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# Existence results for fractional integrodifferential equations with infinite delay and fractional integral boundary conditions

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**Abstract.** In this paper, we study the existence and uniqueness of solutions for the functional and neutral functional integrodifferential equations of fractional order with infinite delay and multi-point multi-term fractional integral boundary conditions by using fixed point theorems. The fractional derivative considered here is in the Liouville-Caputo sense. Examples are provided to illustrate the results.

## 1. Introduction

The theory of fractional differential equations has emerged as an active area of research motivated largely by new applications in many areas of science and engineering. Differential and integral equations involving derivatives of fractional order have proved to be appropriate models for various phenomena arising in diffusion and transport theory, models of earthquake, mathematical physics and engineering, fluid-dynamic traffic model, fluid and continuum mechanics, chemistry, acoustics and psychology. Indeed, it is well known that the analysis of fractional differential equations is more complex than that of classical differential equations due to the fact that fractional derivatives are nonlocal and have weakly singular kernels. For a detailed study of the theory and applications of fractional calculus and fractional differential equations, one can refer to the books [4, 16, 26–28] and the articles [9, 24, 31–33].

In recent years, boundary value problems of fractional differential equations involving a variety of boundary conditions have been investigated by several researchers [1–3, 11, 13, 14, 25, 29, 30, 36, 38]. In particular, integral boundary conditions have various applications in applied fields. Also many practically important problems lead to multi-point boundary value problems which arise in many areas of applied sciences such as heat conduction, electric power networks, elastic stability, telecommunication lines and electric railway systems etc. Existence results for various kinds of fractional differential equations can be found in [5–8, 10, 19–23, 37, 39, 40] and the references therein. However the theory of fractional functional boundary value problems is not fully explored and many aspects of this theory need to be studied.

Benchohra *et al.*[10] established the existence and uniqueness of solutions for fractional functional differential equations with infinite delay by using the nonlinear alternative of Leray-Schauder type and the

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Banach fixed point theorem respectively for

$$\begin{aligned} D_t^\alpha y(t) &= f\Big(t, y_t\Big), \ 0 < \alpha < 1, \ t \in J = [0, b], \\ y(t) &= \phi(t), \ t \in (-\infty, 0], \end{aligned}$$

where $D_t^\alpha$ is the standard Riemann-Liouville fractional derivative, $f : J \times B \to \mathbb{R}$ is a given function, $\phi \in B$, $\phi(0) = 0$ and $B$ is a phase space.

Chauhan *et al.* [12] studied the existence results for an impulsive fractional functional integro-differential equation with infinite delay and integral boundary condition of the form

$$\begin{aligned} {}^{\mathrm{LC}}D_t^\alpha x(t) &= f\Big(t, x_t, Bx(t)\Big), \ 1 < \alpha < 2, \ t \in J = [0, T], \ t \neq t_k, \\ \Delta x(t_k) &= Q_k(x(t_k^-)), \ k = 1, 2, ..., m, \\ \Delta x'(t_k) &= I_k(x(t_k^-)), \ k = 1, 2, ..., m, \\ x(t) &= \phi(t), \ t \in (-\infty, 0], \\ ax(0) + bx'(T) &= \int_0^T q(x(s)) \mathrm{d}s, \end{aligned}$$

where ${}^{\mathrm{LC}}D_t^\alpha$ is the Liouville-Caputo fractional derivative, $a, b \in \mathbb{R}$ such that $a + b \neq 0$, $f : J \times \mathcal{B}_h \times X \to X$ and $q : X \to X$ are given functions with $\mathcal{B}_h$ to be the phase space. Here $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $Q_k$, $I_k \in C(X, X)$, $(k = 1, 2, ..., m)$, are bounded functions, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$ and $Bx(t) = \int_0^t K(t, s)\, x(s) \mathrm{d}s$. The results are proved by applying the well known fixed point theorems.

Dabas and Gautam [15] investigated the existence results for an impulsive neutral fractional integro-differential equation with state dependent delays and integral boundary condition of the form

$$\begin{aligned} D_t^\alpha &\left[ x(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s, x_s)}\right) \mathrm{d}s \right] \\ &= f\left(t, x_{\rho(t, x_t)}, B(x)(t)\right), \ 1 < \alpha < 2, \ t \in J = [0, T], \ T > \infty, \ t \neq t_k, \\ \Delta x(t_k) &= I_k(x(t_k^-)), \ \Delta x'(t_k) = Q_k(x(t_k^-)), \ k = 1, 2, ..., m, \\ x(t) &= \phi(t), \ t \in [-d, 0], \\ ax(0) + bx'(T) &= \int_0^T q(x(s)) \mathrm{d}s, \ a + b \neq 0, \ b \neq 0, \end{aligned}$$

where $f : J \times PC_0 \times X \to X$, $g : J \times PC_0 \to X$ and $q : X \to X$ are given continuous functions with $PC_0 = PC([-d, 0], X)$. Here $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $Q_k$, $I_k \in C(X, X)$, $(k = 1, 2, ..., m)$, are continuous and bounded functions. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$ and $Bx(t) = \int_0^t K(t, s)\, x(s) \mathrm{d}s$. The results are proved by using the classical fixed point theorems. Srivastava et al.[34, 35] discussed the solutions of fractional differential equations with different Laplacian operators.

Motivated by the above works, in the first part of this paper, we study the existence and uniqueness results for the following fractional functional integrodifferential equation with infinite delay of the form

$$\left.\begin{aligned} &{}^{\mathrm{LC}}D_t^q x(t) = f\left(t, x_t, \int_0^t k\Big(t, s, x_s\Big) \mathrm{d}s\right), \ 1 < q \leq 2, \ t \in J = [0, T], \\ &x(t) = \phi(t), \ t \in (-\infty, 0], \\ &x(T) = \sum_{i=1}^m a_i (I_{0+}^{p_i} x)(\eta_i), \ 0 < \eta_1 < \eta_2 < ... < \eta_m < T, \end{aligned}\right\} \quad (1)$$

where the functions $f : J \times \mathcal{B} \times X \to X$, $k : \Omega \times \mathcal{B} \to X$ are continuous with $X$ as a Banach space and $\phi \in \mathcal{B}$, a phase space to be defined later. $I_{0+}^{p_i}$ is the Riemann-Liouville fractional integral of order $p_i > 0$ and $a_i$ are suitably chosen real constants, for $i = 1, 2, ..., m$. Here $\Omega = \{(t, s) : 0 \leq s \leq t \leq T\}$.

For any continuous function $x$ defined on $(-\infty, T]$ and any $t \in J$, we denote by $x_t$ the element of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t + \theta), \ \theta \in (-\infty, 0].$$

Here $x_t(\cdot)$ represents the history of the state from time $-\infty$ up to the present time $t$. We assume that the histories $x_t$ belong to some abstract phase space $\mathcal{B}$.

In the second part of this paper, we consider the fractional neutral integrodifferential boundary value problem

$$\left.\begin{aligned} {}^{\mathrm{LC}}D_t^q\left[x(t)-\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}g\left(s,x_s,\int_0^s k_1(s,\tau,x_\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right]&\\ =f\left(t,x_t,\int_0^t k_2(t,s,x_s)\,\mathrm{d}s\right),\ 1<q\leq 2,\ t\in J,&\\ x(t)=\phi(t),\ t\in(-\infty,0],&\\ x(T)=\sum_{i=1}^m a_i(I_{0+}^{p_i}x)(\eta_i),\ 0<\eta_1<\eta_2<\ldots<\eta_m<T,& \end{aligned}\right\}\tag{2}$$

where the functions $f,g: J\times\mathcal{B}\times X\to X$, $k_1,k_2:\Omega\times\mathcal{B}\to X$ are continuous.

The paper is organized as follows: In Section 2, some basic definitions, notions and results are recalled in support of the subsequent sections. In Section 3, we present our main results on existence and uniqueness of solutions to the fractional functional integrodifferential equation using Krasnoselskii's fixed point theorem and Banach contraction principle respectively. In Section 4, existence and uniqueness results for the fractional neutral integrodifferential equation are discussed. Examples are presented to illustrate the applicability of the imposed conditions. It is worth mentioning that no contributions exist in the literature studying fractional functional and neutral integrodifferential equations with infinite delay and multipoint multiterm fractional integral boundary conditions and hence this paper attempts to fill this gap in the existing literature.

## 2. Preliminaries

In this section, we state some basic definitions, notations and lemmas [26] which will be used throughout the work. Let $X$ be a Banach space with norm $\|\cdot\|$ and $C(J,X)$ denote the Banach space of all continuous functions from $J\to X$ endowed with the topology of uniform convergence with the norm denoted by $\|\cdot\|_C$.

**Definition 2.1.** *The Riemann-Liouville fractional integral of a function $f\in L^1(\mathbb{R}^+)$ of order $q>0$ is defined by*

$$I_{0+}^q f(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s)\mathrm{d}s,$$

*provided the integral exists.*

**Definition 2.2.** *The Liouville-Caputo fractional derivative of order $n-1<q\leq n$ is defined by*

$${}^{\mathrm{LC}}D_t^q f(t)=\frac{1}{\Gamma(n-q)}\int_0^t(t-s)^{n-q-1}f^{(n)}(s)\mathrm{d}s,$$

*where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$. In particular, if $0<q\leq 1$,*

$${}^{\mathrm{LC}}D_t^q f(t)=\frac{1}{\Gamma(1-q)}\int_0^t\frac{f'(s)}{(t-s)^q}\mathrm{d}s,$$

*where $f'(s)=Df(s)=\frac{df(s)}{\mathrm{d}s}$.*

For brevity of notation, $I_{0+}^q$ is taken as $I^q$ and ${}^{\mathrm{LC}}D_t^q$ is taken as ${}^{\mathrm{LC}}D^q$.

**Lemma 2.3.** *[26] Let $p,q\geq 0$, $f\in L^1[a,b]$. Then $I^pI^qf(t)=I^{p+q}f(t)=I^qI^pf(t)$ and ${}^{\mathrm{LC}}D^qI^qf(t)=f(t)$, for all $t\in[a,b]$.*

In this paper, we assume that the state space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $X$ and satisfying the following fundamental axioms which were introduced by Hale and Kato [18].

(A) If $x : (-\infty, T] \to X$, is continuous on $[0, T]$ and $x_0 \in \mathcal{B}$, then, for every $t \in [0, T]$, the following conditions hold:
  (i) $x_t$ is in $\mathcal{B}$,
  (ii) $\|x(t)\| \le H\|x_t\|_{\mathcal{B}}$,
  (iii) $\|x_t\|_{\mathcal{B}} \le M_1(t) \sup\{\|x(s)\| : 0 \le s \le t\} + M_2(t)\|x_0\|_{\mathcal{B}}$,
  where $H \ge 0$ is a constant, $M_1 : [0,\infty) \to [0,\infty)$ is continuous, $M_2 : [0,\infty) \to [0,\infty)$ is locally bounded and $H$, $M_1$, $M_2$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in $(A)$, $x_t$ is a $\mathcal{B}$-valued continuous function on $[0, T]$.

(C) The space $\mathcal{B}$ is complete.

Let $M_1^* = \sup_{0\le t\le T} M_1(t)$ and $M_2^* = \sup_{0\le t\le T} M_2(t)$.

## 3. Boundary Value Problem of Fractional Order

Let the space $\bar{\Omega} = \Big\{x : (-\infty, T] \to X : x\,|_{(-\infty,0]} \in \mathcal{B}$ and $x\,|_{[0,T]}$ is continuous$\Big\}$ and take $Kx(t) = \int_0^t k\big(t, s, x_s\big)ds$.

**Definition 3.1.** *A function $x \in \bar{\Omega}$ is said to be a solution of* (1) *if it satisfies the equation*

$$^{LC}D^q x(t) = f\,(t, x_t, Kx(t))$$

*on $J$ and the boundary conditions*

$$\begin{aligned} x(t) &= \phi(t),\ t \in (-\infty, 0], \\ x(T) &= \sum_{i=1}^{m} a_i(I^{p_i}x)(\eta_i),\ 0 < \eta_1 < \eta_2 < \dots < \eta_m < T. \end{aligned}$$

To study the nonlinear problem (1), first we consider the linear problem and obtain its solution.

**Lemma 3.2.** *For $f(t) \in C(J, X)$, the unique solution of the fractional boundary value problem*

$$\left.\begin{aligned} {}^{LC}D^q x(t) &= f\big(t\big),\ 1 < q \le 2,\ t \in J, \\ x(t) &= \phi(t),\ t \in (-\infty, 0], \\ x(T) &= \sum_{i=1}^{m} a_i(I^{p_i}x)(\eta_i),\ 0 < \eta_1 < \eta_2 < \dots < \eta_m < T, \end{aligned}\right\} \tag{3}$$

*is given by*

$$x(t) = \begin{cases} \phi(t),\ t \in (-\infty, 0], \\ I^q f(t) + \frac{t}{A}\left(\sum\limits_{i=1}^{m} a_i I^{p_i+q} f(\eta_i) - I^q f(T)\right) + \phi(0)\left[1 + \frac{t}{A}\left(\sum\limits_{i=1}^{m} \frac{a_i\eta_i^{p_i}}{\Gamma(p_i+1)} - 1\right)\right],\ t \in J, \end{cases} \tag{4}$$

*where* $A = T - \sum_{i=1}^{m} \frac{a_i\eta_i^{p_i+1}}{\Gamma(p_i+2)} \neq 0.$

*Proof.* For some vector constants $c_0, c_1 \in X$, the general solution of (3) can be written as [26]

$$x(t) = I^q f(t) + c_0 + c_1 t. \tag{5}$$

Using the boundary condition $x(t) = \phi(t)$ in (5), we have

$$c_0 = \phi(0). \tag{6}$$

Next, using the boundary condition $x(T) = \sum_{i=1}^{m} a_i (I^{p_i} x)(\eta_i)$, we have

$$c_1 = \frac{1}{\left[T - \sum_{i=1}^{m} \frac{a_i \eta_i^{p_i+1}}{\Gamma(p_i+2)}\right]} \left\{ \sum_{i=1}^{m} a_i I^{p_i+q} f(\eta_i) + \phi(0) \left[ \sum_{i=1}^{m} \frac{a_i \eta_i^{p_i}}{\Gamma(p_i+1)} - 1 \right] - I^q f(T) \right\}. \tag{7}$$

Substituting the above values of $c_0$ and $c_1$ in (5), we get

$$x(t) = I^q f(t) + \frac{t}{A} \left( \sum_{i=1}^{m} a_i I^{p_i+q} f(\eta_i) - I^q f(T) \right) + \phi(0) \left[ 1 + \frac{t}{A} \left( \sum_{i=1}^{m} \frac{a_i \eta_i^{p_i}}{\Gamma(p_i+1)} - 1 \right) \right].$$

□

*3.1. Main Results*

In addition to the conditions stated for the problem (1) we assume that the following conditions:

(A1) There exist positive constants $L_f$ and $L_k$ such that
(i) $\|f(t,\phi_1,x_1) - f(t,\phi_2,x_2)\|_X \le L_f \left(\|\phi_1 - \phi_2\|_{\mathcal{B}} + \|x_1 - x_2\|_X\right)$, $t \in J$, $\phi_1, \phi_2 \in \mathcal{B}$, $x_1, x_2 \in X$,
(ii) $\left\|k(t,s,\psi_1) - k(t,s,\psi_2)\right\|_X \le L_k \|\psi_1 - \psi_2\|_{\mathcal{B}}$, $t, s \in J$, $\psi_1, \psi_2 \in \mathcal{B}$.

(A2) For $p_i \in L^1(J, \mathbb{R}^+)$, $i = 1,2,3$, we have
(i) $\|f(t,\phi,x)\|_X \le p_1(t)\|\phi\|_{\mathcal{B}} + p_2(t)\|x\|_X$, $(t,\phi,x) \in J \times \mathcal{B} \times X$,
(ii) $\|k(t,s,\psi)\|_X \le p_3(t)\|\psi\|_{\mathcal{B}}$, $(t,s,\psi) \in \Omega \times \mathcal{B}$.

(A3) Let $\Lambda = L_f M_1^* \{\theta_1 + L_k \theta_2\} < 1$ where
$\theta_1 = \left(1 + \frac{T}{|A|}\right)\gamma_1 + \frac{T}{|A|}\gamma_3$ and $\theta_2 = \left(1 + \frac{T}{|A|}\right)\gamma_2 + \frac{T}{|A|}\gamma_4$ with $\gamma_1 = \frac{T^q}{\Gamma(q+1)}$, $\gamma_2 = \frac{T^{q+1}}{\Gamma(q+2)}$,
$\gamma_3 = \sum_{i=1}^{m} |a_i| \frac{\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}$ and $\gamma_4 = \sum_{i=1}^{m} |a_i| \frac{\eta_i^{p_i+q+1}}{\Gamma(p_i+q+2)}$.

We prove the existence of solutions to (1) by applying Krasnoselskii's fixed point theorem.

**Lemma 3.3 (Krasnoselskii Theorem).** *[17] . Let S be a closed, convex, nonempty subset of a Banach space X. Let $\mathcal{P}, \mathcal{Q}$ be two operators such that*

*(i) $\mathcal{P}x + \mathcal{Q}y \in S$ whenever $x, y \in S$,*

*(ii) $\mathcal{P}$ is compact and continuous,*

*(iii) $\mathcal{Q}$ is a contraction mapping.*

*Then there exists $z \in S$ such that $z = \mathcal{P}z + \mathcal{Q}z$.*

**Theorem 3.4.** *Suppose that the assumptions (A1) and (A2) hold with*

$$L = \frac{T}{|A|} L_f M_1^* \{(\gamma_1 + \gamma_3) + L_k(\gamma_2 + \gamma_4)\} < 1. \tag{8}$$

*Then the boundary value problem (1) has at least one solution on $(-\infty, T]$.*

*Proof.* In view of Lemma 3.2, we transform (1) into a fixed point problem. Consider the operator $N : \bar{\Omega} \to \bar{\Omega}$ defined by

$$(Nx)(t) = \begin{cases} \phi(t),\ t \in (-\infty, 0], \\ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x_s, Kx(s))ds + \frac{t}{A}\left( \sum_{i=1}^{m} a_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} f(s, x_s, Kx(s))ds \right. \\ \left. - \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x_s, Kx(s))ds \right) + \phi(0)\left[1 + \frac{t}{A}\left( \sum_{i=1}^{m} \frac{a_i \eta_i^{p_i}}{\Gamma(p_i+1)} - 1 \right)\right],\ t \in J. \end{cases} \tag{9}$$

Let $y(\cdot) : (-\infty, T] \to X$ be the function defined by

$$y(t) = \begin{cases} \phi(t),\ t \in (-\infty, 0], \\ 0,\ t \in J. \end{cases}$$

Then $y_0 = \phi$. For each $z \in C(J, X)$ with $z(0) = 0$, we denote

$$\bar{z}(t) = \begin{cases} 0,\ t \in (-\infty, 0], \\ z(t),\ t \in J. \end{cases}$$

If $x(\cdot)$ satisfies (9), then we can decompose $x(\cdot)$ as $x(t) = y(t) + \bar{z}(t)$, for $t \in J$, which implies $x_t = y_t + \bar{z}_t$ for $t \in J$ and the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\mathrm{d}s + \frac{t}{A}\left(\sum_{i=1}^m a_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)}\right. \\ & \left. \times f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\mathrm{d}s - \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\mathrm{d}s\right) \\ & + \phi(0)\left[1 + \frac{t}{A}\left(\sum_{i=1}^m \frac{a_i \eta_i^{p_i}}{\Gamma(p_i+1)} - 1\right)\right]. \end{aligned}$$

Set $E_0 = \{z \in C(J, X) : z_0 = 0\}$ and let $\|\cdot\|_{E_0}$ be the seminorm in $E_0$ defined by

$$\|z\|_{E_0} = \sup_{t \in J} \|z(t)\|_X + \|z_0\|_{\mathcal{B}} = \sup_{t \in J} \|z(t)\|_X,\ z \in E_0.$$

Thus $(E_0, \|\cdot\|_{E_0})$ is a Banach space. Let the operator $P : E_0 \to E_0$ be defined by

$$\begin{aligned} (Pz)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\mathrm{d}s + \frac{t}{A}\left(\sum_{i=1}^m a_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)}\right. \\ & \left. \times f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\mathrm{d}s - \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\mathrm{d}s\right) \\ & + \phi(0)\left[1 + \frac{t}{A}\left(\sum_{i=1}^m \frac{a_i \eta_i^{p_i}}{\Gamma(p_i+1)} - 1\right)\right],\ t \in J. \end{aligned}$$

It is clear that the operator $N$ has a fixed point if and only if $P$ has a fixed point. So we prove that $P$ has a fixed point.

Define $B_r = \{z \in E_0 : \|z\|_{E_0} \le r\}$, then $B_r$ is a bounded, closed, convex subset of $E_0$. For any positive constant $h$, let $h < r$ where $h = \|p\|_{L^1} r^* \left[\left(1 + \frac{T}{|A|}\right)(\gamma_1 + \gamma_2) + \frac{T}{|A|}(\gamma_3 + \gamma_4)\right] + \|\phi(0)\| \left[1 + \frac{T}{|A|}\left(\sum_{i=1}^m \frac{|a_i| \eta_i^{p_i}}{\Gamma(p_i+1)} + 1\right)\right]$. Now, for $t \in J$, we decompose $P$ as $P_1 + P_2$ on $B_r$ where

$$\begin{aligned} (P_1 z)(t) = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\mathrm{d}s, \\ (P_2 z)(t) = & \frac{t}{A}\left(\sum_{i=1}^m a_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\mathrm{d}s\right. \\ & \left. - \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\mathrm{d}s\right) \\ & + \phi(0)\left[1 + \frac{t}{A}\left(\sum_{i=1}^m \frac{a_i \eta_i^{p_i}}{\Gamma(p_i+1)} - 1\right)\right]. \end{aligned}$$

Now for $z, z^* \in B_r$ and $t \in J$, we find that

$$\begin{aligned}
\|(P_1z)(t) + (P_2z^*)(t)\|_X \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\|_X \mathrm{d}s + \frac{T}{|A|}\left(\sum_{i=1}^m |a_i|\right.\\
& \times \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i + q)}\|f(s, y_s + \bar{z}^*_s, K(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s\\
& \left. + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\|f(s, y_s + \bar{z}^*_s, K(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s\right)\\
& + \|\phi(0)\|\left[1 + \frac{T}{|A|}\left(\sum_{i=1}^m \frac{|a_i|\eta_i^{p_i}}{\Gamma(p_i+1)} + 1\right)\right]\\
\leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\left[p_1(s)\|y_s + \bar{z}_s\|_{\mathcal{B}} + p_2(s)\|K(y(s) + \bar{z}(s))\|_X\right] \mathrm{d}s\\
& + \frac{T}{|A|}\left(\sum_{i=1}^m |a_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i + q)}\left[p_1(s)\|y_s + \bar{z}^*_s\|_{\mathcal{B}} s + p_2(s)\right.\right.\\
& \times \|K(y(s) + \bar{z}^*(s))\|_X] \mathrm{d}s + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\left[p_1(s)\|y_s + \bar{z}^*_s\|_{\mathcal{B}} + p_2(s)\right.\\
& \times \|K(y(s) + \bar{z}^*(s))\|_X] \mathrm{d}s) + \|\phi(0)\|\left[1 + \frac{T}{|A|}\left(\sum_{i=1}^m \frac{|a_i|\eta_i^{p_i}}{\Gamma(p_i+1)} + 1\right)\right]\\
\leq & \|p\|_{L^1} r^*\left[\left(1 + \frac{T}{|A|}\right)(\gamma_1 + \gamma_2) + \frac{T}{|A|}(\gamma_3 + \gamma_4)\right]\\
& + \|\phi(0)\|\left[1 + \frac{T}{|A|}\left(\sum_{i=1}^m \frac{|a_i|\eta_i^{p_i}}{\Gamma(p_i+1)} + 1\right)\right] := h,
\end{aligned}$$

which implies that $\|P_1z + P_2z^*\|_{E_0} \leq h$. Here $p(t) = \max\{p_1(t), p_2(t)p_3(t)\}$ and

$$\begin{aligned}
\|y_s + \bar{z}_s\|_{\mathcal{B}} \leq & \|y_s\|_{\mathcal{B}} + \|\bar{z}_s\|_{\mathcal{B}}\\
\leq & M_1(s) \sup_{0\leq\tau\leq s}\|y(\tau)\| + M_2(s)\|y_0\|_{\mathcal{B}} + M_1(s) \sup_{0\leq\tau\leq s}\|\bar{z}(\tau)\| + M_2(s)\|\bar{z}_0\|_{\mathcal{B}}\\
\leq & M_1^* r + M_2^*\|\phi\|_{\mathcal{B}} \leq r^*.
\end{aligned}$$

Thus $P_1z + P_2z^* \in B_r$. Next we prove that $P_2$ is a contraction. For $z, z^* \in B_r$ and $t \in J$, we have

$$\begin{aligned}
\|(P_2z)(t) - (P_2z^*)(t)\|_X \leq & \frac{T}{|A|}\Big(\sum_{i=1}^m |a_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i + q)}\|f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s)))\\
& - f(s, y_s + \bar{z}^*_s, K(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\\
& \times \|f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s))) - f(s, y_s + \bar{z}^*_s, K(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s\Big)\\
\leq & \frac{T}{|A|}\Big[\sum_{i=1}^m |a_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i + q)} L_f\left[\|(y_s + \bar{z}_s) - (y_s + \bar{z}^*_s)\|_{\mathcal{B}}\right.\\
& + \|K(y(s) + \bar{z}(s)) - K(y(s) + \bar{z}^*(s))\|_X] \mathrm{d}s\\
& + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} L_f\left[\|(y_s + \bar{z}_s) - (y_s + \bar{z}^*_s)\|_{\mathcal{B}}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\|K(y(s)+\bar{z}(s))-K(y(s)+\bar{z}^*(s))\|_X\big]\,\mathrm{d}s\Big]\\
\le\ &\frac{T}{|A|}\Big\{\sum_{i=1}^{m}|a_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}L_f\,[\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}}+L_k\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}}s]\,\mathrm{d}s\\
&+\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}L_f\,[\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}}+L_k\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}}s]\,\mathrm{d}s\Big\}\\
\le\ &\frac{T}{|A|}\Big\{\sum_{i=1}^{m}|a_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}L_f\Big[M_1^*\sup_{s\in[0,t]}\|z(s)-z^*(s)\|\\
&+L_kM_1^*\sup_{\tau\in[0,s]}\|z(\tau)-z^*(\tau)\|s\Big]\mathrm{d}s+\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}\\
&\times L_f\Big[M_1^*\sup_{s\in[0,t]}\|z(s)-z^*(s)\|+L_kM_1^*\sup_{\tau\in[0,s]}\|z(\tau)-z^*(\tau)\|s\Big]\mathrm{d}s\Big\}\\
\le\ &\frac{T}{|A|}L_fM_1^*\left\{\sum_{i=1}^{m}|a_i|\frac{\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}+L_k\sum_{i=1}^{m}|a_i|\frac{\eta_i^{p_i+q+1}}{\Gamma(p_i+q+2)}\right.\\
&\left.+\frac{T^q}{\Gamma(q+1)}+L_k\frac{T^{q+1}}{\Gamma(q+2)}\right\}\|z-z^*\|_{E_0}\\
\le\ &\frac{T}{|A|}L_fM_1^*\,\{(\gamma_1+\gamma_3)+L_k(\gamma_2+\gamma_4)\}\,\|z-z^*\|_{E_0}.
\end{aligned}
$$

Thus $\|P_2z-P_2z^*\|_{E_0}\le L\|z-z^*\|_{E_0}$.
Hence $P_2$ is a contraction. Continuity of $f$ and $k$ implies that the operator $P_1$ is continuous. Also $P_1$ is uniformly bounded on $B_r$ as

$$
\begin{aligned}
\|(P_1z)(t)\|_X &\le \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,y_s+\bar{z}_s,K(y(s)+\bar{z}(s)))\|_X\mathrm{d}s\\
&\le \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\,[p_1(s)\|y_s+\bar{z}_s\|_{\mathcal{B}}+p_2(s)\|K(y(s)+\bar{z}(s))\|_X]\,\mathrm{d}s\\
&\le \|p\|_{L^1}r^*\,(\gamma_1+\gamma_2)\,.
\end{aligned}
$$

To prove that $P_1$ is compact, it remains to show that $P_1$ is equicontinuous. Now, for any $t_1,\ t_2\in J$ with $t_1<t_2$ and $z\in B_r$, we have

$$
\begin{aligned}
\|(P_1z)(t_2)-(P_1z)(t_1)\|_X &\le \int_0^{t_1}\frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}\|f(s,y_s+\bar{z}_s,K(y(s)+\bar{z}(s)))\|_X\mathrm{d}s\\
&\quad+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}\|f(s,y_s+\bar{z}_s,K(y(s)+\bar{z}(s)))\|_X\mathrm{d}s\\
&\le \int_0^{t_1}\frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}\,[p_1(s)\|y_s+\bar{z}_s\|_{\mathcal{B}}\\
&\quad+p_2(s)\|K(y(s)+\bar{z}(s))\|_X]\,\mathrm{d}s+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}\,[p_1(s)\|y_s+\bar{z}_s\|_{\mathcal{B}}\\
&\quad+p_2(s)\|K(y(s)+\bar{z}(s))\|_X]\,\mathrm{d}s\\
&\le \|p\|_{L^1}r^*\left[\int_0^{t_1}\frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}(1+s)\mathrm{d}s\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}(1+s)\mathrm{d}s\right].
\end{aligned}
$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero independent of $x \in B_r$. Thus $P_1$ is equicontinuous. By Arzela-Ascoli's Theorem, $P_1$ is compact on $B_r$. Hence, by the Krasnoselskii fixed point theorem, there exists a fixed point $z \in E_0$ such that $Pz = z$ which is a solution to the fractional boundary value problem (1). □

The next uniqueness result is based on Banach contraction principle.

**Theorem 3.5.** *Assume that the hypotheses* $(A1)$ *and* $(A3)$ *hold. Then the boundary value problem* (1) *has a unique solution on* $(-\infty, T]$.

*Proof.* Consider $B_r = \{z \in E_0 : \|z\|_{E_0} \le r\}$. Let $z, z^* \in E_0$. For $t \in J$, we have

$$\begin{aligned}
\|(Pz)(t) - (Pz^*)(t)\|_X \le & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s))) \\
& - f(s, y_s + \bar{z}^*_s, K(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s + \frac{T}{|A|}\Big(\sum_{i=1}^m |a_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} \\
& \times \|f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s))) - f(s, y_s + \bar{z}^*_s, K(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s \\
& + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \|f(s, y_s + \bar{z}_s, K(y(s) + \bar{z}(s))) \\
& - f(s, y_s + \bar{z}^*_s, K(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s\Big) \\
\le & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} L_f \left[\|\bar{z}_s - \bar{z}^*_s\|_{\mathcal{B}} + L_k \|\bar{z}_\tau - \bar{z}^*_\tau\|_{\mathcal{B}} s\right] \mathrm{d}s + \frac{T}{|A|}\Big(\sum_{i=1}^m |a_i| \\
& \times \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} L_f \left[\|\bar{z}_s - \bar{z}^*_s\|_{\mathcal{B}} + L_k \|\bar{z}_\tau - \bar{z}^*_\tau\|_{\mathcal{B}} s\right] \mathrm{d}s \\
& + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} L_f \left[\|\bar{z}_s - \bar{z}^*_s\|_{\mathcal{B}} + L_k \|\bar{z}_\tau - \bar{z}^*_\tau\|_{\mathcal{B}} s\right] \mathrm{d}s\Big) \\
\le & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} L_f \Big[M_1^* \|z - z^*\|_{E_0} + L_k M_1^* \|z - z^*\|_{E_0} s\Big] \mathrm{d}s \\
& + \frac{T}{|A|}\Big(\sum_{i=1}^m |a_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} L_f \Big[M_1^* \|z - z^*\|_{E_0} \\
& + L_k M_1^* \|z - z^*\|_{E_0} s\Big] \mathrm{d}s + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \\
& \times L_f \Big[M_1^* \|z - z^*\|_{E_0} + L_k M_1^* \|z - z^*\|_{E_0} s\Big] \mathrm{d}s\Big) \\
\le & L_f M_1^* \Bigg\{\Bigg[\frac{T^q}{\Gamma(q+1)} + \frac{T}{|A|} \sum_{i=1}^m |a_i| \frac{\eta_i^{p_i+q}}{\Gamma(p_i+q+1)} + \frac{T^{q+1}}{|A|\Gamma(q+1)}\Bigg] \\
& + L_k \Bigg[\frac{T^{q+1}}{\Gamma(q+2)} + \frac{T}{|A|} \sum_{i=1}^m |a_i| \frac{\eta_i^{p_i+q+1}}{\Gamma(p_i+q+2)} + \frac{T^{q+2}}{|A|\Gamma(q+2)}\Bigg]\Bigg\} \|z - z^*\|_{E_0} \\
\le & L_f M_1^* \{\theta_1 + L_k \theta_2\} \|z - z^*\|_{E_0}.
\end{aligned}$$

Thus $\|P(z) - P(z^*)\|_{E_0} \le \Lambda \|z - z^*\|_{E_0}$.
Here $\Lambda$ depends only on the parameters involved in the problem. By assumption $(A3)$, $\Lambda<1$ and therefore the map $P$ is a contraction. Hence, by the Banach contraction principle, the problem (1) has a unique solution on $(-\infty, T]$. □

*3.2. Example*

Consider the following fractional functional integrodifferential equation

$$\begin{aligned} {}^{LC}D^{3/2}x(t) &= \frac{e^{-\gamma t}}{(5t+3)^2}\frac{|x_t|}{1+|x_t|} + \frac{1}{9}\int_0^t \frac{e^{-\gamma t}ts}{15(1+s)}\frac{\sin(x_s)}{1+\sin(x_s)}ds,\ t\in[0,1],\\ x(t) &= \phi(t),\ t\in(-\infty,0],\\ x(1) &= \sum_{i=1}^{5} a_i(I^{p_i}x)(\eta_i),\ 0<\eta_1<\eta_2<\eta_3<\eta_4<\eta_5<1. \end{aligned} \tag{10}$$

Let $\gamma$ be a positive real constant and

$$B_\gamma = \left\{x \in C\left((-\infty,0],\mathbb{R}\right) : \lim_{\delta\to-\infty} e^{\gamma\delta}x(\delta) \text{ exists in } \mathbb{R}\right\}.$$

The norm of $B_\gamma$ is given by

$$\|x\|_\gamma = \sup_{-\infty<\delta\le 0} e^{\gamma\delta}|x(\delta)|.$$

Let $x:(-\infty,T]\to\mathbb{R}$ be such that $x_0=\phi\in B_\gamma$. Then

$$\lim_{\delta\to-\infty} e^{\gamma\delta}x_t(\delta) = \lim_{\delta\to-\infty} e^{\gamma\delta}x(t+\delta) = \lim_{\delta\to-\infty} e^{\gamma(\delta-t)}x(\delta) = e^{-\gamma t}\lim_{\delta\to-\infty} e^{\gamma\delta}x_0(\delta) < \infty.$$

Hence $x_t\in B_\gamma$. Take $M_1=M_2=1$ and $H=1$ and then prove that

$$\|x_t\|_\gamma \le M_1(t)\sup\{|x(s)| : 0\le s\le t\} + M_2(t)\|x_0\|_\gamma,$$

We have $|x_t(\delta)| = |x(t+\delta)|$.

If $\delta+t\le 0$, we get

$$|x_t(\delta)| \le \sup\{|x(s)| : -\infty<s\le 0\}.$$

For $t+\delta\ge 0$, then we have

$$|x_t(\delta)| \le \sup\{|x(s)| : 0<s\le t\}.$$

Thus, for all $t+\delta\in[0,T]$, we get

$$|x_t(\delta)| \le \sup\{|x(s)| : -\infty<s\le 0\} + \sup\{|x(s)| : 0\le s\le t\}.$$

Then

$$\|x_t\|_\gamma \le \|x_0\|_\gamma + \sup\{|x(s)| : 0\le s\le t\}.$$

It is clear that $(B_\gamma, \|\cdot\|_\gamma)$ is a Banach space. We conclude that $B_\gamma$ is a phase space. Here $q=\frac{3}{2}$, $m=5$,

$a_1=\frac{1}{5},\ a_2=\frac{1}{7},\ a_3=3,\ a_4=\frac{1}{11},\ a_5=\frac{1}{17},$

$\eta_1=\frac{1}{9},\ \eta_2=\frac{1}{3},\ \eta_3=\frac{3}{7},\ \eta_4=\frac{1}{2},\ \eta_5=\frac{3}{4},$

$p_1=\frac{1}{2},\ p_2=\frac{3}{4},\ p_3=\frac{5}{4},\ p_4=\frac{3}{2},\ p_5=\frac{5}{3}.$

From the above data, we see that
$A=0.7949$, $\gamma_1=0.75225$, $\gamma_2=0.3009$, $\gamma_3=0.07702$, $\gamma_4=0.00889$, $\theta_1=1.79549$ and $\theta_2=0.69062$.

(i) From (10), we have
$f(t,x_t,Kx(t)) = \frac{e^{-\gamma t}}{(5t+3)^2}\frac{|x_t|}{1+|x_t|} + \frac{1}{9}Kx(t)$ where $Kx(t) = \int_0^t \frac{e^{-\gamma t}ts}{15(1+s)}\frac{\sin(x_s)}{1+\sin(x_s)}ds$. Now, for $x_t, y_t\in B_\gamma$, we have

$$\begin{aligned} \left|k(t,s,x_s)-k(t,s,y_s)\right| &= \left|\frac{e^{-\gamma t}ts}{15(1+s)}\frac{\sin(x_s)}{1+\sin(x_s)} - \frac{e^{-\gamma t}ts}{15(1+s)}\frac{\sin(y_s)}{1+\sin(y_s)}\right|\\ &\le \frac{1}{15}\|x-y\|_\gamma. \end{aligned}$$

and

$$\begin{aligned}|f(t,x_t,Kx(t))-f(t,y_t,Ky(t))| &\leq \frac{e^{-\gamma t}}{(5t+3)^2}\frac{|x_t-y_t|}{(1+|x_t|)(1+|y_t|)}+\frac{1}{9}|Kx(t)-Ky(t)|\\ &\leq \frac{1}{9}\left[\|x-y\|_\gamma+\frac{1}{15}\|x-y\|_\gamma\right].\end{aligned}$$

Also $|f(t,\phi,\psi)| = \left|\frac{e^{-\gamma t}}{(5t+3)^2}\frac{|\phi_t|}{1+|\phi_t|}+\frac{1}{9}\int_0^t \frac{e^{-\gamma t}ts}{15(1+s)}\frac{\sin(\psi_s)}{1+\sin(\psi_s)}\mathrm{d}s\right| \leq \frac{1}{9}+\frac{1}{135}$ and $|k(t,s,\varphi)| = |\frac{e^{-\gamma t}ts}{15(1+s)}\frac{\sin(\varphi_s)}{1+\sin(\varphi_s)}| \leq \frac{1}{15}$.

(ii) Taking $f(t,x_t,Kx(t)) = \dfrac{e^{-\gamma t-t}}{26+e^t}\dfrac{\sin 2\pi x_t}{4\pi}+\dfrac{1}{54}Kx(t)$ where $Kx(t) = \displaystyle\int_0^t \frac{e^{-\gamma t}}{2}\frac{|x_s|}{1+|x_s|}\mathrm{d}s$ in (10), we have, for $x_t, y_t \in B_\gamma$,

$$\left|k(t,s,x_s)-k(t,s,y_s)\right| \leq \frac{1}{2}\|x-y\|_\gamma.$$

and

$$|f(t,x_t,Kx(t))-f(t,y_t,Ky(t))| \leq \frac{1}{54}\left[\|x-y\|_\gamma+\frac{1}{2}\|x-y\|_\gamma\right].$$

The condition $(A1)$ is satisfied with $L_f = \frac{1}{54}$ and $L_k = \frac{1}{2}$. Computing the value of $\Lambda$, we have $\Lambda = 0.03964 < 1$ thereby satisfying the condition $(A3)$. Thus all the assumptions of the Theorem 3.5 are satisfied. Hence the problem (10) with the given function $f$ has a unique solution on $(-\infty, T]$.

## 4. Neutral Boundary Value Problem of Fractional Order

Let $\bar{\Omega}$ be the space as defined in Section (3).

**Definition 4.1.** *A function $x \in \bar{\Omega}$ is said to be a solution of* (2) *if it satisfies the equation*

$${}^{\mathrm{LC}}D^q\left[x(t)-\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}g(s,x_s,K_1x(s))\,\mathrm{d}s\right] = f(t,x_t,K_2x(t))$$

*on $J$ and the boundary conditions*

$$\begin{aligned}x(t) &= \phi(t),\ t \in (-\infty,0],\\ x(T) &= \sum_{i=1}^m a_i(I^{p_i}x)(\eta_i),\ 0<\eta_1<\eta_2<\ldots<\eta_m<T,\end{aligned}$$

where $K_1x(t) = \int_0^t k_1(t,s,x_s)\,\mathrm{d}s$ and $K_2x(t) = \int_0^t k_2(t,s,x_s)\,\mathrm{d}s$. Equation (2) is equivalent to the following integral equation

$$x(t) = \begin{cases}\phi(t),\ t\in(-\infty,0],\\ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x_s,K_2x(s))\mathrm{d}s+\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}g(s,x_s,K_1x(s))\mathrm{d}s+\frac{t}{A}\Big(\sum_{i=1}^m a_i\\ \times\int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}f(s,x_s,K_2x(s))\mathrm{d}s+\sum_{i=1}^m a_i\int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}g(s,x_s,K_1x(s))\mathrm{d}s\\ -\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f(s,x_s,K_2x(s))\mathrm{d}s-\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}g(s,x_s,K_1x(s))\mathrm{d}s\Big)\\ +\phi(0)\left[1+\frac{t}{A}\left(\sum_{i=1}^m\frac{a_i\eta_i^{p_i}}{\Gamma(p_i+1)}-1\right)\right],\ t\in J,\end{cases} \tag{11}$$

where $A = T-\sum_{i=1}^m \frac{a_i\eta_i^{p_i+1}}{\Gamma(p_i+2)} \neq 0$.

*4.1. Existence Results*

Assume that the following conditions hold:

(A4) There exist positive constants $L_f$, $L_g$, $L_{k_1}$ and $L_{k_2}$ such that
(i) $\|f(t,\phi_1,x_1) - f(t,\phi_2,x_2)\|_X \leq L_f\left(\|\phi_1-\phi_2\|_{\mathcal{B}} + \|x_1-x_2\|_X\right)$, $t \in J$, $\phi_1,\phi_2 \in \mathcal{B}$, $x_1,x_2 \in X$,
(i) $\|g(t,\phi_1,x_1) - g(t,\phi_2,x_2)\|_X \leq L_g\left(\|\phi_1-\phi_2\|_{\mathcal{B}} + \|x_1-x_2\|_X\right)$, $t \in J$, $\phi_1,\phi_2 \in \mathcal{B}$, $x_1,x_2 \in X$,
(iii) $\left\|k_1(t,s,\psi_1) - k_1(t,s,\psi_2)\right\|_X \leq L_{k_1}\|\psi_1-\psi_2\|_{\mathcal{B}}$, $t,s \in J$, $\psi_1,\psi_2 \in \mathcal{B}$,
(iv) $\left\|k_2(t,s,\psi_1) - k_2(t,s,\psi_2)\right\|_X \leq L_{k_2}\|\psi_1-\psi_2\|_{\mathcal{B}}$, $t,s \in J$, $\psi_1,\psi_2 \in \mathcal{B}$.

(A5) For $p_i \in L^1(J,\mathbb{R}^+)$, $i = 1,2,...,6$, we have
(i) $\|f(t,\phi,x)\|_X \leq p_1(t)\|\phi\|_{\mathcal{B}} + p_2(t)\|x\|_X$, $(t,\phi,x) \in J\times\mathcal{B}\times X$,
(ii) $\|g(t,\phi,x)\|_X \leq p_3(t)\|\phi\|_{\mathcal{B}} + p_4(t)\|x\|_X$, $(t,\phi,x) \in J\times\mathcal{B}\times X$,
(iii) $\|k_1(t,s,\psi)\|_X \leq p_5(t)\|\psi\|_{\mathcal{B}}$, $(t,s,\psi) \in \Omega\times\mathcal{B}$,
(iv) $\|k_2(t,s,\psi)\|_X \leq p_6(t)\|\psi\|_{\mathcal{B}}$, $(t,s,\psi) \in \Omega\times\mathcal{B}$.

(A6) Let $\Lambda^* = L_f M_1^*\left[\theta_1 + L_{k_2}\theta_2\right] + L_g M_1^*\left[\theta_1 + L_{k_1}\theta_2\right] < 1$ where $\theta_1 = \left(1+\frac{T}{|A|}\right)\gamma_1 + \frac{T}{|A|}\gamma_3$ and $\theta_2 = \left(1+\frac{T}{|A|}\right)\gamma_2 + \frac{T}{|A|}\gamma_4$ with $\gamma_1 = \frac{T^q}{\Gamma(q+1)}$, $\gamma_2 = \frac{T^{q+1}}{\Gamma(q+2)}$, $\gamma_3 = \sum_{i=1}^{m}|a_i|\frac{\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}$ and $\gamma_4 = \sum_{i=1}^{m}|a_i|\frac{\eta_i^{p_i+q+1}}{\Gamma(p_i+q+2)}$.

We prove the existence of solutions to (2) by applying Krasnoselskii's fixed point theorem.

**Theorem 4.2.** *Suppose that the assumptions* (*A*4) *and* (*A*5) *hold with*

$$L^* = \frac{TM_1^*}{|A|}\left\{L_f\left[(\gamma_1+\gamma_3) + L_{k_2}(\gamma_2+\gamma_4)\right] + L_g\left[(\gamma_1+\gamma_3) + L_{k_1}(\gamma_2+\gamma_4)\right]\right\} < 1. \tag{12}$$

*Then the boundary value problem* (2) *has at least one solution on* $(-\infty, T]$.

*Proof.* Consider the operator $F : \bar{\Omega} \to \bar{\Omega}$ defined by

$$(Fx)(t) = \begin{cases} \phi(t),\ t \in (-\infty,0], \\ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x_s,K_2x(s))\mathrm{d}s + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s,x_s,K_1x(s))\mathrm{d}s + \frac{t}{A}\Bigg(\sum_{i=1}^{m} a_i \\ \times \int_0^{\eta_i} \frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)} f(s,x_s,K_2x(s))\mathrm{d}s + \sum_{i=1}^{m} a_i \int_0^{\eta_i} \frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)} g(s,x_s,K_1x(s))\mathrm{d}s \\ - \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x_s,K_2x(s))\mathrm{d}s - \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} g(s,x_s,K_1x(s))\mathrm{d}s\Bigg) \\ +\phi(0)\left[1 + \frac{t}{A}\left(\sum_{i=1}^{m}\frac{a_i\eta_i^{p_i}}{\Gamma(p_i+1)} - 1\right)\right],\ t \in J. \end{cases} \tag{13}$$

In analogy with Theorem 3.4, we consider the operator $Q : E_0 \to E_0$ defined by,

$$\begin{aligned}(Qz)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K_2(y(s) + \bar{z}(s)))\mathrm{d}s \\ &+ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s, y_s + \bar{z}_s, K_1(y(s) + \bar{z}(s)))\mathrm{d}s \\ &+ \frac{t}{A}\Bigg(\sum_{i=1}^m a_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} f(s, y_s + \bar{z}_s, K_2(y(s) + \bar{z}(s)))\mathrm{d}s \\ &+ \sum_{i=1}^m a_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} g(s, y_s + \bar{z}_s, K_1(y(s) + \bar{z}(s)))\mathrm{d}s \\ &- \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K_2(y(s) + \bar{z}(s)))\mathrm{d}s \\ &- \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} g(s, y_s + \bar{z}_s, K_1(y(s) + \bar{z}(s)))\mathrm{d}s\Bigg) \\ &+ \phi(0)\left[1 + \frac{t}{A}\left(\sum_{i=1}^m \frac{a_i \eta_i^{p_i}}{\Gamma(p_i+1)} - 1\right)\right], \ t \in J.\end{aligned}$$

Define $B_{\bar{r}} = \{z \in E_0 : \|z\|_{E_0} \le \bar{r}\}$. For any positive constant $h^*$, let $h^* < \bar{r}$ where $h^* = 2\|p^*\|_{L^1}\bar{r}^* [\theta_1 + \theta_2] + \|\phi(0)\| \left[1 + \frac{T}{|A|}\left(\sum_{i=1}^m \frac{|a_i|\eta_i^{p_i}}{\Gamma(p_i+1)} + 1\right)\right]$. Now, for $t \in J$, we decompose $Q$ as $Q_1 + Q_2$ on $B_{\bar{r}}$ where

$$\begin{aligned}(Q_1 z)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K_2(y(s) + \bar{z}(s)))\,\mathrm{d}s \\ &+ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s, y_s + \bar{z}_s, K_1(y(s) + \bar{z}(s)))\mathrm{d}s, \\ (Q_2 z)(t) = &\frac{t}{A}\Bigg(\sum_{i=1}^m a_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} f(s, y_s + \bar{z}_s, K_2(y(s) + \bar{z}(s)))\mathrm{d}s \\ &+ \sum_{i=1}^m a_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} g(s, y_s + \bar{z}_s, K_1(y(s) + \bar{z}(s)))\mathrm{d}s \\ &- \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, y_s + \bar{z}_s, K_2(y(s) + \bar{z}(s)))\mathrm{d}s \\ &- \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} g(s, y_s + \bar{z}_s, K_1(y(s) + \bar{z}(s)))\mathrm{d}s\Bigg) \\ &+ \phi(0)\left[1 + \frac{t}{A}\left(\sum_{i=1}^m \frac{a_i \eta_i^{p_i}}{\Gamma(p_i+1)} - 1\right)\right].\end{aligned}$$

Now, for $z, z^* \in B_{\bar{r}}$ and $t \in J$, we find that

$$\begin{aligned}
\|(Q_1 z)(t) + (Q_2 z^*)(t)\|_X \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s, y_s + \bar{z}_s, K_2(y(s) + \bar{z}(s)))\|_X \mathrm{d}s \\
& + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|g(s, y_s + \bar{z}_s, K_1(y(s) + \bar{z}(s)))\|_X \mathrm{d}s + \frac{T}{|A|} \Bigg( \sum_{i=1}^m |a_i| \\
& \times \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} \|f(s, y_s + \bar{z}^*_s, K_2(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s \\
& + \sum_{i=1}^m |a_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} \|g(s, y_s + \bar{z}^*_s, K_1(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s \\
& + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \|f(s, y_s + \bar{z}^*_s, K_2(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s \\
& + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \|g(s, y_s + \bar{z}^*_s, K_1(y(s) + \bar{z}^*(s)))\|_X \mathrm{d}s \Bigg) \\
& + \|\phi(0)\| \left[ 1 + \frac{T}{|A|} \left( \sum_{i=1}^m \frac{|a_i| \eta_i^{p_i}}{\Gamma(p_i+1)} + 1 \right) \right] \\
\leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left[ p_1(s) \|y_s + \bar{z}_s\|_{\mathcal{B}} + p_2(s) \|K_2(y(s) + \bar{z}(s))\|_X \right. \\
& \left. + p_3(s) \|y_s + \bar{z}_s\|_{\mathcal{B}} + p_4(s) \|K_1(y(s) + \bar{z}(s))\|_X \right] \mathrm{d}s + \frac{T}{|A|} \Bigg\{ \sum_{i=1}^m |a_i| \\
& \times \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} \left[ p_1(s) \|y_s + \bar{z}^*_s\|_{\mathcal{B}} + p_2(s) \|K_2(y(s) + \bar{z}^*(s))\|_X \right. \\
& \left. + p_3(s) \|y_s + \bar{z}^*_s\|_{\mathcal{B}} + p_4(s) \|K_1(y(s) + \bar{z}^*(s))\|_X \right] \mathrm{d}s + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \\
& \times \left[ p_1(s) \|y_s + \bar{z}^*_s\|_{\mathcal{B}} + p_2(s) \|K_2(y(s) + \bar{z}^*(s))\|_X + p_3(s) \|y_s + \bar{z}^*_s\|_{\mathcal{B}} \right. \\
& \left. + p_4(s) \|K_1(y(s) + \bar{z}^*(s))\|_X \right] \mathrm{d}s \Bigg\} \\
& + \|\phi(0)\| \left[ 1 + \frac{T}{|A|} \left( \sum_{i=1}^m \frac{|a_i| \eta_i^{p_i}}{\Gamma(p_i+1)} + 1 \right) \right] \\
\leq & \; 2 \|p^*\|_{L^1} \bar{r}^* (\theta_1 + \theta_2) + \|\phi(0)\| \left[ 1 + \frac{T}{|A|} \left( \sum_{i=1}^m \frac{|a_i| \eta_i^{p_i}}{\Gamma(p_i+1)} + 1 \right) \right] := h^*.
\end{aligned}$$

which implies that $\|Q_1 z + Q_2 z^*\|_{E_0} \leq h^*$. Thus $Q_1 z + Q_2 z^* \in B_{\bar{r}}$. Here

$$\|y_s + \bar{z}_s\|_{\mathcal{B}} \leq M_1^* \bar{r} + M_2^* \bar{r} \|\phi\|_{\mathcal{B}} \leq \bar{r}^*.$$

Next we prove that $Q_2$ is a contraction. For $z, z^* \in B_{\bar{r}}$ and $t \in J$, we have

$$\begin{aligned}\|(Q_2z)(t)-(Q_2z^*)(t)\|_X \le{}& \frac{T}{|A|}\Big\{\sum_{i=1}^m |a_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)} [\|f(s,y_s+\bar{z}_s,K_2(y(s)+\bar{z}(s)))\\ &-f(s,y_s+\bar{z}^*_s,K_2(y(s)+\bar{z}^*(s)))\|_X+\|g(s,y_s+\bar{z}_s,K_1(y(s)+\bar{z}(s)))\\ &-g(s,y_s+\bar{z}^*_s,K_1(y(s)+\bar{z}^*(s)))\|_X]\,\mathrm{d}s+\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\\ &[\|f(s,y_s+\bar{z}_s,K_2(y(s)+\bar{z}(s)))-f(s,y_s+\bar{z}^*_s,K_2(y(s)+\bar{z}^*(s)))\|_X\\ &+\|g(s,y_s+\bar{z}_s,K_1(y(s)+\bar{z}(s)))-g(s,y_s+\bar{z}^*_s,K_1(y(s)+\bar{z}^*(s)))\|_X]\mathrm{d}s\Big\}\\ \le{}& \frac{T}{|A|}\Big\{\sum_{i=1}^m |a_i| \int_0^{\eta_i} \frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}\Big[L_f\,(\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}}+L_{k_2}\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}}s)\\ &+L_g\,(\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}}+L_{k_1}\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}}s)\Big]\mathrm{d}s+\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}\Big[L_f\,(\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}}\\ &+L_{k_2}\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}}s)+L_g\,(\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}}+L_{k_1}\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}}s)\Big]\mathrm{d}s\Big\}\\ \le{}& \frac{TM_1^*}{|A|}\Big\{L_f\,[(\gamma_1+\gamma_3)+L_{k_2}\,(\gamma_2+\gamma_4)]+L_g\,[(\gamma_1+\gamma_3)\\ &+L_{k_1}\,(\gamma_2+\gamma_4)]\Big\}\|z-z^*\|_{E_0}.\end{aligned}$$

Thus $\|Q_2z-Q_2z^*\|_{E_0} \le L^*\|z-z^*\|_{E_0}$. Hence $Q_2$ is a contraction.
Continuity of $f$, $g$, $k_1$ and $k_2$ implies that the operator $Q_1$ is continuous on $B_{\bar{r}}$. Also $Q_1$ is uniformly bounded on $B_{\bar{r}}$ as

$$\begin{aligned}\|(Q_1z)(t)\|_X \le{}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s,y_s+\bar{z}_s,K_2(y(s)+\bar{z}(s)))\|_X\mathrm{d}s\\ &+\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|g(s,y_s+\bar{z}_s,K_1(y(s)+\bar{z}(s)))\|_X\mathrm{d}s\\ \le{}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\,[p_1(s)\|y_s+\bar{z}_s\|_{\mathcal{B}}+p_2(s)\|K_2(y(s)+\bar{z}(s))\|_X]\,\mathrm{d}s\\ &+\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\,[p_3(s)\|y_s+\bar{z}_s\|_{\mathcal{B}}+p_4(s)\|K_1(y(s)+\bar{z}(s))\|_X]\,\mathrm{d}s\\ \le{}& 2\|p^*\|_{L^1}\bar{r}^*\,(\gamma_1+\gamma_2)\,,\end{aligned}$$

where $p^*(t)=\max\{p_1(t),p_2(t)p_6(t),p_3(t),p_4(t)p_5(t)\}$. To prove that $Q_1$ is compact, it remains to show that $Q_1$ is equicontinuous. Now, for any $t_1$, $t_2 \in J$ with $t_1 < t_2$ and $z \in B_{\bar{r}}$, we have

$$\begin{aligned}\|(Q_1z)(t_2)-(Q_1z)(t_1)\|_X \le{}& \int_0^{t_1} \frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}\|f(s,y_s+\bar{z}_s,K_2(y(s)+\bar{z}(s)))\|_X\mathrm{d}s\\ &+\int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)}\|f(s,y_s+\bar{z}_s,K_2(y(s)+\bar{z}(s)))\|_X\mathrm{d}s\\ &+\int_0^{t_1} \frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}\|g(s,y_s+\bar{z}_s,K_1(y(s)+\bar{z}(s)))\|_X\mathrm{d}s\\ &+\int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)}\|g(s,y_s+\bar{z}_s,K_1(y(s)+\bar{z}(s)))\|_X\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{t_1} \frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}\left[p_1(s)\bar{r}^* + p_2(s)p_6(s)\bar{r}^* s\right.\\
&\quad \left.+p_3(s)\bar{r}^* + p_4(s)p_5(s)\bar{r}^* s\right] ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)}\left[p_1(s)\bar{r}^*\right.\\
&\quad \left.+p_2(s)p_6(s)\bar{r}^* s + p_3(s)\bar{r}^* + p_4(s)p_5(s)\bar{r}^* s\right] ds\\
&\leq 2\|p^*\|_{L^1}\bar{r}^*\left[\int_0^{t_1}\frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}(1+s)ds\right.\\
&\quad \left.+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}(1+s)ds\right].
\end{aligned}
$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero independent of $x \in B_{\bar{r}}$. Thus $Q_1$ is equicontinuous. By Arzela-Ascoli's Theorem, $Q_1$ is compact on $B_{\bar{r}}$. Hence, by the Krasnoselskii fixed point theorem, there exists a fixed point $z \in E_0$ such that $Qz = z$ which is a solution to the fractional boundary value problem (2). □

The next uniqueness result is based on the Banach contraction principle.

**Theorem 4.3.** *Assume that the hypotheses* (*A*4) *and* (*A*6) *hold. Then the boundary value problem* (2) *has a unique solution on* $(-\infty, T]$.

*Proof.* Consider $B_{\bar{r}} = \{z \in E_0 : \|z\|_{E_0} \leq \bar{r}\}$. Let $z, z^* \in E_0$. For $t \in J$, we have

$$
\begin{aligned}
\|(Qz)(t)-(Qz^*)(t)\|_X \leq{}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f(s, y_s+\bar{z}_s, K_2(y(s)+\bar{z}(s)))\\
&-f(s, y_s+\bar{z}^*_s, K_2(y(s)+\bar{z}^*(s)))\|_X ds + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\\
&\times\|g(s, y_s+\bar{z}_s, K_1(y(s)+\bar{z}(s))) - g(s, y_s+\bar{z}^*_s, K_1(y(s)+\bar{z}^*(s)))\|_X ds\\
&+\frac{T}{|A|}\left\{\sum_{i=1}^m |a_i| \int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}\|f(s, y_s+\bar{z}_s, K_2(y(s)+\bar{z}(s)))\right.\\
&-f(s, y_s+\bar{z}^*_s, K_2(y(s)+\bar{z}^*(s)))\|_X ds + \sum_{i=1}^m |a_i| \int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}\\
&\times\|g(s, y_s+\bar{z}_s, K_1(y(s)+\bar{z}(s))) - g(s, y_s+\bar{z}^*_s, K_1(y(s)+\bar{z}^*(s)))\|_X ds\\
&+\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\|f(s, y_s+\bar{z}_s, K_2(y(s)+\bar{z}(s)))\\
&-f(s, y_s+\bar{z}^*_s, K_2(y(s)+\bar{z}^*(s)))\|_X ds + \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\\
&\left.\times\|g(s, y_s+\bar{z}_s, K_1(y(s)+\bar{z}(s))) - g(s, y_s+\bar{z}^*_s, K_1(y(s)+\bar{z}^*(s)))\|_X ds\right\}\\
\leq{}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\left\{L_f\Big[\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}} + L_{k_2}\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}} s\Big] + L_g\Big[\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}}\right.\\
&\left.+L_{k_1}\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}} s\Big]\right\}ds + \frac{T}{|A|}\left\{\sum_{i=1}^m |a_i| \int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}\Big[L_f\Big(\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}}\right.\\
&+L_{k_2}\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}} s\Big) + L_g\Big(\|\bar{z}_s-\bar{z}^*_s\|_{\mathcal{B}} + L_{k_1}\|\bar{z}_\tau-\bar{z}^*_\tau\|_{\mathcal{B}} s\Big)\Big]ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\Big[L_f\Big(\|\bar{z}_s-\bar{z}_s^*\|_{\mathcal{B}}+L_{k_2}\|\bar{z}_\tau-\bar{z}_\tau^*\|_{\mathcal{B}}s\Big)\\
&+L_g\Big(\|\bar{z}_s-\bar{z}_s^*\|_{\mathcal{B}}+L_{k_1}\|\bar{z}_\tau-\bar{z}_\tau^*\|_{\mathcal{B}}s\Big)\Big]\mathrm{d}s\Big\}\\
\leq&\ \Big\{L_fM_1^*\Big[\Big(\frac{T^q}{\Gamma(q+1)}+\frac{T}{|A|}\sum_{i=1}^m|a_i|\frac{\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}+\frac{T^{q+1}}{|A|\Gamma(q+1)}\Big)\\
&+L_{k_2}\Big(\frac{T^{q+1}}{\Gamma(q+2)}+\frac{T}{|A|}\sum_{i=1}^m|a_i|\frac{\eta_i^{p_i+q+1}}{\Gamma(p_i+q+2)}+\frac{T^{q+2}}{|A|\Gamma(q+2)}\Big)\Big]\\
&+L_gM_1^*\Big[\Big(\frac{T^q}{\Gamma(q+1)}+\frac{T}{|A|}\sum_{i=1}^m|a_i|\frac{\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}+\frac{T^{q+1}}{|A|\Gamma(q+1)}\Big)\\
&+L_{k_1}\Big(\frac{T^{q+1}}{\Gamma(q+2)}+\frac{T}{|A|}\sum_{i=1}^m|a_i|\frac{\eta_i^{p_i+q+1}}{\Gamma(p_i+q+2)}+\frac{T^{q+2}}{|A|\Gamma(q+2)}\Big)\Big]\Big\}\|z-z^*\|_{E_0}\\
\leq&\ \Big\{L_fM_1^*\left[\theta_1+L_{k_2}\theta_2\right]+L_gM_1^*\left[\theta_1+L_{k_1}\theta_2\right]\Big\}\|z-z^*\|_{E_0}.
\end{aligned}
$$

Thus $\|Q(z)-Q(z^*)\|_{E_0}\leq\Lambda^*\|z-z^*\|_{E_0}$.
Here $\Lambda^*$ depends only on the parameters involved in the problem. By assumption $(A6)$, $\Lambda^*<1$ and therefore the map $Q$ is a contraction. Hence, by the Banach contraction principle, the problem (2) has a unique solution on $(-\infty,T]$. □

### 4.2. Example

Consider the following fractional neutral integrodifferential equation

$$
\begin{aligned}
&{}^{\mathrm{LC}}D^{3/2}\left[x(t)-\int_0^t 2\sqrt{\frac{t-s}{\pi}}\left(\frac{e^{-\gamma s}x_\tau^2}{16(1+x_\tau^2)}+\frac{1}{16}\int_0^s\frac{e^{-\gamma s}}{3}\ln(1+x_\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\\
&\quad=\frac{(1+e^{-t})}{(35+e^t)}\frac{e^{-\gamma t}|x_t|}{(1+|x_t|)}+\frac{1}{18}\int_0^t e^{-\gamma t}\sin\left(\frac{x_s}{2}\right)\mathrm{d}s,\ t\in[0,1],\\
&\quad x(t)=\phi(t),\ t\in(-\infty,0],\\
&\qquad x(1)=\sum_{i=1}^{5}a_i(I^{p_i}x)(\eta_i),\ 0<\eta_1<\eta_2<\eta_3<1.
\end{aligned}
\tag{14}
$$

We consider the phase space $B_\gamma$ as defined in Section 3.2.

Here $q=\frac{3}{2}$, $m=3$,

$a_1=\frac{1}{2}$, $a_2=3$, $a_3=\frac{1}{5}$,

$\eta_1=\frac{1}{8}$, $\eta_2=\frac{1}{2}$, $\eta_3=\frac{3}{4}$,

$p_1=\frac{1}{9}$, $p_2=\frac{1}{5}$, $p_3=\frac{1}{3}$.

From the above data, we see that
$A=0.84473$, $\gamma_1=0.75225$, $\gamma_2=0.3009$, $\gamma_3=0.04238$, $\gamma_4=0.00245$, $\theta_1=1.69295$ and $\theta_2=0.66001$.

(i) From (14), we have
$f(t,x_t,K_2x(t))=\frac{(1+e^{-t})}{(35+e^t)}\frac{e^{-\gamma t}|x_t|}{1+|x_t|}+\frac{1}{18}K_2x(t)$ where $K_2x(t)=\int_0^t e^{-\gamma t}\sin\left(\frac{x_s}{2}\right)\mathrm{d}s$ and
$g(t,x_t,K_1x(t))=\frac{e^{-\gamma t}x_t^2}{16(1+x_t^2)}+\frac{1}{16}K_1x(t)$ where $K_1x(t)=\int_0^t\frac{e^{-\gamma t}}{3}\ln(1+x_s)\mathrm{d}s$ . Now, for $x_t,y_t\in B_\gamma$, we

have

$$\begin{aligned}
\left|k_1(t,s,x_s)-k_1(t,s,y_s)\right| &\leq \frac{1}{3}\|x-y\|_\gamma,\\
\left|k_2(t,s,x_s)-k_2(t,s,y_s)\right| &\leq \frac{1}{2}\|x-y\|_\gamma,\\
|f(t,x_t,K_2x(t))-f(t,y_t,K_2y(t))| &\leq \frac{1}{18}\left[\|x-y\|_\gamma+\frac{1}{2}\|x-y\|_\gamma\right],\\
|g(t,x_t,K_1x(t))-g(t,y_t,K_1y(t))| &\leq \frac{1}{16}\left[\|x-y\|_\gamma+\frac{1}{3}\|x-y\|_\gamma\right].
\end{aligned}$$

Also $|f(t,\phi_1,\psi_1)| \leq \frac{1}{18}+\frac{1}{36}\|\psi_1\|_\gamma$, $|k_1(t,s,\varphi_1)| \leq \frac{1}{3}\|\varphi_1\|_\gamma$, $|g(t,\phi_2,\psi_2)| \leq \frac{1}{16}+\frac{1}{48}\|\psi_2\|_\gamma$ and $|k_2(t,s,\varphi_2)|_X \leq \frac{1}{2}|\varphi_2|_\gamma$. The condition $(A4)$ is satisfied with $L_f=\frac{1}{18}$, $L_g=\frac{1}{16}$, $L_{k_1}=\frac{1}{3}$ and $L_{k_2}=\frac{1}{2}$. The condition $(A5)$ is satisfied with $p_1(t)=1$, $\|\phi_1\|_\gamma=\frac{1}{18}$, $p_2(t)=\frac{1}{36}$, $p_3(t)=1$, $\|\phi_2\|_\gamma=\frac{1}{16}$, $p_4(t)=\frac{1}{48}$, $p_5(t)=\frac{1}{3}$ and $p_6(t)=\frac{1}{2}$. Computing the value of $L^*$, we have $L^*=0.1285<1$ thereby satisfying the condition (12). Thus all the assumptions of the Theorem 4.2 are satisfied. Hence the problem (14) has at least one solution on $(-\infty,T]$.

(ii) Taking $f(t,x_t,K_2x(t))=\sqrt{(12+5\sin 2t)}+\dfrac{e^{-\gamma t}}{5}\tan^{-1}x_t+\dfrac{1}{5}K_2x(t)$ where $K_2x(t)=\displaystyle\int_0^t\frac{e^{-\gamma t}(ts)^2}{(1+s^2)}e^{-\frac{x_s}{15}}ds$ and $g(t,x_t,K_1x(t))=\dfrac{e^{-\gamma t}(t+1)^2}{(2t+5)^2}\dfrac{x_t^3}{1+x_t^3}+\dfrac{1}{25}K_1x(t)$ where $K_1x(t)=\displaystyle\int_0^t e^{-\gamma t}\frac{s}{2}\sin(x_s)ds$ in (14), we have for $x_t,y_t\in B_\gamma$,

$$\begin{aligned}
\left|k_1(t,s,x_s)-k_1(t,s,y_s)\right| &\leq \frac{1}{2}\|x-y\|_\gamma,\\
\left|k_2(t,s,x_s)-k_2(t,s,y_s)\right| &\leq \frac{1}{15}\|x-y\|_\gamma,\\
|f(t,x_t,K_2x(t))-f(t,y_t,K_2y(t))| &\leq \frac{1}{58}\left[\|x-y\|_\gamma+\frac{1}{15}\|x-y\|_\gamma\right],\\
|g(t,x_t,K_1x(t))-g(t,y_t,K_1y(t))| &\leq \frac{1}{25}\left[\|x-y\|_\gamma+\frac{1}{2}\|x-y\|_\gamma\right].
\end{aligned}$$

The condition $(A4)$ is satisfied with $L_f=\frac{1}{5}$, $L_g=\frac{1}{25}$, $L_{k_1}=\frac{1}{2}$ and $L_{k_2}=\frac{1}{15}$. Computing the value of $\Lambda^*$, we have $\Lambda^*=0.42831<1$ thereby satisfying the condition $(A6)$. Thus all the assumptions of the Theorem 4.3 are satisfied. Hence the problem (14) with the given function $f$ has a unique solution on $(-\infty,T]$.

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