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Core and strongly core orthogonal elements in rings with involution

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Abstract. In this paper, we present a new concept of the core orthogonality for two core invertible elements a and b in rings with involution. a is said to be core orthogonal to b, if $a^{\#\!\!\!\bigcirc}b = 0$ and $ba^{\#\!\!\!\bigcirc} = 0$, where $a^{\#\!\!\!\bigcirc}$ is the core inverse of a. The characterizations of the core orthogonality and the core additivity are provided. By using the matrix representations of the core orthogonal elements, the connection between the core orthogonality and the core partial order is also given. Moreover, the strongly core orthogonality is defined and characterized.

1. Introduction

As we all know, there are two forms of the orthogonality: one-sided or two-sided orthogonality. Let $R(A)$ and $R(B)$ denote the ranges of A and B, respectively. It is called that $R(A)$ and $R(B)$ are orthogonal if $A^*B = 0$. If $AB = 0$ and $BA = 0$, then A and B are orthogonal, denoted as $A \perp B$. Notice that, when $A^\#$ exists and $AB = 0$, where $A^\#$ is the group inverse of A, we have $A^\#B = A^\#AA^\#B = (A^\#)^2AB = 0$. And it is obvious that $A^\#B = 0$ implies $AB = 0$. Thus, when $A^\#$ exists, $A \perp B$ if and only if $A^\#B = 0$ and $BA^\# = 0$ (i.e. A and B are #-orthogonal, denoted as $A \perp_\# B$).

In [1], Hestenes gave the concept of $*$-orthogonality: let $A, B \in \mathbb{C}^{m\times n}$, if $A^*B = 0$ and $BA^* = 0$ hold, then A is $*$-orthogonal to B, denoted by $A \perp_* B$. For matrices, Hartwig and Styan proved in [2] that if A, B satisfy the dagger additivity (i.e. $(A+B)^\dagger = A^\dagger + B^\dagger$, where $A^\dagger$ is the Moore-penrose inverse of A) and the rank additivity (i.e. $rank(A+B) = rank(A) + rank(B)$), then A is $*$-orthogonal to B.

Ferreyra and Malik introduced the core and strongly core orthogonal matrices by using the core inverse in [3]. Let $A, B \in \mathbb{C}^{m\times n}$ with $\mathrm{Ind}(A) \le 1$, where $\mathrm{Ind}(A)$ is the index of A, if $A^{\#\!\!\!\bigcirc}B = 0$ and $BA^{\#\!\!\!\bigcirc} = 0$, then A is core orthogonal to B, denoted as $A \perp_{\#\!\!\!\bigcirc} B$. The notion of the strongly core orthogonality also has been given by Ferreyra and Malik in [3]. $A, B \in \mathbb{C}^{m\times n}$ with $\mathrm{Ind}(A) \le 1$ and $\mathrm{Ind}(B) \le 1$ are strongly core orthogonal matrices (denoted as $A \perp_{s,\#\!\!\!\bigcirc} B$) if $A \perp_{\#\!\!\!\bigcirc} B$ and $B \perp_{\#\!\!\!\bigcirc} A$. In [3], we can see that $A \perp_{S,\#\!\!\!\bigcirc} B$ implies $(A+B)^{\#\!\!\!\bigcirc} = A^{\#\!\!\!\bigcirc} + B^{\#\!\!\!\bigcirc}$ (core additivity).

In [4], Liu et al. proved that $A, B \in \mathbb{C}^{m\times n}$ with $\mathrm{Ind}(A) \le 1$ and $\mathrm{Ind}(B) \le 1$ are strongly core orthogonal if and only if $(A+B)^{\#\!\!\!\bigcirc} = A^{\#\!\!\!\bigcirc} + B^{\#\!\!\!\bigcirc}$ and $A^{\#\!\!\!\bigcirc}B = 0$ (or $BA^{\#\!\!\!\bigcirc} = 0$) instead of $A \perp_{\#\!\!\!\bigcirc} B$. The proof is more concise than Theorem 7.3 in [3]. And Ferreyra and Malik in [3] have proved that if A is strongly core orthogonal to B, then $rank(A+B) = rank(A) + rank(B)$ and $(A+B)^{\#\!\!\!\bigcirc} = A^{\#\!\!\!\bigcirc} + B^{\#\!\!\!\bigcirc}$. But whether the reverse holds is an open

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question. In [4], Liu, Wang and Wang solved the problem completely. Furthermore, they also gave some new equivalent conditions for the strongly core orthogonality, which are related to the minus partial order and some Hermitian matrices.

On the basis of the core orthogonal matrix, Mosić et al. extended the concept of the core orthogonality and present the new concept of the core-EP orthogonality in [5]. Let A and B be two Drazin invertible operators in a bounded linear Hilbert space. A is said to be core-EP orthogonal to B if $A^{\textcircled{D}}B = 0$ and $BA^{\textcircled{D}} = 0$, where $A^{\textcircled{D}}$ is the core-EP inverse of A. A number of characterizations for core-EP orthogonality were proved, including simultaneous canonical forms, connections with the core partial order and core additivity in [5]. Applying the core-EP orthogonality, the concept and characterizations of the strong core-EP orthogonality were introduced in [5].

Motivated by these reserches, we give the concept of the core orthogonality to rings with involution by using the core inverse in this paper. We discuss their characterizations. We also study the equivalent conditions of the core additivity and give their matrix representations when a and b are core orthogonal. The connection between the core partial order and the core orthogonality has been given. And we study some characterizations of the core orthogonal element when a is EP. Moreover, we study that two arbitrary complementary projections are strongly core orthogonal.

2. Preliminaries

Throughout this paper, R is a unital ring with involution, i.e. all $a, b \in R$ satisfy $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a + b)^* = a^* + b^*$. $R^{(1)} = \{a \in R : a \in aRa\}$ denotes the set of all regular elements.

For $a, x \in R$, we consider the following equations:
1. $axa = a$;
2. $xax = x$;
3. $(ax)^* = ax$;
4. $(xa)^* = xa$;
5. $ax = xa$;
6. $xa^2 = a$;
7. $ax^2 = x$;
8.(8) $a^2x = a$;
9. $ax^2 = x$;
10. $xa^{m+1} = a^m$ for some positive integer m.

The set of all elements $x \in R$ which satisfies equations $i, j, ..., k$ in equations 1-10 are denoted as $a\{i, j, ..., k\}$. If there exists

$$a^{\dagger} \in a\{1, 2, 3, 4\},$$

then a is said to be Moore-Penrose invertible, in this case, $a^{\dagger}$ is unique and called the Moore-Penrose inverse of a. It is introduced by Moore [6] and improved by Bjerhammar [7] and Penrose [8]. If there exists

$$a^{\#} \in a\{1, 2, 5\},$$

then a is said to be group invertible, in this case, $a^{\#}$ is unique and called the group inverse of a [9]. In [10], Rakić, Dinčić and Djordjević considered the core inverses in the setting of arbitrary $*$-ring, which is shown that x is the core inverse of a if and only if

$$axa = a, xR = aR, Rx = Ra^*, \tag{1}$$

in this case, x is unique. If there exists

$$a^{\textcircled{\#}} \in a\{1, 2, 3, 6, 7\},$$

then a is said to be core invertible, in this case, $a^{\textcircled{\#}}$ is unique and called the core inverse of a [11]. And if there exists

$$a^{\textcircled{\dagger}} \in a\{3, 9, 10\},$$

then a is said to be pseudo-core invertible, in this case, $a^{\textcircled{\dagger}}$ is unique and called the pseudo-core inverse of a [12]. $R^{\dagger}$, $R^{\#}$, $R^{\textcircled{\#}}$ and $R^{\textcircled{\dagger}}$ denote the sets of all Moore-Penrose, group, core and core-EP invertible elements of R, respectively.

Based on this, we review the definition of the partial order.

Definition 2.1. *[13] Let $a, b \in R$,*
(1) if there exists $a^{(1)} \in a\{1\}$ such that,

$$aa^{(1)} = ba^{(1)}, a^{(1)}a = a^{(1)}b.$$

then a is below b under the minus partial order (written as $a <^{-} b$).
(2) a is below b under the star partial order (written as $a <^{*} b$) if

$$aa^{*} = ba^{*}, a^{*}a = a^{*}b.$$

(3) if $a, b \in R^{\#}$, then a is below b under the sharp partial order (written as $a <^{\#} b$) if

$$aa^{\#} = ba^{\#}, a^{\#}a = a^{\#}b.$$

(4) if $a, b \in R^{\textcircled{\#}}$, then a is below b under the core partial order (written as $a <^{\textcircled{\#}} b$) if

$$aa^{\textcircled{\#}} = ba^{\textcircled{\#}}, a^{\textcircled{\#}}a = a^{\textcircled{\#}}b.$$

We denote aR and Ra as the right and left ideals generated by a, respectively, that is, $aR = \{ax : x \in R\}$ and $Ra = \{xa : x \in R\}$. Also $aRb = \{axb : x \in R\}$. The right annihilator of a is denoted by a° and is defined by $a^{\circ} = \{x \in R : ax = 0\}$. Similarly, the left annihilator of a is the set ${}^{\circ}a = \{x \in R : xa = 0\}$. In [14], Koliha and Patrício introduced the concept of the *EP* element in rings with involution.

Definition 2.2. *[14] Let $a \in R^{\dagger}$, a is an EP element if and only if $a^{*}R = aR$.*

Lemma 2.3. *[15] Let $a \in R^{\textcircled{\dagger}}$, $p = aa^{\textcircled{\dagger}}$, and $a = a_1 + a_2$ be the core-EP decomposition of a, where a_2 is the nilpotent. Then,*

$$a_1 = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p \times p} \text{ and } a_2 = \begin{bmatrix} 0 & 0 \\ 0 & a_2 \end{bmatrix}_{p \times p},$$

where $t \in pRp$ is invertible.

3. The core orthgonality and its consequences

Firstly, we give the concept of the core orthogonality in rings with involution.

Definition 3.1. *Let $a, b \in R^{\textcircled{\#}}$. If*

$$a^{\textcircled{\#}}b = 0, ba^{\textcircled{\#}} = 0,$$

then a is core orthogonal to b, denoted as $a \perp_{\textcircled{\#}} b$.

If $a, b \in R$, then

$$ab = 0 \Leftrightarrow bR \subset a^{\circ} \Leftrightarrow Ra \subset {}^{\circ}b. \tag{2}$$

We explore the relations between the right ideal and right cancellable elements of core orthogonal elements in rings with involution, and the relations between the left ideal and left cancellable elements of core orthogonal elements can be obtained similarly.

Lemma 3.2. *[10] Let $a, b \in R$.*
(1) $aR \subset bR$ implies ${}^{\circ}b \subset {}^{\circ}a$ and the converse is valid whenever b is regular; (2) $Ra \subset Rb$ implies $b^{\circ} \subset a^{\circ}$ and the converse is valid whenever b is regular.

Remark 3.3. *By 1, we have $a^{\#}R \subset aR$ and $Ra^{\#} \subset Ra^*$, and it follows from Lemma 3.2 that*

$$(a^{\#})^{\circ} = (a^*)^{\circ} \tag{3}$$

and

$$^{\circ}(a^{\#}) = {}^{\circ}a. \tag{4}$$

Theorem 3.4. *Let $a, b \in R^{\#}$, then the following are equivalent:*
(1) $a \perp_{\#} b$;
(2) $bR \subset (a^{\#})^{\circ}$, $a^{\#}R \subset b^{\circ}$;
(3) $bR \subset (a^*)^{\circ}$, $aR \subset b^{\circ}$;
(4) $aR \subset (b^*)^{\circ}$, $b^*R \subset (a^*)^{\circ}$;
(5) $aR \subset (b^{\#})^{\circ}$, $b^*R \subset (a^{\#})^{\circ}$;
(6) $a^{\#}R \subset (b^*)^{\circ}$, $b^*R \subset (a^{\#})^{\circ}$;
(7) $a^*b = 0$, $ba = 0$;
(8) $b^*a = 0$, $a^*b^* = 0$;
(9) $b^{\#}a = 0$, $a^{\#}b^* = 0$;
(10) $b^*a^{\#} = 0$, $a^{\#}b^* = 0$.

Proof. (1) ⇔ (2) From (2), we have

$$a^{\#}b = 0, ba^{\#} = 0 \Leftrightarrow bR \subset (a^{\#})^{\circ}, a^{\#}R \subset b^{\circ}.$$

(2) ⇔ (3) From (3) and (1), we have $(a^{\#})^{\circ} = (a^*)^{\circ}$ and $a^{\#}R = aR$. Then

$$bR \subset (a^{\#})^{\circ}, a^{\#}R \subset b^{\circ} \Leftrightarrow bR \subset (a^*)^{\circ}, aR \subset b^{\circ}.$$

(3) ⇔ (7) From (1), we have

$$bR \subset (a^*)^{\circ}, aR \subset b^{\circ} \Leftrightarrow a^*b = 0, ba = 0.$$

(7) ⇔ (8) Transposition of (7) and (8).

According to (1), (2) and (3), we can prove (4) ⇔ (5), (5) ⇔ (6), (8) ⇔ (6), (5) ⇔ (9) and (6) ⇔ (10). □

Theorem 3.5. *Let $a, b \in R^{\#}$, then $a \perp_{\#} b$ if and only if $a \perp_{\#} b^*$.*

Proof. It follows from (10) in Theorem 3.4. □

Theorem 3.6. *Let $a, b \in R^{\#}$, and $e = a^{\#}a$, $e' = aa^{\#}$, $f = b^{\#}b$, $f' = bb^{\#}$. Then $a \perp_{\#} b$ if and only if $e' \perp f'$ and $fe = 0$.*

Proof. Only if: If $a \perp_{\circledast} b$, i.e. $a^{\circledast}b = 0$, $ba^{\circledast} = 0$, then

$$fe = b^{\circledast}ba^{\circledast}a = b^{\circledast}(ba^{\circledast})a = 0,$$
$$e'f' = aa^{\circledast}bb^{\circledast} = a(a^{\circledast}b)b^{\circledast} = 0.$$

By (9) in Theorem 3.5, we have $b^{\circledast}a = 0$. Then

$$f'e' = bb^{\circledast}aa^{\circledast} = b(b^{\circledast}a)a^{\circledast} = 0.$$

Consequently, we get $e' \perp f'$.

If: If $fe = b^{\circledast}ba^{\circledast}a = 0$, then pre-multiplying by b and post-multiplying by $a^{\circledast}$, we obtain $ba^{\circledast} = 0$. Since $e' \perp f'$, $e'f' = aa^{\circledast}bb^{\circledast} = 0$. And pre-multiplying by $a^{\circledast}$ and post-multiplying by b, we obtain $a^{\circledast}b = 0$. Then we get $a \perp_{\circledast} b$. □

Lemma 3.7. *Let $a, b \in R^{\circledast}$. If $ab = 0$, then*
(1) $aR \cap bR = \{0\}$;
(2) $a^*R \cap b^*R = \{0\}$;
(3) $(a+b)^{\circ} = a^{\circ} \cap b^{\circ}$;
(4) $(a^* + b^*)^{\circ} = (a^*)^{\circ} \cap (b^*)^{\circ}$.

Proof. (1) From (1), we have $ab = 0 \Leftrightarrow bR \subset a^{\circ}$. Then

$$aR \cap bR \subset aR \cap a^{\circ} = \{0\}.$$

(2) Let $ab = 0$, we have $b^*a^* = 0$. By (1), we can prove (2).
(3) Obviously, $a^{\circ} \cap b^{\circ} \subset (a+b)^{\circ}$. Let $x \in (a+b)^{\circ}$, then $(a+b)x = 0$, i.e. $ax = -bx$. Since $ax = a^{\circledast}a^2x = a^{\circledast}a(-bx) = -a^{\circledast}abx = 0$, i.e. $bx = 0$, we get $x \in a^{\circ} \cap b^{\circ}$, which implies $(a+b)^{\circ} \subset a^{\circ} \cap b^{\circ}$. Then $(a+b)^{\circ} = a^{\circ} \cap b^{\circ}$.
(4) Let $ab = 0$, we have $b^*a^* = 0$. By (3), we can prove (4).
□

Theorem 3.8. *Let $a, b \in R^{\circledast}$. If $a \perp_{\circledast} b$, then*
(1) $aR \cap bR = \{0\}$;
(2) $a^*R \cap b^*R = \{0\}$;
(3) $aR \cap b^*R = \{0\}$;
(4) $a^*R \cap bR = \{0\}$;
(5) $(a+b)^{\circ} = a^{\circ} \cap b^{\circ}$;
(6) $(a^* + b^*)^{\circ} = (a^*)^{\circ} \cap (b^*)^{\circ}$;
(7) $(a + b^*)^{\circ} = a^{\circ} \cap (b^*)^{\circ}$;
(8) $(a^* + b)^{\circ} = (a^*)^{\circ} \cap b^{\circ}$.

Proof. Let $a \perp_{\circledast} b$. It follows from (7), (8) in Theorem 3.5 that we get $a^*b = 0$, $ba = 0$, $b^*a = 0$ and $a^*b^* = 0$. And by Lemma 3.7, we can get Theorem 3.8. □

When a is an *EP* element, we have a more refined result which reduces to the well-known characterizations of the orthogonality in the usual sense.

Theorem 3.9. *Let $a \in R$. If a is an EP element, then the following are equivalent.*
(1) $a \perp_{\circledast} b$;
(2) $a \perp_* b$;
(3) $a \perp b$;
(4) $a \leq^{\circledast} a + b$;
(5) $a \leq^* a + b$;
(6) $a \leq^{\#} a + b$.

Proof. (1) ⇔ (2) When $a \in R^{\#}$ is an *EP* element, we have $aa^{\#} = a^{\#}a$. Let $a \perp_{\#} b$, then by (7) in Theorem 3.4, i.e. $a^*b = 0$ and $ba = 0$, we have

$$ba^* = b(aa^{\#}a)^* = b(a^{\#}a)^*a^* = b(aa^{\#})^*a^* = baa^{\#}a^* = 0.$$

Then, $a \perp_* b$.
On the contrary, if $a \perp_* b$, i.e. $a^*b = ba^* = 0$, then we have

$$ba = b(aa^{\#}a) = b(aa^{\#})^*a = b(a^{\#}a)^*a = ba^*(a^{\#})^*a = 0.$$

By (7) in Theorem 3.4, we have $a \perp_{\#} b$.
(2) ⇔ (3) Let $a \perp_* b$, we have $a^*b = ba^* = 0$. Then

$$ab = (aa^{\#}a)b = a(aa^{\#})b = a(aa^{\#})^*b = a(a^{\#})^*a^*b = 0$$

and

$$ba = b(aa^{\#}a) = b(aa^{\#})^*a = b(a^{\#}a)^*a = ba^*(a^{\#})^*a = 0.$$

Thus, $a \perp b$.
On the contrary, if $a \perp b$, i.e. $ab = ba = 0$, then we have

$$a^*b = (aa^{\#}a)^*b = a^*(aa^{\#})^*b = a^*aa^{\#}b = a^*a^{\#}ab = 0$$

and

$$ba^* = b(aa^{\#}a)^* = b(a^{\#}a)^*a^* = b(aa^{\#})^*a^* = baa^{\#}a^* = 0.$$

Then $a \perp_* b$.
(1) ⇔ (4) Let $a \leq^{\#} a + b$, then

$$(a + b)\, a^{\#} = aa^{\#} + ba^{\#} = aa^{\#},$$

and

$$a^{\#}\,(a + b) = a^{\#}a + a^{\#}b = a^{\#}a.$$

Thus, we have $a^{\#}b = 0$ and $ba^{\#} = 0$, i.e. $a \perp_{\#} b$.
On the contrary, if $a \perp_{\#} b$, i.e. $a^{\#}b = 0$ and $ba^{\#} = 0$, then we have

$$(a + b)\, a^{\#} = aa^{\#},$$

$$a^{\#}\,(a + b) = a^{\#}a.$$

Then $a \leq^{\#} a + b$.
(2) ⇔ (5) It can be proved similarly with (1) ⇔ (4).
(1) ⇔ (6) Let $a \in R$ is an *EP* element, and by [16], we have $a^{\#} = a^{\#}$. Then, it can be proved similarly with (1) ⇔ (4).
□

Remark 3.10. *In Theorem 3.9, we give a condition that $a \in R^{\#}$ is an EP element instead of b, which is more concise than Corollary 4.8 in [3]. If $a \perp_{\#} b$, by (4) in Theorem 3.4, we have $aR \subset (b^*)^\circ$ and $b^*R \subset (a^*)^\circ$. When $a, b \in R^{\#}$ are EP elements, $aR = a^*R$ and $bR = b^*R$ hold. Thus, $a^*R \subset (b^*)^\circ$ and $bR \subset (a^*)^\circ$, i.e. $b^*a^* = 0$ and $a^*b = 0$. Therefore, by (8) in Theorem 3.4, $b \perp_{\#} a$. In other words, when $a, b \in R^{\#}$ are EP elements, $a \perp_{\#} b$ is equivalent to that $b \perp_{\#} a$.*

Theorem 3.11. *Let $a, b \in R$. If a, b are EP elements with $a \perp_{\#} b$, then $a + b$ is the EP element.*

Proof. Let $a, b \in R$ be *EP* elements, we have that $a^\circ = (a^*)^\circ$, $b^\circ = (b^*)^\circ$. Since $a \perp_{\circledR} b$, by (5) and (6) in Theorem 3.9, we have

$$(a+b)^\circ = a^\circ \cap b^\circ = (a^*)^\circ \cap (b^*)^\circ = (a^* + b^*)^\circ = ((a+b)^*)^\circ .$$

Then $a + b$ is an *EP* element. □

Furthermore, we study the matrix forms of the two elements that are core orthogonal. At first, we review the ring factorization. An equation $1 = e_1 + e_2 + \cdots + e_n$, where $e_i, i = 1, 2, \cdots n$ are idempotents from R such that $e_i e_j = 0$ for $i \neq j$ is called the decomposition of the identity of the ring R. And we denote $e : \{e_1, e_2, \cdots, e_n\}$. If $1 = e_1 + e_2 + \cdots + e_n$ and $1 = f_1 + f_2 + \cdots + f_n$ are two decompositions of the identity of the ring R, then for any $x \in R$, we have

$$x = 1 \cdot x \cdot 1 = (e_1 + e_2 + \cdots + e_n)\, x\, (f_1 + f_2 + \cdots + f_n) = \textstyle\sum_{i,j=1}^{n} e_i x f_j.$$

We may write x as a matrix

$$x = \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{n1} & \ldots & x_{nn} \end{bmatrix}_{e\times f},$$

where $x_{i,j} = e_i x f_j$.

Lemma 3.12. *Let $a \in R^{\circledR}$, $p = aa^{\circledR}$, then*

$$a = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p}$$

is the core-EP decomposition of a, where $t \in pRp$ is invertible.

Proof. Let $a \in R^{\circledR}$ and $p = aa^{\circledR}$, then $t = pap = a^2 a^{\circledR}$ is invertible, $s = pa(1-p) = a - a^2 a^{\circledR}$, and $a_2 = (1-p)a(1-p) = 0$. Thus,

$$a_1 = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p} \text{ and } a_2 = \begin{bmatrix} 0 & 0 \\ 0 & a_2 \end{bmatrix}_{p\times p} = 0.$$

From Lemma 2.3, we have

$$a = a_1 + a_2 = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p}$$

is the core-EP decomposition of a, where $t \in pRp$ is invertible. □

Theorem 3.13. *Let $a, b \in R^{\circledR}$. Then, the following are equivalent.*
(1) $a \perp_{\circledR} b$;
(2) $a \leq^{\circledR} a + b$;
(3) there exists $1 = e_1 + e_2 + e_3$ which is a decomposition of identity of the ring R, where $e_1 = p = p^2 = p^*$, $e_2 = q = q^*$,$e_3 = 1 - p - q$, and then

$$a = \begin{bmatrix} t_1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

where $t_1 \in pRp$ and $t_2 \in qRq$ are invertible, respectively.

Proof. (1) ⇒ (2) Let $a \perp_{\circledast} b$. Then

$$(a+b)\,a^{\circledast} = aa^{\circledast} + ba^{\circledast} = aa^{\circledast}$$

and

$$a^{\circledast}(a+b) = a^{\circledast}a + a^{\circledast}b = a^{\circledast}a.$$

Thus we have $a \leq^{\circledast} a+b$.
(2) ⇒ (3) Let

$$a = \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p\times p}$$

be the core-EP decomposition of a, where $t_1 \in pRp$ is invertible, $p = aa^{\circledast}$. Since

$$x = \begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}$$

satisfies $xax = x$, $axa = a$, $(ax)^* = ax$, $ax^2 = x$ and $xa^2 = a$, we get $x = a^{\circledast}$. And let the decomposition of b be

$$b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{p\times p}.$$

By (2), we have that $a \leq^{\circledast} a+b$ if and only if $a^{\circledast}b = 0$ and $ba^{\circledast} = 0$. Then, by

$$a^{\circledast}b = \begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{p\times p} = \begin{bmatrix} t_1^{-1}b_{11} & t_1^{-1}b_{12} \\ 0 & 0 \end{bmatrix}_{p\times p} = 0,$$

we get $b_{11} = 0$ and $b_{12} = 0$. And by

$$ba^{\circledast} = \begin{bmatrix} 0 & 0 \\ b_{21} & b_{22} \end{bmatrix}_{p\times p}\begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} 0 & 0 \\ b_{21}t_1^{-1} & 0 \end{bmatrix}_{p\times p} = 0,$$

we get $b_{21} = 0$. Then $b = \begin{bmatrix} 0 & 0 \\ 0 & b_{22} \end{bmatrix}_{p\times p}$.

Let the core-EP decomposition of b_{22} be

$$b_{22} = \begin{bmatrix} t_2 & s_2 \\ 0 & 0 \end{bmatrix}_{q\times q}$$

where $t_2 \in qRq$ is invertible, $q = b_{22}b_{22}^{\circledast}$. Then

$$b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

Let $1-p = q + (1-p-q)$, we get $1 = p + (1-p) = p + q + (1-p-q)$. And there exits $s_1 = r + s$ with $r \in pRq$ and $s \in pR(1-p-q)$. So, we have

$$a = \begin{bmatrix} t_1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

(3) ⇒ (1) We know that

$$y = \begin{bmatrix} t_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{p\times p}$$

is core inverse of $a = \begin{bmatrix} t_1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}$, i.e. $y = a^{\#}$. Then $a^{\#}b = 0$ and $ba^{\#} = 0$, i.e. $a \perp_{\#} b$.

□

In Theorem 5.1 and Corollary 5.1 of [19], Xu, Chen and Benítez pointed out the conditions for $(B-A)^{\#} = B^{\#} - A^{\#}$, where $A, B \in C^{m\times n}$. Using the core-EP decomposition of a which is core orthogonal, we apply a different method to prove that this theorem and corollary also hold in rings with involution as follows.

Theorem 3.14. *Let $a, b-a \in R^{\#} \cap R^{\dagger}$ and $a \leq^{\#} b$. Then, b is core invertible. In this case,*

$$b^{\#} = a^{\#} + (b-a)^{\#} - a^{\#}a(b-a)^{\#}.$$

Moreover, if $(aa^{\dagger} - aa^{\#})b(1-aa^{\dagger}) = 0$, then
(1) $(b-a)^{\#} = b^{\#} - a^{\#}$;
(2) $(b-a) \leq^{\#} b$.

Proof. From $a \leq^{\#} b = a + (b-a)$ and (3) in Theorem 3.13, we have that there exists $1 = e_1 + e_2 + e_3$, which is a decomposition of identity of the ring R, where $e_1 = p = p^2 = p^*$, $e_2 = q = q^*$ and $e_3 = 1 - p - q$. Then,

$$a = \begin{bmatrix} t_1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, b-a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, \tag{5}$$

where t_1 and t_2 are invertible in the ring eRe and qRq, respectively. Thus,

$$a^{\#} = \begin{bmatrix} t_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, (b-a)^{\#} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \tag{6}$$

and

$$b = (b-a) + a = \begin{bmatrix} t_1 & r & s \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}. \tag{7}$$

Let

$$x = a^{\#} + (b-a)^{\#} - a^{\#}a(b-a)^{\#} = \begin{bmatrix} t_1^{-1} & -t_1^{-1}rt_2^{-1} & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

Then

$$bx = \begin{bmatrix} t_1 & r & s \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} t_1^{-1} & -t_1^{-1}rt_2^{-1} & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

which implies

$$bxb = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} t_1 & r & s \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} t_1 & r & s \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = b.$$

In the same way, we can prove that $xbx = x$, $(bx)^* = bx$, $bx^2 = x$ and $xb^2 = b$ also hold. Then b is core invertible and

$$b^{\#\!\!\!\!\bigcirc} = x = a^{\#\!\!\!\!\bigcirc} + (b-a)^{\#\!\!\!\!\bigcirc} - a^{\#\!\!\!\!\bigcirc} a(b-a)^{\#\!\!\!\!\bigcirc} = \begin{bmatrix} t_1^{-1} & -t_1^{-1} r t_2^{-1} & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}. \tag{8}$$

Let

$$a^{\dagger} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{p\times p}.$$

Then

$$aa^{\dagger} = \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{p\times p} = \begin{bmatrix} t_1 a_{11} + s_1 a_{21} & t_1 a_{12} + s_1 a_{22} \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

From $(aa^{\dagger})^* = aa^{\dagger}$, we have $t_1 a_{12} + s_1 a_{22} = 0$, i.e.

$$aa^{\dagger} = \begin{bmatrix} t_1 a_{11} + s_1 a_{21} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

And from $aa^{\dagger}a = a$, we have

$$\begin{bmatrix} t_1 a_{11} t_1 + s_1 a_{21} t_1 & t_1 a_{11} s_1 + s_1 a_{21} s_1 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p\times p},$$

which implies $t_1 a_{11} t_1 + s_1 a_{21} t_1 = t_1$, i.e. $t_1 a_{11} + s_1 a_{21} = 1$. Then

$$aa^{\dagger} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

Let

$$a^{\#} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}_{p\times p}.$$

Then

$$aa^{\#} = \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}_{p\times p} = \begin{bmatrix} t_1 y_{11} + s_1 y_{21} & t_1 y_{12} + s_1 y_{22} \\ 0 & 0 \end{bmatrix}_{p\times p}$$

and

$$a^{\#}a = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}_{p\times p} \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} y_{11} t_1 & y_{11} s_1 \\ y_{21} t_1 & y_{21} s_1 \end{bmatrix}_{p\times p}.$$

From $aa^{\#} = a^{\#}a$, we have $y_{11}s_1 = t_1y_{12} + s_1y_{22}$ and $y_{21} = 0$, i.e.

$$y_{12} = t_1^{-1}y_{11}s_1 + t_1^{-1}s_1y_{22} \tag{9}$$

and

$$aa^{\#} = \begin{bmatrix} t_1y_{11} & t_1y_{12} + s_1y_{22} \\ 0 & 0 \end{bmatrix}_{p\times p}. \tag{10}$$

From $aa^{\#}a = a$ and $a^{\#}aa^{\#} = a^{\#}$, we have

$$\begin{bmatrix} t_1y_{11}t_1 & t_1y_{11}s_1 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p\times p}$$

and

$$\begin{bmatrix} y_{11}t_1y_{11} & y_{11}t_1y_{22} + y_{11}s_1y_{22} \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{bmatrix}_{p\times p},$$

which imply $y_{11} = t_1^{-1}$ and $y_{22} = 0$. Then from (9), we have

$$y_{12} = t_1^{-1}y_{11}s_1 + t_1^{-1}s_1y_{22} = t_1^{-2}s_1.$$

From (10), we obtain that

$$aa^{\#} = \begin{bmatrix} 1 & t_1^{-1}s_1 \\ 0 & 0 \end{bmatrix}_{p\times p},$$

i.e.

$$aa^{\#} = \begin{bmatrix} 1 & t_1^{-1}r & t_1^{-1}s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

Then we have

$$(aa^{\dagger} - aa^{\#})b = \begin{bmatrix} 0 & -t_1^{-1}rt_2 & -t_1^{-1}rs_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \tag{11}$$

and

$$(aa^{\dagger} - aa^{\#})baa^{\dagger} = \begin{bmatrix} 0 & -t_1^{-1}rt_2 & -t_1^{-1}rs_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = 0. \tag{12}$$

If $(aa^{\dagger} - aa^{\#})b(1 - aa^{\dagger}) = 0$, i.e. $(aa^{\dagger} - aa^{\#})b = (aa^{\dagger} - aa^{\#})baa^{\dagger}$, from (11) and (12), we have

$$\begin{bmatrix} 0 & -t_1^{-1}rt_2 & -t_1^{-1}rs_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = 0.$$

Then we have $-t_1^{-1}rt_2 = 0$ and $-t_1^{-1}rs_2 = 0$, which imply $r = 0$. And from (7) and (8), we have

$$b = \begin{bmatrix} t_1 & 0 & s \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, b^{\circledast} = \begin{bmatrix} t_1^{-1} & 0 & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}. \tag{13}$$

And

$$(b-a)^{\oplus} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \tag{14}$$

$$= \begin{bmatrix} t_1^{-1} & 0 & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} - \begin{bmatrix} t_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}$$

$$= b^{\oplus} - a^{\oplus}.$$

Since

$$(b-a)^{\oplus}(b-a) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & t_2^{-1}s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}$$

and

$$(b-a)^{\oplus}b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} t_1 & 0 & s \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & t_2^{-1}s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

we get $(b-a)^{\oplus}(b-a) = (b-a)^{\oplus}b$. In the same way, we get $(b-a)(b-a)^{\oplus} = b(b-a)^{\oplus}$. Then $(b-a) \leq^{\oplus} b$. □

Corollary 3.15. *Let $a, b-a \in R^{\oplus}$. If $a \leq^{\oplus} b$, then $a^2 = ab$ if and only if $(b-a)^{\oplus} = b^{\oplus} - a^{\oplus}$.*

Proof. Let the decompositions of a, $b-a$, $a^{\oplus}$, $(b-a)^{\oplus}$, b and $b^{\oplus}$ be as in (5), (6), (7) and (8), respectively.

Only if: Let $a^2 = ab$, we have

$$a^2 = \begin{bmatrix} t_1^2 & t_1 r & t_1 s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} t_1^2 & t_1 r + r t_2 & t_1 s + r s_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = ab.$$

Then we get $t_1 r = t_1 r + r t_2$, which implies $r t_2 = 0$. Since t_2 is nonsingular, we have $r = 0$. From (13) and (14), we get $(b-a)^{\oplus} = b^{\oplus} - a^{\oplus}$.

If: Let $(b-a)^{\oplus} = b^{\oplus} - a^{\oplus}$, we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} t_1^{-1} & -t_1^{-1} r t_2^{-1} & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} - \begin{bmatrix} t_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}$$

$$= \begin{bmatrix} 0 & -t_1^{-1} r t_2^{-1} & 0 \\ 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

which implies $-t_1^{-1} r t_2^{-1} = 0$. Since t_1 and t_2 are nonsingular, we have $r = 0$. Then

$$a = \begin{bmatrix} t_1 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

From (13), we have

$$a^2 = \begin{bmatrix} t_1^2 & 0 & t_1 s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} t_1 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} t_1 & 0 & s \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = ab.$$

□

4. The strongly core orthgonality and its consequences

A special orthogonality is mentioned in Remark 3.2 in the previous section, that is, $a \perp_{\textcircled{\#}} b$ and $a \perp_{\textcircled{\#}} b$. Now we give the definition.

Definition 4.1. *Let $a, b \in R^{\textcircled{\#}}$. If*

$$a \perp_{\textcircled{\#}} b,\ b \perp_{\textcircled{\#}} a,$$

then a and b are said to be strongly core orthogonal, denoted as

$$a \perp_{s,\textcircled{\#}} b.$$

Remark 4.2. *$a \perp_{s,\textcircled{\#}} b$ if and only if $a^{\textcircled{\#}}b = 0$, $ba^{\textcircled{\#}} = 0$, $b^{\textcircled{\#}}a = 0$ and $ab^{\textcircled{\#}} = 0$. By (9) in Theorem 3.4, $a^{\textcircled{\#}}b = 0$ is equivalent to $b^{\textcircled{\#}}a = 0$. Then, $a \perp_{s,\textcircled{\#}} b$ if and only if $a^{\textcircled{\#}}b = 0$, $ba^{\textcircled{\#}} = 0$ and $ab^{\textcircled{\#}} = 0$, i.e. $a \perp_{\textcircled{\#}} b$ and $ab^{\textcircled{\#}} = 0$, or $b \perp_{\textcircled{\#}} a$ and $ba^{\textcircled{\#}} = 0$.*

Theorem 4.3. *Let $a, b \in R^{\textcircled{\#}}$. Then, the following statements are equivalent.*
(1) $a \perp_{s,\textcircled{\#}} b$;
(2) $a \perp_{\textcircled{\#}} b$, $ab = 0$;
(3) $a \perp b$, $a^*b = 0$;
(4) $a^* \perp b^*$, $b^*a = 0$.

Proof. (1) ⇔ (2) According to (7) in Theorem 3.5, we have that $ab^{\textcircled{\#}} = 0$ is equivalent to $ab = 0$. Then, by remark 4.3, we have that $a \perp_{s,\textcircled{\#}} b$ if and only if $a \perp_{\textcircled{\#}} b$ and $ab = 0$.
(2) ⇔ (3) From (2), we have $ab = 0$. According to (7) in Theorem 3.5, we get that $a \perp_{\textcircled{\#}} b$ is equivalent to $a^*b = 0$ and $ba = 0$, which imply that $a \perp b$ and $a^*b = 0$.
(3) ⇔ (4) Take the conjugate transpose of a and b. □

Theorem 4.4. *Let $a, b \in R^{\textcircled{\#}}$. Then, $a \perp_{s,\textcircled{\#}} b$ if and only if $a^{\textcircled{\#}} \perp_{s,\textcircled{\#}} b$.*

Proof. From (3) in Theorem 4.3, $a^{\textcircled{\#}} \perp_{\textcircled{\#}} b$ if and only if $a^{\textcircled{\#}} \perp b$ and $(a^{\textcircled{\#}})^*b = 0$. Then, by Lemma 3.7, we get $a^{\textcircled{\#}}R \cap bR = \{0\}$ and $(a^{\textcircled{\#}})^*R \cap bR = \{0\}$. And by $a^{\textcircled{\#}}R = aR$, we have $aR \cap bR = \{0\}$ and $a^*R \cap bR = \{0\}$, i.e. $a \perp b$ and $a^*b = 0$. From Theorem 4.3, we have $a \perp_{s,\textcircled{\#}} b$. □

Theorem 4.5. *Let $a, b, c \in R^{\textcircled{\#}}$, $a \leq^{\textcircled{\#}} c$ and $c \perp_{s,\textcircled{\#}} b$. Then, $a \perp_{s,\textcircled{\#}} b$.*

Proof. From Theorem 2.3 in [4], $a \leq^{\textcircled{\#}} c$ if and only if $ca^{\textcircled{\#}}c = a$ and $a^{\textcircled{\#}}ca^{\textcircled{\#}} = a^{\textcircled{\#}}$. By (3) in Theorem 4.3 and $c \perp_{s,\textcircled{\#}} b$, we have $c \perp b$ and $c^*b = 0$. Then

$$ab = ca^{\textcircled{\#}}cb = 0, ba = bca^{\textcircled{\#}}c = 0, a^*b = c^*(a^{\textcircled{\#}})^*c^*b = 0,$$

which implies $a \perp_{s,\textcircled{\#}} b$. □

Theorem 4.6. *Let $a, b \in R^{\textcircled{\#}}$, $e = a^{\textcircled{\#}}a$, $e' = aa^{\textcircled{\#}}$, $f = b^{\textcircled{\#}}b$ and $f' = bb^{\textcircled{\#}}$. Then, $a \perp_{s,\textcircled{\#}} b$ if and only if $e' \perp f'$ and $e \perp f$.*

Proof. From Theorem 3.6, $a \perp_{\textcircled{\#}} b$ if and only if $e' \perp f'$ and $fe = 0$. And $b \perp_{\textcircled{\#}} a$ if and only if $e \perp f$ and $e'f' = 0$. Then $a \perp_{s,\textcircled{\#}} b$ if and only if $e' \perp f'$ and $e \perp f$. □

From the above theorems, it can be seen that the condition of the strong core orthogonality is stronger than that of the core orthogonality and usual orthogonality. Next, we explore the relationship between the strongly core orthogonality and various kinds of orthogonality when $a, b \in R$ are EP elements. In fact, it has been proved in Remark 3.10 that $b \perp_{\textcircled{\#}} a$, when $a, b \in R$ are EP elements and $a \perp_{\textcircled{\#}} b$. So the corollary is as follows.

Corollary 4.7. *Let $a, b \in R$. If a, b are EP elements, Then the following statements are equivalent.*
(1) $a \perp b$;
(2) $a \perp_* b$;
(3) $a \perp_{\textcircled{\#}} b$;
(4) $a \perp_{s,\textcircled{\#}} b$.

In [4], Liu, Wang and Wang pointed out that $A \perp_{s,\textcircled{\#}} B$ if and only if $BA^{\textcircled{\#}} = 0$ (or $A^{\textcircled{\#}}B = 0$) and $(A+B)^{\textcircled{\#}} = A^{\textcircled{\#}} + B^{\textcircled{\#}}$. Based on the result, we prove these theorems also hold in rings with involution and give a different way to prove these theorems as follows.

Theorem 4.8. *Let $a, b \in R^{\textcircled{\#}}$. $a \perp_{s,\textcircled{\#}} b$ if and only if $ba^{\textcircled{\#}} = 0$ and $(a+b)^{\textcircled{\#}} = a^{\textcircled{\#}} + b^{\textcircled{\#}}$.*

Proof. If: Let $ba^{\textcircled{\#}} = 0$ and $(a+b)^{\textcircled{\#}} = a^{\textcircled{\#}} + b^{\textcircled{\#}}$. Then

$$(a+b)(a+b)^{\textcircled{\#}} = (a+b)(a^{\textcircled{\#}} + b^{\textcircled{\#}}) = aa^{\textcircled{\#}} + bb^{\textcircled{\#}} + ab^{\textcircled{\#}}.$$

From $((a+b)(a+b)^{\textcircled{\#}})^* = (a+b)(a+b)^{\textcircled{\#}}$, it follows that

$$aa^{\textcircled{\#}} + bb^{\textcircled{\#}} + (ab^{\textcircled{\#}})^* = aa^{\textcircled{\#}} + bb^{\textcircled{\#}} + ab^{\textcircled{\#}},$$

which implies $(ab^{\textcircled{\#}})^* = ab^{\textcircled{\#}}$.
Since $ba^{\textcircled{\#}} = 0$, we get $ba = ba^{\textcircled{\#}}a^2 = 0$. From $(a+b)(a+b)^{\textcircled{\#}}(a+b) = a+b$, it follows that $aa^{\textcircled{\#}}b + ab^{\textcircled{\#}}a + ab^{\textcircled{\#}}b + bb^{\textcircled{\#}}a = 0$. Left multiplying by b, we obtain $bbb^{\textcircled{\#}}a = 0$, which implies $b^{\textcircled{\#}}(bbb^{\textcircled{\#}}a) = bb^{\textcircled{\#}}a = 0$. Then

$$b^{\textcircled{\#}}a = b^{\textcircled{\#}}(bb^{\textcircled{\#}}a) = 0.$$

From $(a+b)((a+b)^{\textcircled{\#}})^2 = (a+b)^{\textcircled{\#}}$, it follows that $aa^{\textcircled{\#}}b^{\textcircled{\#}} + ab^{\textcircled{\#}}b^{\textcircled{\#}} = 0$. Then

$$(aa^{\textcircled{\#}}b^{\textcircled{\#}} + ab^{\textcircled{\#}}b^{\textcircled{\#}})^* = (aa^{\textcircled{\#}}b^{\textcircled{\#}})^* + (ab^{\textcircled{\#}}b^{\textcircled{\#}})^* = (b^{\textcircled{\#}})^*aa^{\textcircled{\#}} + (b^{\textcircled{\#}})^*ab^{\textcircled{\#}},$$

which implies $(b^{\textcircled{\#}})^*aa^{\textcircled{\#}} + (b^{\textcircled{\#}})^*ab^{\textcircled{\#}} = 0$. Right multiplying by a, we obtain $(b^{\textcircled{\#}})^*a = 0$. Then $(b^{\textcircled{\#}})^*aa^{\textcircled{\#}} = (aa^{\textcircled{\#}}b^{\textcircled{\#}})^* = 0$, i.e. $aa^{\textcircled{\#}}b^{\textcircled{\#}} = 0$. Therefore, we have

$$a^{\textcircled{\#}}b = a^{\textcircled{\#}}(aa^{\textcircled{\#}}b^{\textcircled{\#}})b^2 = 0.$$

From $(a+b)^{\textcircled{\#}}(a+b)(a+b)^{\textcircled{\#}} = (a+b)^{\textcircled{\#}}$, it follows that $a^{\textcircled{\#}}ab^{\textcircled{\#}} = 0$. Then

$$ab^{\textcircled{\#}} = a(a^{\textcircled{\#}}ab^{\textcircled{\#}}) = 0.$$

Above all, we have $ba^{\textcircled{\#}} = a^{\textcircled{\#}}b = ab^{\textcircled{\#}} = 0$, i.e. $a \perp_{s,\textcircled{\#}} b$.
Only if: Let $a \perp_{s,\textcircled{\#}} b$ and $x = a^{\textcircled{\#}} + b^{\textcircled{\#}}$. Then $ba^{\textcircled{\#}} = 0$. And

$$(a+b)x(a+b) = (a+b)(a^{\textcircled{\#}} + b^{\textcircled{\#}})(a+b) = (aa^{\textcircled{\#}} + bb^{\textcircled{\#}})(a+b) = a+b.$$

In the same way, we can also get $x(a+b)x = x$, $((a+b)x)^* = (a+b)x$, $x(a+b)^2 = a+b$ and $(a+b)x^2 = x$. Then x is the core inverse of $a+b$, i.e. $(a+b)^{\textcircled{\#}} = x = a^{\textcircled{\#}} + b^{\textcircled{\#}}$. □

Theorem 4.9. *Let $a, b \in R^{\textcircled{\#}}$. $a \perp_{s,\textcircled{\#}} b$ if and only if $a^{\textcircled{\#}}b = 0$ and $(a+b)^{\textcircled{\#}} = a^{\textcircled{\#}} + b^{\textcircled{\#}}$.*

Proof. If: Let $a^{\textcircled{\#}}b = 0$ and $(a+b)^{\textcircled{\#}} = a^{\textcircled{\#}} + b^{\textcircled{\#}}$. Then, from $(a+b)^{\textcircled{\#}}(a+b)(a+b)^{\textcircled{\#}} = (a+b)^{\textcircled{\#}}$, we have $a^{\textcircled{\#}}ab^{\textcircled{\#}} + baa^{\textcircled{\#}} + b^{\textcircled{\#}}ab^{\textcircled{\#}} + b^{\textcircled{\#}}ba^{\textcircled{\#}} = 0$. Lift multiplying by $a^{\textcircled{\#}}$,we obtain $(a^{\textcircled{\#}})^2ab^{\textcircled{\#}} = 0$, which implies $a^{\textcircled{\#}}ab^{\textcircled{\#}} = a((a^{\textcircled{\#}})^2ab^{\textcircled{\#}}) = 0$. Then

$$ab^{\textcircled{\#}} = a^2(a^{\textcircled{\#}}ab^{\textcircled{\#}}) = 0.$$

And

$$(a+b)(a+b)^{\#} = (a+b)(a^{\#}+b^{\#}) = aa^{\#} + bb^{\#} + ba^{\#}.$$

From $((a+b)(a+b)^{\#})^* = (a+b)(a+b)^{\#}$, it follows that

$$aa^{\#} + bb^{\#} + (ba^{\#})^* = aa^{\#} + bb^{\#} + ba^{\#},$$

which implies $(ba^{\#})^* = ba^{\#}$.
From $(a+b)((a+b)^{\#})^2 = (a+b)^{\#}$, it follows that $b(a^{\#})^2 + bb^{\#}a^{\#} = 0$. Then

$$(b(a^{\#})^2 + bb^{\#}a^{\#})^* = (b(a^{\#})^2)^* + (bb^{\#}a^{\#})^* = (a^{\#})^*ba^{\#} + (a^{\#})^*bb^{\#},$$

which implies $(a^{\#})^*ba^{\#} + (a^{\#})^*bb^{\#} = 0$. Right multiplying by b, we obtain $(a^{\#})^*b = 0$, which implies that $(a^{\#})^*bb^{\#} = (bb^{\#}a^{\#})^* = 0$, i.e. $bb^{\#}a^{\#} = 0$. Then, we have

$$b^{\#}a = b^{\#}(bb^{\#}a^{\#})a^2 = 0.$$

From $(a+b)(a+b)^{\#}(a+b) = a+b$, it follows that $b^{\#}a^{\#}a = 0$. Then

$$ba^{\#} = (b^{\#}a^{\#}a)a^{\#} = 0.$$

Based on the above results, we have $a^{\#}b = ab^{\#} = b^{\#}a = ba^{\#} = 0$, i.e. $a \perp_{s,\#} b$.

Only if: Let $a \perp_{s,\#} b$ and $x = a^{\#} + b^{\#}$. Then $a^{\#}b = 0$. Following the proof of Theorem 4.8, we get x is the core inverse of $a+b$, i.e. $(a+b)^{\#} = x = a^{\#} + b^{\#}$. □

Remark 4.10. *We extend Theorem 3.8 in [4] in a different way and give two new equivalent conditions for the strongly core orthogonality in rings with involution, which is more comprehensive than Theorem 3.8 in [4].*

Theorem 4.11. *Let $a, b \in R^{\#}$. Then, the following statements are equivalent.*
(1) $a \perp_{s,\#} b$;
(2) $ab^{\#} = 0$ and $a^{\#}b = ba^{\#}$;
(3) $ba^{\#} = 0$ and $b^{\#}a = ab^{\#}$;
(4) $a^{\#}b = 0$ and $ab^{\#} = ba^{\#}$.

Proof. (1) ⇒ (2) It is obvious.
(2) ⇒ (3) Let $ab^{\#} = 0$. Then $ab = ab^{\#}b^2 = 0$. From $a^{\#}b = ba^{\#}$, it follows that

$$a^{\#}b = a^{\#}aa^{\#}b = a^{\#}aba^{\#} = 0.$$

Then, we have

$$bb^{\#}aa^{\#} = (bb^{\#})^*(aa^{\#})^* = (aa^{\#}bb^{\#})^* = 0,$$

which implies that $b^{\#}a = b^{\#}(bb^{\#}aa^{\#})a = 0$. Then, $ba^{\#} = 0$ and $b^{\#}a = ab^{\#}$.
(3) ⇒ (4) Let $ba^{\#} = 0$. Then $ba = ba^{\#}a^2 = 0$. From $b^{\#}a = ab^{\#}$, it follows that

$$b^{\#}a = b^{\#}bb^{\#}a = b^{\#}bab^{\#} = 0.$$

Then, we have

$$aa^{\#}bb^{\#} = (aa^{\#})^*(bb^{\#})^* = (bb^{\#}aa^{\#})^* = 0,$$

which implies that $a^{\#}b = a^{\#}(aa^{\#}bb^{\#})b = 0$. Then, $a^{\#}b = 0$ and $ab^{\#} = ba^{\#}$.
(4) ⇒ (1) Let $a^{\#}b = 0$ and $ab^{\#} = ba^{\#}$, it follows that

$$ab^{\#} = ab^{\#}bb^{\#} = ba^{\#}bb^{\#} = 0.$$

Then, $b^{\#}a = a^{\#}b = ab^{\#} = 0$. Applying Remark 4.2, we have $a \perp_{s,\#} b$.
□

Theorem 4.12. *Let $a, a^{\oplus} \in R^{\oplus}$ be two projections. Then, a is strongly core orthogonal to its complementary idempotent element $1-a$, i.e. $a \perp_{s,\oplus} 1-a$.*

Proof. Let $p = a$ and $q = a^{\oplus}$. Then $b = p - qp = a - a^{\oplus}a$. If $x = a^{\oplus} - aa^{\oplus}$, we have

$$bxb = (a - a^{\oplus}a)(a^{\oplus} - aa^{\oplus})(a - a^{\oplus}a) = (aa^{\oplus} - a^{\oplus})(a - a^{\oplus}a) = a - a^{\oplus}a = b$$

and

$$xbx = (a^{\oplus} - aa^{\oplus})(a - a^{\oplus}a)(a^{\oplus} - aa^{\oplus}) = (a^{\oplus}a - a)(a^{\oplus} - aa^{\oplus}) = a^{\oplus} - aa^{\oplus} = x.$$

Since $a, a^{\oplus} \in R^{\oplus}$ are two projections, we have $a^{\oplus} = a(a^{\oplus})^2 = aa^{\oplus}$ and $a = a^{\oplus}a^2 = a^{\oplus}a$. Then

$$bx = (a - a^{\oplus})(a^{\oplus} - aa^{\oplus}) = aa^{\oplus} - a^{\oplus} = 0$$

and

$$xb = (a^{\oplus} - aa^{\oplus})(a - a^{\oplus}) = a^{\oplus}a - a = 0.$$

We have $bx = xb$, $(bx)^* = bx$ and $(xb)^* = xb$. Then, $b = p - qp = a - a^{\oplus}a \in R^{EP}$.

From Theorem 4.1 in [20], we have $pq = qp$, i.e. $aa^{\oplus} = a^{\oplus}a$. Then

$$a^{\oplus}(1 - a^{\oplus}a) = a^{\oplus} - a^{\oplus}a^{\oplus}a = a^{\oplus} - a^{\oplus}aa^{\oplus} = 0.$$

Obviously, $(1 - a^{\oplus}a)a^{\oplus} = 0$. Then, $a \perp_{s,\oplus} 1 - a^{\oplus}a$. From Theorem 4.1 in [20], we have

$$1 - a^{\oplus}a = (1 - aa^{\oplus})^{\oplus} = 1 - aa^{\oplus}a = 1 - a.$$

Then $a \perp_{s,\oplus} 1 - a$. □

Theorem 4.13. *Let $a, b \in R^{\oplus}$. Then, the following statements are equivalent.*
(1) $a \perp_{s,\oplus} b$;
(2) $a \leq^{\oplus} a + b$, $b \leq^{\oplus} a + b$;
(3) there exist $1 = e_1 + e_2 + e_3$ which is a decomposition of the identity of the ring R, where $e_1 = p = p^2 = p^*$, $e_2 = q = q^*$ and $e_3 = 1 - p - q$, and then

$$a = \begin{bmatrix} t_1 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

where t_1 and t_2 are invertible in the ring eRe and qRq, respectively.

Proof. (1) ⇒ (2) It is obvious.
(2) ⇒ (3) By Theorem 3.13, we have that $a \leq^{\oplus} a + b$ if and only if

$$a = \begin{bmatrix} t_1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

where t_1 and t_2 are invertible in the ring eRe and qRq respectively.

Since $b \leq^{\oplus} a + b$, we have $b \perp_{\oplus} a$. From (7) in Theorem 3.4, we get $ab = 0$ and $b^*a = 0$. Then,

$$ab = \begin{bmatrix} t_1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & rt_2 & rs_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = 0,$$

which implies that $r = 0$. And $b^*a = 0$ is obvious. Then,

$$a = \begin{bmatrix} t_1 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

where t_1 and t_2 are invertible in the ring eRe and qRq, respectively.
(3) ⇒ (1) Let

$$z = \begin{bmatrix} t_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

We can check that z satisfies $zaz = z$, $aza = a$, $(az)^* = az$, $az^2 = z$ and $za^2 = a$. Then $z = a^{\#}$. Thus, we have

$$a^{\#}b = \begin{bmatrix} t_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = 0,$$

$$ba^{\#} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} t_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = 0$$

and

$$ab = \begin{bmatrix} t_1 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_2 & s_2 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = 0.$$

Then $a \perp_{s,\#} b$. □

Lemma 4.14. *Let $a, b \in R^{\#}$, and $a \leq^- b$. If $b = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p}$, then $a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p\times p}$, where $a_1 = a_1t^{-1}a_1$ and $a_2 = a_1t^{-1}s$.*

Proof. By $a \leq^- b$, we have

$$bb^{\#}a = bb^{\#}aa^{\dagger}a = bb^{\#}ba^{\dagger}a = ba^{\dagger}a = aa^{\dagger}a = a, \tag{15}$$

$$ab^{\#}b = aa^{\dagger}ab^{\#}b = aa^{\dagger}bb^{\#}b = aa^{\dagger}b = aa^{\dagger}a = a \tag{16}$$

and

$$ab^{\#}a = aa^{\dagger}ab^{\#}aa^{\dagger}a = aa^{\dagger}bb^{\#}ba^{\dagger}a = aa^{\dagger}ba^{\dagger}a = aa^{\dagger}aa^{\dagger}a = a. \tag{17}$$

Let the core-EP decomposition of a be

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{p\times p}.$$

If $b = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p}$, then

$$b^{\#} = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}$$

and

$$bb^{\#}a = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{p\times p} = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

From (15), we have

$$\begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{p\times p},$$

which implies $a_3 = a_4 = 0$. Then

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

And it follows from (16) that

$$ab^{\#}b = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} a_1 & a_1t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p\times p},$$

which implies $a_2 = a_1t^{-1}s$. Furthermore, from (17), we get

$$\begin{aligned} ab^{\#}a &= \begin{bmatrix} a_1 & a_1t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} a_1 & a_1t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p} \\ &= \begin{bmatrix} a_1t^{-1}a_1 & a_1t^{-1}a_1t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p} \\ &= \begin{bmatrix} a_1 & a_1t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p}, \end{aligned}$$

which implies $a_1 = a_1t^{-1}a_1$. Then

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p\times p},$$

where $a_1 = a_1t^{-1}a_1$ and $a_2 = a_1t^{-1}s$. □

Theorem 4.15. *Let $a, b \in R^{\#}$. Then, the following statements are equivalent:*
(1) $a \perp_{s,\#} b$;
(2) $a \leq^{-} a + b$ and $(a + b)^{\#} = a^{\#} + b^{\#}$.

Proof. (1) ⇒ (2) It follows from Theorem 4.9 and Theorem 4.13.
(2) ⇒ (1) Let the core-EP decomposition of $a + b$ be

$$a + b = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

From $a \leq^{-} a + b$ and Lemma 4.14, we get

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p\times p},$$

where $a_1 = a_1 t^{-1} a_1$ and $a_2 = a_1 t^{-1} s$. Then

$$b = \begin{bmatrix} t - a_1 & s - a_2 \\ 0 & 0 \end{bmatrix}_{p \times p}, b^{\circledS} = \begin{bmatrix} (t - a_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p}.$$

Furthermore, let

$$a_1 = \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p \times p}$$

be the core-EP decomposition of a_1. Then

$$a_1^{\circledS} = \begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p}.$$

And let the core-EP decomposition of t^{-1} be

$$t^{-1} = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}_{p \times p}.$$

By $a_1 = a_1 t^{-1} a_1$, we have

$$\begin{aligned} a_1 t^{-1} a_1 &= \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p \times p} \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}_{p \times p} \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p \times p} \\ &= \begin{bmatrix} t_1 f_1 t_1 + s_1 f_3 t_1 & t_1 f_1 s_1 + s_1 f_3 s_1 \\ 0 & 0 \end{bmatrix}_{p \times p} = \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p \times p}, \end{aligned}$$

which implies $t_1 f_1 t_1 + s_1 f_3 t_1 = t_1$. Then

$$t_1 f_1 + s_1 f_3 = 1. \tag{18}$$

By $(a + b)^{\circledS} = a^{\circledS} + b^{\circledS}$, we get

$$\begin{bmatrix} a_1^{\circledS} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p} + \begin{bmatrix} (t - a_1)^{\circledS} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p} = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p},$$

which implies

$$(t - a_1)^{\circledS} = t^{-1} - a_1^{\circledS}. \tag{19}$$

Write

$$x := t_1 f_2 + s_1 f_4. \tag{20}$$

By $(t - a_1)(t - a_1)^{\circledS}(t - a_1) = t - a_1$, we have

$$\begin{aligned} (t - a_1)(t - a_1)^{-1}(t - a_1) &= (t - a_1)(t^{-1} - a_1^{-1})(t - a_1) \\ &= t - 2a_1 - t a_1^{\circledS} t + t a_1^{\circledS} a_1 + a_1 a_1^{\circledS} t \\ &= t - a_1. \end{aligned}$$

Then

$$-a_1 - t a_1^{\circledS} t + t a_1^{\circledS} a_1 + a_1 a_1^{\circledS} t = 0.$$

Pre-multiplying the above equation and post-multiplying the above equation by t^{-1}, we have

$$-t^{-1}a_1t^{-1} - a_1^{\#} + a_1^{\#}a_1t^{-1} + t^{-1}a_1a_1^{\#} = 0,$$

i.e.

$$\begin{bmatrix} 0 & t_1^{-1}x - f_1x \\ 0 & f_3x \end{bmatrix}_{p\times p} = 0.$$

Then

$$t_1^{-1}x = f_1x, f_3x = 0. \tag{21}$$

By $(t-a_1)((t-a_1)^{\#})^2 = (t-a_1)^{\#}$, we have

$$-a_1 - ta_1^{\#}t^{-1} + t(a_1^{\#})^2 - a_1(t^{-1})^2 + a_1t^{-1}a_1^{\#} - a_1a_1^{\#}t^{-1} = 0.$$

Pre-multiplying the above equation by t^{-1}, we have

$$-t^{-1}a_1 - a_1^{\#}t^{-1} + (a_1^{\#})^2 - t^{-1}a_1(t^{-1})^2 + t^{-1}a_1t^{-1}a_1^{\#} - t^{-1}a_1a_1^{\#}t^{-1} = 0,$$

i.e.

$$\begin{bmatrix} -t_1^{-1}f_1 - f_1xf_3 + (t^{-1})^2 & -t_1^{-1}f_2 - f_1xf_4 \\ 0 & 0 \end{bmatrix}_{p\times p} = 0.$$

Then

$$-t_1^{-1}f_1 - f_1xf_3 + (t^{-1})^2 = 0, -t_1^{-1}f_2 - f_1xf_4 = 0.$$

From (21), we have

$$-t_1^{-1}f_1 - f_1xf_3 + (t^{-1})^2 = -t_1^{-1}f_1 - t_1^{-1}xf_3 + (t^{-1})^2 = t_1^{-1}(-f_1 - xf_3 + t^{-1}),$$

which implies

$$-f_1 - xf_3 + t^{-1} = 0.$$

Then

$$f_1 = t^{-1} - xf_3, t_1^{-1}f_2 = f_1xf_4. \tag{22}$$

Applying $((t-a_1)(t-a_1)^{\#})^* = (t-a_1)(t-a_1)^{\#}$, we have

$$(ta_1^{\#} + a_1t^{-1})^* = ta_1^{\#} + a_1t^{-1}. \tag{23}$$

Since

$$ta_1^{\#} + a_1t^{-1} = t(a_1^{\#} + t^{-1}a_1t^{-1})(t^{-1})^*t^*,$$

we get $((a_1^{\#} + t^{-1}a_1t^{-1})(t^{-1})^*)^* = (a_1^{\#} + t^{-1}a_1t^{-1})(t^{-1})^*$ by (23). Then,

$$(a_1^{\#} + t^{-1}a_1t^{-1})(t^{-1})^* = \begin{bmatrix} t_1^{-1}f_1^* + f_1f_1^* + f_1xf_2^* & t_1^{-1}f_3^* + f_1f_3^* + f_1xf_4^* \\ f_3f_1^* & f_3f_3^* \end{bmatrix}_{p\times p} = 0,$$

which implies

$$t_1^{-1}f_3^* + f_1f_3^* + f_1xf_4^* = (f_3f_1^*)^*.$$

So,

$$f_3 = -f_4 x^* f_1^* t_1^*. \tag{24}$$

From (21) and (22), we have

$$-f_3 x = f_4 x^* f_1^* t_1^* x = f_4 x^* (t^{-1} - x f_3)^* t_1^* x = f_4 x^* x = 0.$$

When $f_4 \neq 0$, it implies $x = 0$. Then from (18) and (20), we get

$$\begin{aligned} a_1^{\circledast}(t - a_1) &= (a_1^{\circledast} - a_1^{\circledast} a_1 t^{-1})t \\ &= (\begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} - \begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}_{p\times p})t \\ &= (\begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} - \begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t_1 f_1 + s_1 f_3 & t_1 f_2 + s_1 f_4 \\ 0 & 0 \end{bmatrix}_{p\times p})t \\ &= (\begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} - \begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times p})t \\ &= 0. \end{aligned}$$

When $f_4 = 0$, it implies $f_2 = f_3 = 0$ and $f_1 = t^{-1}$ by (21), (22) and (24). And from (18), we have $t_1 t^{-1} = 1$. Then

$$\begin{aligned} a_1^{\circledast}(t - a_1) &= (a_1^{\circledast} - a_1^{\circledast} a_1 t^{-1})t \\ &= (\begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} - \begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t_1 & s_1 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p})t \\ &= (\begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} - \begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t_1 t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p})t \\ &= (\begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} - \begin{bmatrix} t_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{p\times p})t \\ &= 0. \end{aligned}$$

Therefore, by Lemma 4.14, we get

$$\begin{aligned} a^{\circledast} b &= \begin{bmatrix} a^{\circledast}(t - a_1) & a^{\circledast}(s - a_2) \\ 0 & 0 \end{bmatrix}_{p\times p} \\ &= \begin{bmatrix} a^{\circledast}(t - a_1) & a^{\circledast}(t - a_1)t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p} \\ &= 0. \end{aligned}$$

From Theorem 4.9, we have $a \perp_{s,\circledast} b$. □

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