begin{aligned} \left\| (\mathbb{L}Y)(t) - (\mathbb{L}U)(t) \right\|_2 & \leq \left( \frac{2 \lambda^{\beta+\alpha} \mathcal{V}_3}{\Gamma(\beta+\alpha+1)} + n \sup_{i=\overline{1,n}} |\lambda_i| \right) \|Y - U\|_{C^\gamma} \\ & \leq \Upsilon_1 \, \|Y - U\|_{C^\gamma} . \end{aligned} \tag{13}$$

Also, we observe that

$$\begin{aligned} & \left\| \mathcal{D}_{0^+}^{\gamma} (\mathbb{L}Y)(t) - \mathcal{D}_{0^+}^{\gamma} (\mathbb{L}U)(t) \right\|_2 \\ \leq & \left( \frac{\lambda^{\beta+\alpha-\gamma}}{\Gamma(\beta+\alpha-\gamma+1)} + \frac{\Gamma(\beta+1) \lambda^{\beta+\alpha-\gamma}}{\Gamma(\beta-\gamma+1) \Gamma(\beta+\alpha+1)} \right) \|G_Y(s) - G_U(s)\|_2 \\ & + \frac{\Gamma(\beta+1) \lambda^{-\gamma}}{\Gamma(\beta-\gamma+1)} \left( n \sup_{i=\overline{1,n}} |\lambda_i| \, \|Y - U\|_C \right) \\ \leq & \left\{ \frac{\lambda^{\beta+\alpha-\gamma} \mathcal{V}_3}{\Gamma(\beta+\alpha-\gamma+1)} + \frac{\Gamma(\beta+1) \lambda^{\beta+\alpha-\gamma} \mathcal{V}_3}{\Gamma(\beta-\gamma+1) \Gamma(\beta+\alpha+1)} \right. \\ & \left. + \frac{n \Gamma(\beta+1) \lambda^{-\gamma}}{\Gamma(\beta-\gamma+1)} \sup_{i=\overline{1,n}} |\lambda_i| \right\} \|Y - U\|_{C^\gamma} \\ \leq & \Upsilon_2 \, \|Y - U\|_{C^\gamma} \end{aligned} \tag{14}$$

By (13) and (14), we constat that

$$\|\mathbb{L}Y - \mathbb{L}U\|_{C^\gamma} \leq \max \{\Upsilon_1, \Upsilon_2\} \, \|Y - U\|_{C^\gamma} .$$

Therefore, we conclude that $\mathbb{L}$ is contraction..
Thanks to **A** and **B**, and by applying Banach fixed point theorem, there exists a unique solution-stochastic process of the problem (2). □

### 3.2. Criteria For Existence of a Solution-Stochastic Process

**Theorem 3.2.** *Assume that* $(\mathbb{A}_3)$ *is satisfied. So, the problem (2) has at least one solution-stochastic process* $Y(t)$ *on* $J$.

*Proof.* **First step:** First of all, we show that the operator $\mathbb{L}$ is completely continuous on $C^\gamma$.

Let $Y_n, Y \in \mathcal{U}_r$, so we have

$$\begin{aligned} & \left\|(\mathbb{L}Y_n)(t) - (\mathbb{L}Y)(t)\right\|_2 \\ \leq & \left\| \frac{1}{\Gamma(\beta+\alpha)} \int_0^t (t-s)^{\beta+\alpha-1} \left[G_{Y_n}(s) - G_Y(s)\right] ds \right. \\ & - \frac{t^\beta}{T^\beta} \frac{1}{\Gamma(\beta+\alpha)} \int_0^\lambda (\lambda-s)^{\beta+\alpha-1} \left[G_{Y_n}(s) - G_Y(s)\right] ds \\ & \left. - \frac{t^\beta}{\lambda^\beta} \sum_{i=\overline{1,n}} \lambda_i \left[Y_n - Y\right](\zeta_i) \right\|_2 . \end{aligned} \tag{15}$$

Substituting (12) in (15), we get

$$\sup_{t \in J} \left\|(\mathbb{L}Y_n)(t) - (\mathbb{L}Y)(t)\right\|_2 \leq \Upsilon_1 \left\|Y_n - Y\right\|_{C^\gamma} . \tag{16}$$

Also

$$\sup_{t \in J} \left\|\mathcal{D}_{0^+}^\gamma (\mathbb{L}Y_n)(t) - \mathcal{D}_{0^+}^\gamma (\mathbb{L}Y)(t)\right\|_2 \leq \Upsilon_2 \left\|Y_n - Y\right\|_{C^\gamma} . \tag{17}$$

By (16) and (17), it yields that

$$\left\|\mathbb{L}Y_n - \mathbb{L}Y\right\|_{C^\gamma} \leq \max\{\Upsilon_1, \Upsilon_2\} \left\|Y_n - Y\right\|_{C^\gamma} .$$

So, we have

$$\left\|\mathbb{L}Y_n - \mathbb{L}Y\right\|_{C^\gamma} \to 0, \quad \text{as } n \to \infty.$$

Consequently, $\mathbb{L}$ is continuous on $C^\gamma$.

**Second step:** $\mathbb{L}$ maps bounded sets into bounded sets in $C^\gamma$. Indeed, it is enough to show that for any $r > 0$, there exists a positive constant $l$ such that for each
$Y \in \mathcal{U}_r$ one has $\|Y\|_{C^\gamma} \leq l$.

Let $Z \in \mathcal{U}_r$. We put:

$$l = \max\left\{(\Lambda_1 + \Lambda_2) r + 2\|Y^*\|_2, (\Lambda_3 + \Lambda_4) r + \frac{\Gamma(\beta+1)\lambda^{-\gamma}\|Y^*\|_2}{\Gamma(\beta-\gamma+1)}\right\}.$$

By $(\mathbb{A}_2)$, $(\mathbb{A}_3)$, (8) and (9), for all $t \in J$, we obtain,

$$\|\mathbb{L}Z\|_C \leq (\Lambda_1 + \Lambda_2) r + 2\|Y^*\|_2$$

and

$$\left\|\mathcal{D}_{0^+}^\gamma \mathbb{L}Z\right\|_C \leq (\Lambda_3 + \Lambda_4) r + \frac{\lambda^{-\gamma}\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)} \|Y^*\|_2 .$$

Hence,

$$\begin{aligned} & \|\mathbb{L}Z\|_{C^\gamma} \\ \leq & \max\left\{(\Lambda_1 + \Lambda_2) r + 2\|Y^*\|_2, (\Lambda_3 + \Lambda_4) r + \frac{\lambda^{-\gamma}\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)} \|Y^*\|_2\right\}. \end{aligned}$$

Consequently, $\mathbb{L}$ is uniformly bounded on $\mathcal{U}_r$.

**Third step:** $\mathbb{L}$ maps bounded sets into equicontinuous sets of $C^\gamma$. The functions $Y, f, g$ are continuous, hence the operator $\mathbb{L}$ is continuous. For any $Y \in \mathcal{U}_r$ and $t_1, t_2 \in [0, \lambda]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
& \|(\mathbb{L}Y)(t_2) - (\mathbb{L}Y)(t_1)\|_2 \\
= \; & \Bigg\| \frac{1}{\Gamma(\beta+\alpha)} \int_0^{t_2} (t_2 - s)^{\beta+\alpha-1} G_Y(s) ds \\
& - \frac{1}{\Gamma(\beta+\alpha)} \int_0^{t_1} (t_1 - s)^{\beta+\alpha-1} G_Y(s) ds \\
& - \frac{t_2^\beta - t_1^\beta}{\lambda^\beta} \frac{1}{\Gamma(\beta+\alpha)} \int_0^{\lambda} (\lambda - s)^{\beta+\alpha-1} G_Y(s) ds \\
& + \frac{t_2^\beta - t_1^\beta}{\lambda^\beta} \left( \sum_{i=\overline{1,n}} \lambda_i Y(\zeta_i) - Y^* \right) \Bigg\|_2 \\
\leq \; & \frac{\|G_Y(s)\|_2}{\Gamma(\beta+\alpha+1)} \left( t_2^{\beta+\alpha} - t_1^{\beta+\alpha} \right) + \\
& \left( \frac{\lambda^\alpha \|G_Y(s)\|_2}{\Gamma(\beta+\alpha+1)} + n\lambda^{-\beta} \sup_{i=\overline{1,n}} |\lambda_i| \, r + \lambda^{-\beta} \|Y^*\|_2 \right) \left( t_2^\beta - t_1^\beta \right).
\end{aligned}$$

By (7), we obtain

$$\begin{aligned}
& \|(\mathbb{L}Y)(t_2) - (\mathbb{L}Y)(t_1)\|_2 \\
\leq \; & \frac{(\mathcal{V}_1 + \mathcal{V}_2) \|Y\|_{C^\gamma}}{\Gamma(\beta+\alpha+1)} \left( t_2^{\beta+\alpha} - t_1^{\beta+\alpha} \right) \\
& + \left( \frac{\lambda^\alpha (\mathcal{V}_1 + \mathcal{V}_2) \|Y\|_{C^\gamma}}{\Gamma(\beta+\alpha+1)} + n\lambda^{-\beta} \sup_{i=\overline{1,n}} |\lambda_i| \, r + \lambda^{-\beta} \|Y^*\|_2 \right) \left( t_2^\beta - t_1^\beta \right).
\end{aligned}$$

Hence,

$$\|(\mathbb{L}Y)(t_2) - (\mathbb{L}Y)(t_1)\|_2 \to 0, \text{ as } t_2 \to t_1.$$

Also

$$\begin{aligned}
& \left\| \left( \mathcal{D}_{0^+}^\gamma \mathbb{L}Y \right)(t_2) - \left( \mathcal{D}_{0^+}^\gamma \mathbb{L}Y \right)(t_1) \right\|_2 \\
= \; & \Bigg\| \frac{1}{\Gamma(\beta+\alpha-\gamma)} \int_0^{t_2} (t_2 - s)^{\beta+\alpha-\gamma-1} G_Y(s) ds \\
& - \frac{1}{\Gamma(\beta+\alpha-\gamma)} \int_0^{t_1} (t_1 - s)^{\beta+\alpha-\gamma-1} G_Y(s) ds \\
& - \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)\Gamma(\beta+\alpha)} \frac{\left( t_2^{\beta-\gamma} - t_1^{\beta-\gamma} \right)}{\lambda^\beta} \int_0^{\lambda} (\lambda - s)^{\beta+\alpha-1} G_Y(s) ds \\
& + \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)} \frac{\left( t_2^{\beta-\gamma} - t_1^{\beta-\gamma} \right)}{\lambda^\beta} (Y(\lambda) - Y^*) \Bigg\|_2 .
\end{aligned}$$

By (7), we obtain

$$\begin{aligned}
&\left\|\left(\mathcal{D}_{0^+}^{\gamma}\mathbb{L}Y\right)(t_2)-\left(\mathcal{D}_{0^+}^{\gamma}\mathbb{L}Y\right)(t_1)\right\|_2\\
\leq\ & \frac{(\mathcal{V}_1+\mathcal{V}_2)\|Y\|_{C^\gamma}}{\Gamma(\beta+\alpha-\gamma+1)}\left(t_2^{\beta+\alpha-\gamma}-t_1^{\beta+\alpha-\gamma}\right)\\
&+\left\{\frac{\lambda^{\alpha}\Gamma(\beta+1)(\mathcal{V}_1+\mathcal{V}_2)\|Y\|_{C^\gamma}}{\Gamma(\beta-\gamma+1)\Gamma(\beta+\alpha+1)}\right.\\
&\left.+\frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}\left(n\lambda^{-\beta}\sup_{i=\overline{1,n}}|\lambda_i|\,r+\lambda^{-\beta}\|Y^*\|_2\right)\right\}\left(t_2^{\beta-\gamma}-t_1^{\beta-\gamma}\right).
\end{aligned}$$

Hence,

$$\left\|\left(\mathcal{D}_{0^+}^{\gamma}\mathbb{L}Y\right)(t_2)-\left(\mathcal{D}_{0^+}^{\gamma}\mathbb{L}Y\right)(t_1)\right\|_2\to 0,\ \text{ as }\ t_2\to t_1.$$

As a consequence of the above three steps, together with the Arzela–Ascoli theorem, we conclude that $\mathbb{L}$ is completely continuous.

**Forth step:** The set defined by $\mathbb{D}=\{Y\in C^\gamma : Y=\sigma\mathbb{L}Y, 0<\sigma<1\}$ is bounded: let $W\in\mathbb{D}$, then $W=\sigma\mathbb{L}W$, hence, we can write

$$\begin{aligned}
\|W\|_{C^\gamma} &= \|\sigma\mathbb{L}W\|_{C^\gamma}=\sigma\|\mathbb{L}W\|_{C^\gamma}\\
&\leq \sigma\max\left\{(\Lambda_1+\Lambda_2)r+2\|Y^*\|_2,(\Lambda_3+\Lambda_4)r+\frac{\lambda^{-\gamma}\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}\|Y^*\|_2\right\}\\
&\leq \sigma r<+\infty.
\end{aligned}$$

The set is thus bounded.

As a consequence of Schaefer fixed point theorem, we deduce that $\mathbb{L}$ has a fixed point which is a solution-stochastic process of (2). The proof of Theorem 3.2 is thus completely achieved. □

### 3.3. *An Example*

**Example 3.3.** *Consider the following problem*

$$\mathcal{D}_{0^+}^{0,6}\left(\mathcal{D}_{0^+}^{0,9}Y(t)\right)=\frac{Y(t)}{10^{1000}}+\frac{1}{15}Af(t,Y(t),\mathcal{D}_{0^+}^{0,5}Y(t))+\frac{1}{20}Bg(t,Y(t),\mathcal{I}_{0^+}^{0,5}Y(t)),t\in J=[0,2] \tag{18}$$

$$\begin{aligned}
\|Y^*\|_2 &= \frac{1}{3},\ \|A\|_2=\frac{1}{5},\ \textit{and}\ \ \|B\|_2=\frac{3}{25},\\
f(t,Y(t),U(t)) &= \frac{Y(t)}{(9+t)(1+\|Y(t)\|_2)}+\frac{U(t)}{9(1+t^4)(1+\|U(t)\|_2)},\\
g(t,Y(t),U(t)) &= \frac{Y(t)}{(2+e^t)(1+\|Y(t)\|_2)}+\frac{U(t)}{(3+\sin t)(1+\|U(t)\|_2)}.
\end{aligned}$$

*The two $f,g$ satisfy*

$$\left\|f(t,Y(t),U(t))\right\|_2\leq\frac{\|Y(t)\|_2}{(1+\|Y\|_{C^\gamma})(9+t)}+\frac{\|U(t)\|_2}{9(1+\|Y\|_{C^\gamma})(1+t^4)}$$

$$\left\|g(t,Y(t),U(t))\right\|_2\leq\frac{\|Y(t)\|_2}{(1+\|Y\|_{C^\gamma})(2+e^t)}+\frac{\|U(t)\|_2}{(1+\|U\|_{C^\gamma})(3+\sin t)}.$$

*For any $Y,U\in L^2(\Omega)$ and $t\in J$, we have*

$$\left\|f(t,Y(t),U(t))-f(t,X(t),V(t))\right\|_2\leq\frac{1}{9}\left(\|Y(t)-U(t)\|_2+\|X(t)-V(t)\|_2\right),$$

*and*

$$\left\|g(t,Y(t),U(t))-g(t,X(t),V(t))\right\|_2 \le \frac{1}{3}\left(\|Y(t)-U(t)\|_2+\|X(t)-V(t)\|_2\right).$$

*By Theorem 3.1, and Theorem 3.2, the problem (18) has an unique solution-stochastic process on J.*

**Example 3.4.** *Consider the following problem*

$$\mathcal{D}^{0,7}_{0^+}\left(\mathcal{D}^{0,8}_{0^+}Y(t)\right)=\frac{10}{99}Y(t)+\frac{1}{25}\left[f(t,Y(t),\mathcal{D}^{0,5}_{0^+}Y(t))+g(t,Y(t),\mathcal{I}^{0,5}_{0^+}Y(t))\right], t\in J=[0,1] \tag{19}$$

$$\begin{aligned}
\|Y^*\|_2 &= \frac{3}{7},\ \|A\|_2=\|B\|_2=\frac{1}{100}, a=b=4\\
f(t,Y(t),U(t)) &= \frac{Y(t)}{(6+e^t)(5+\|Y(t)\|_2)}+\frac{U(t)}{7(1+t^4)(3+\|U(t)\|_2)},\\
g(t,Y(t),U(t)) &= \frac{Y(t)}{2\left(1+e^{2t^2}\right)(1+\|Y(t)\|_2)}+\frac{U(t)}{(5-\cos t)(1+\|U(t)\|_2)}.
\end{aligned}$$

*The two $f,g$ satisfy*

$$\left\|f(t,Y(t),U(t))\right\|_2 \le \frac{\|Y(t)\|_2}{(6+e^t)(7+\|Y\|_{C^\gamma})}+\frac{\|U(t)\|_2}{7(1+t^4)(9+\|U\|_{C^\gamma})},$$

$$\left\|g(t,Y(t),U(t))\right\|_2 \le \frac{\|Y(t)\|_2}{2\left(1+e^{2t^2}\right)(1+\|Y\|_{C^\gamma})}+\frac{\|U(t)\|_2}{(5-\cos t)(1+\|Y\|_{C^\gamma})}.$$

*For any $Y,U\in L^2(\Omega)$ and $t\in J$, we have*

$$\left\|f(t,Y(t),U(t))-f(t,X(t),V(t))\right\|_2 \le \frac{1}{7}\left(\|Y(t)-U(t)\|_2+\|X(t)-V(t)\|_2\right),$$

*and*

$$\left\|g(t,Y(t),U(t))-g(t,X(t),V(t))\right\|_2 \le \frac{1}{4}\left(\|Y(t)-U(t)\|_2+\|X(t)-V(t)\|_2\right).$$

*By Theorem 3.1, and Theorem 3.2, the problem (18) has an unique solution-stochastic process on J.*

**Availability of data and material**

Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

**Competing Interests**

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