Filomat 38:25 (2024), 8701–8713
https://doi.org/10.2298/FIL2425701Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the class of unbounded-U-Dunford-Pettis operators

Larbi Zraoula[a,*], Ouba Issam[b], Jawad H'michane[c], Bouazza El Wahbi[d]

[a]*Departement of Mathematics, EREAM Team, LaREAMI-Lab, CRMEF-Kénitra, "Moroccan Association of Ordered Structures, Operators Theory, Applications and Sustainable Development (MAOSOTA)", Morocco*
[b]*Faculty of Sciences, Math department, LAGA Lab, Ibn Tofail University, Kenitra, Morocco*
[c]*Engineering Sciences Lab, ENSA, B.P 241, Ibn Tofail University, Kénitra, "Moroccan Association of Ordered Structures, Operators Theory, Applications and Sustainable Development (MAOSOTA)", Morocco*
[d]*Faculty of Sciences, Math department, LAGA Lab, Ibn Tofail University, Kenitra, Morocco*

Abstract. The aim of this paper is to introduce and study a new class of operators between Banach lattices based on the unbounded norm topology, nominated "unbounded-U-Dunford-Pettis operators". Namely, we give some characterizations of this class, we study the relations between this class and other classes of operators known in the literature of operator theory on Banach lattices, and we study the duality problem for this class of operators.

1. Introduction

In [10], based on the unbounded norm topology the authors introduce and study the class of σ-un-Dunford-Pettis operators which generalize that of Dunford-Pettis. Namely, an operator $T : X \longrightarrow F$ from a Banach space X to a Banach lattice F is said to be σ-un-Dunford-Pettis if T sends weakly null sequences of X to unbounded norm null sequences of F. We recall that an operator $T : X \longrightarrow Y$ between two Banach spaces is said to be Dunford-Pettis if $T(x_n) \xrightarrow{\|\cdot\|} 0$ whenever $x_n \xrightarrow{w} 0$. Following the same approach, we introduce and study a new class of operators which generalize U-Dunford-Pettis operators. To be specific, an operator $T : E \longrightarrow F$ between two Banach lattices is said to be un-U-Dunford-Pettis if $T(x_n) \xrightarrow{un} 0$ for every order bounded weakly null sequence of E. We recall from [1] that an operator $T : E \longrightarrow Y$ from a Banach lattice E to a Banach space Y is said to be U-Dunford-Pettis if $T(x_n) \xrightarrow{\|\cdot\|} 0$ for every order bounded weakly null sequence (x_n) of E.

This paper will be organized as follows, after a preliminary section in which the definitions, the concepts and the notations used throughout this work are presented. We introduce the class of un-U-Dunford-Pettis (Definition 3.1),we present some characterizations of this class via lattices properties (Proposition 3.2 and 3.3) and we give the characterization of un-U-Dunford-Pettis operators in terms of sets (Theorem 3.7), after that we study the ideal of un-U-Dunford-Pettis operators (Proposition 3.5). In the third section, we

2020 *Mathematics Subject Classification*. Primary 46B42; Secondary 47B60, 47B65.
Keywords. Order continuous Banach lattice, atomic Banach lattice, order unit, U-Dunford-Pettis operator, un-U-Dunford-Pettis operator
Received: 08 December 2023; Accepted: 03 May 2024
Communicated by Dragan S. Djordjević
* Corresponding author: L. Zraoula
Email addresses: zraoularbi@yahoo.fr (Larbi Zraoula), contact.issam.ouba@gmail.com (Ouba Issam), jawad.hmichane@uit.ac.ma (Jawad H'michane), bouazza.elwahbi@uit.ac.ma (Bouazza El Wahbi)

present some results about the relations between the class of un-U-Dunford-Pettis operators and other classes of operators (Theorem 3.12, Theorem 3.17, Theorem 3.18, Theorem 3.19, Theorem 3.21). The last section of this work is devoted to study the duality property of un-U-Dunford-Pettis operators (Theorem 3.24 and Theorem 3.27). As consequences, we fined new characterizations of order continuous Banach lattices (Corollary 3.25 and 3.26) and of σ-Dedekind complete Banach lattices (Corollary 3.28).

2. Preliminaries

We recall from [9] that a net (x_α) in a Banach lattice E is unbounded norm convergent to x (abb. un-norm convergent) if $\||x_\alpha - x| \wedge u\| \longrightarrow 0$ (abb. $x_\alpha \xrightarrow{un} x$), for each $u \in E^+$. We note that the norm convergence implies the un-norm convergence and that un-norm convergence coincides with norm convergence on a Banach lattice with order unit. A net (x_α) is unbounded absolutely weakly convergent (abb. uaw-convergent) to x if $(|x_\alpha - x| \wedge u)$ converges weakly to zero for every $u \in E^+$; we write $x_\alpha \xrightarrow{uaw} x$ ([16]).

To state our results, we need to fix some notations and recall some definitions. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x; y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A vector lattice E is σ-Dedekind complete if every majorized countable nonempty subset of E has a supremum. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A Banach lattice E is order continuous, if for each net (x_α) such that $x_\alpha \downarrow 0$ in E, the net (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the net (x_α) is decreasing and $\inf(x_\alpha) = 0$. Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. A Banach lattice E is said to be KB-space, if every increasing norm bounded sequence of E^+ is norm convergent. The lattice operations in E are weakly sequentially continuous, if the sequence $(|x_n|)$ converges to 0 in the weak topology, whenever the sequence (x_n) converges weakly to 0 in E. A positive element e in a vector lattice E is called an order unit of E if $E_e = E$, where $E_e = \{x \in E;\ \exists\lambda > 0$ such that $|x| \leq \lambda e\}$ is the order ideal generated by e. A nonzero element x of a vector lattice E is discrete (atom) if the order ideal generated by x equals the vector subspace generated by x. The vector lattice E is discrete (atomic), if it admits a complete disjoint system of discrete elements. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $inf(x, y) = 0$, we have $\|x + y\| = max\{\|x\|, \|y\|\}$.

We will use the term operator to mean a bounded linear mapping. A linear mapping T from a vector lattice E into another F is order bounded if it carries order bounded set of E into order bounded set of F. A linear mapping between two vector lattices E and F is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ between two Banach lattices is positive then, its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

3. Main Results

3.1. *Space of un-U-Dunford-Pettis operators*

Definition 3.1. *An operator T between two Banach lattices E and F is said to be unbounded U-Dunford-Pettis (un-U-DP, for short) if $T(x_n) \xrightarrow{un} 0$ for every order bounded weakly null sequence (x_n) of E.*

The collection of all unbounded U-Dunford-Pettis operators of $L(E, F)$ will be denoted by $unDP_u(E, F)$. That is,

$$unDP_u(E, F) = \{T \in L(E, F) : \text{ T is un-U-Dunford-Pettis}\}$$

and we note by

$$DP_u(E, F) = \{T \in L(E, F) : \text{T is U-Dunford-Pettis}\}$$

We note by:

$$\sigma - unDP(E, F) = \{T \in L(E, F)\ : T\ is\ \ \sigma\text{-un-Dunford-Pettis operator }\}$$

We note that a un-U-Dunford-Pettis is not in general σ-un-Dunford-Pettis. Indeed, the canonical injection $i : c_0 \to \ell^\infty$ is a un-U-Dunford-Pettis operator, but it is not σ-un-Dunford-Pettis. If E has an order unit, then every un-U-Dunford-Pettis T defined on E is σ-un-Dunford-Pettis.

In the following result, we give some characterizations of un-U-DP operators.

Proposition 3.2. *Let F be a Banach lattice. The following statements are equivalent:*

1. *Every operator from E into F is un-U-DP, for every Banach lattice E.*
2. *The identity operator of the Banach lattice F is un-U-DP.*
3. *F is atomic and order continuous.*
4. *Every operator from E into F is σ-un-DP, for every Banach lattice E.*

Proof. 1) $\Rightarrow$ 2) Obvious.
2) $\Rightarrow$ 3) Let (x_n) be an order bounded sequence in F such that $x_n \xrightarrow{w} 0$. Since Id_F is un-U-DP, then $x_n \xrightarrow{un} 0$, and hence (x_n) converge in norm to 0. So, it follows from [3, Corollary 2.3] that F is an atomic order continuous Banach lattice.
3) $\Rightarrow$ 4) Let (x_n) be a weakly null sequence of E, we have $T(x_n) \xrightarrow{w} 0$. Since F is an atomic order continuous Banach lattice, then by [11, Proposition 4.16] we infer that $T(x_n) \xrightarrow{un} 0$. So T is σ-un-DP operator.
4) $\Rightarrow$ 1) Obvious.
□

We recall from [12] that an operator T from a Banach lattice E into a Banach space Y is said to be AM-compact if $T(A)$ is relatively compact for every order bounded set A of E.

An other characterizations of un-U-DP operators are given in the following proposition.

Proposition 3.3. *Let E be a Banach lattice. The following statements are equivalent:*

1. *Every operator from E into F is un-U-DP, for every Banach lattice F.*
2. *The identity operator of the Banach lattice E is un-U-DP.*
3. *E is atomic and order continuous.*
4. *Every operator from E into F is AM-compact, for every Banach lattice F.*
5. *Every operator from E into F is U-Dunford-Pettis, for every Banach lattice F.*

Proof. The implications 1) $\Rightarrow$ 2), 4) $\Rightarrow$ 5) and 5) $\Rightarrow$ 1) are obvious.
2) $\Rightarrow$ 3) It is similar to the implication 2) $\Rightarrow$ 3) of the above proposition.
3) $\Rightarrow$ 4) Let E be an atomic order continuous Banach lattice, then it follows from [15, Theorem 6.1] that $[-x, x]$ is compact for each $x \in E^+$, and hence $T([-x, x])$ is relatively compact, that is T is AM-comapct. □

Remark 3.4. *The product of two un-U-Dunford-Pettis operators is not in general un-U-Dunford-Pettis. Indeed, let E and F be two Banach lattices such that E and F are not order continuous and F is σ-Dedekind complete. We will construct two un-U-Dunfrord-Pettis operators S and R such that the product $T = R \circ S$ is not un-U-Dunford-Pettis. Since F is not order continuous, then by [12, Theorem 2.4.2] there exists a positive order bounded disjoint sequence (y_n) of F such that $\|y_n\| = 1$ and $y_n \leq y$ for all n and some $y \in F^+$; and since E is not order continuous then it follows from [12, Theorem 2.4.2] and [6, Lemma 3.4] that there exist a positive order bounded disjoint sequence (x_n) of E satisfying $\|x_n\| = 1$ for all n and a positive disjoint sequence $(g_n) \in E'$ with $\|g_n\| \leq 1$, $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for every $n \neq m$. On the other hand, since (x_n) is an order bounded disjoint sequence then we infer that $x_n \xrightarrow{w} 0$.*
Now, we consider the following operators,

$$\begin{array}{rcl} S : E & \to & c_0 \\ x & \mapsto & (g_n(x)) \end{array} \qquad \begin{array}{rcl} R : c_0 & \to & F \\ (\lambda_n)_n & \mapsto & \sum_{n=1}^{\infty} \lambda_n y_n. \end{array}$$

We note that S and R are two well defined operators which are un-U-Dunford-Pettis (because c_0 is an atomic order continuous Banach lattice), but the operator $T = R \circ S$ is not un-U-Dunford-Pettis. Indeed, we have

$$\begin{aligned} \||T(x_n)| \wedge y\| &= \||R \circ S(x_n)| \wedge y\| \\ &= \||R(e_n)| \wedge y\| \\ &= \||y_n| \wedge y\| \\ &= \|y_n\| = 1, \end{aligned}$$

where (e_n) is the unit basis of c_0.

Recall from [8] that an operator T between two Banach lattices E and F is said to be unbounded norm continuous (un-continuous, for short) whenever $x_\alpha \xrightarrow{un} 0$ implies $T(x_\alpha) \xrightarrow{un} 0$, for each norm bounded net $(x_\alpha) \subset E$. If $T(x_n) \xrightarrow{un} 0$ for every norm bounded sequence $(x_n) \subset E$ such that $x_n \xrightarrow{un} 0$, then T is called σ-unbounded norm continuous operator (σ-un-continuous, for short).

Proposition 3.5. *Let E, F and G be Banach lattices. We have the following assertions:*

1. *If $R : F \longrightarrow G$ is a un-U-Dunford-Pettis operator and F has an order unit, then the product operator $R \circ S$ is un-U-Dunford-Pettis for every operator $S : E \longrightarrow F$.*
2. *If $S : E \longrightarrow F$ is a un-U-Dunford-Pettis operator and $R : F \longrightarrow G$ is a σ-un-continuous operator, then the product operator $R \circ S$ is un-U-Dunford-Pettis.*
3. *If $S : E \longrightarrow F$ is an order bounded operator and $R : F \longrightarrow G$ is a un-U-Dunford-Pettis operator, then the product operator $R \circ S$ is un-U-Dunford-Pettis.*

Proof.
1. Let (x_n) be an order bounded weakly null sequence in E. We have $S(x_n) \xrightarrow{w} 0$ in F. Since F has order unit, then $(S(x_n))$ is an order bounded sequence in F. As R is a un-U-Dunford-Pettis operator, then $R \circ S(x_n) \xrightarrow{un} 0$ in G, which means that $R \circ S$ is a un-U-Dunford-Pettis operator.
2. Let (x_n) be an order bounded weakly null sequence in E. Since S is un-U-Dunford-Pettis, then $S(x_n) \xrightarrow{un} 0$ in F, and since R is σ-un-continuous then $R \circ S(x_n) \xrightarrow{un} 0$, that is $R \circ S$ is a un-U-Dunford-Pettis operator.
3. Let (x_n) be an order bounded weakly null sequence in E. Since S is an order bouned operator then $(S(x_n))$ is an order bounded weakly null sequence in F. As R is a un-U-Dunford-Pettis operator, then $R \circ S$ is a un-U-Dunford-Pettis operator.

□

Remark 3.6.
1. *The condition $T : F \longrightarrow G$ is a σ-un-continuous operator is essential in the seconde assertion of the Proposition 3.5. Indeed, let E be a Banach lattice such that E is not order continuous. So, we consider the un-U-Dunford-Pettis operator S defined in the Remark 3.4,*

$$\begin{array}{rcl} S : E & \to & c_0 \\ x & \mapsto & (g_n(x)) \end{array} ;$$

and we consider the operator $T : E \xrightarrow{S} c_0 \xrightarrow{i} \ell^\infty$, where i is the canonical injection from c_0 to ℓ^∞. We note that i is not σ-un-continuous. In fact, let (e_n) be the standard basis of c_0; we have $e_n \xrightarrow{uaw} 0$, and since c_0 is order continuous then it follow from [16, Theorem 4] that $e_n \xrightarrow{un} 0$ in c_0, but $i(e_n) = e_n$ is not un-convergent in ℓ^∞. Now, for the sequence (x_n) used in the construction of the operator S which is an order bounded weakly null sequence, we have $T(x_n) = e_n$. Hence, the operator T is not un-U-Dunford-Pettis.
2. *The condition $S : E \longrightarrow F$ is an order bounded operator is essential in the third assertion of the Proposition 3.5. Indeed, let E be an infinite dimensional Banach lattice with order unit. We consider the same operator S mentioned in the above assertion. We have S is not order weakly compact, hence S is a non-weakly compact operator (because E has order unit). On the other hand, S is not order bounded. In fact, if S is order bounded*

then $S(B_E)$ will be order bounded in c_0, where B_E is the closed unit ball of E. As c_0 is order continuous, it follows from [2, Theorem 4.9] that $S(B_E)$ is weakly compact, and then S is a weakly compact operator, which is a contradiction.
Now, we consider the following composed operator

$$T : E \xrightarrow{S} c_0 \xrightarrow{i} \ell^\infty.$$

The canonical injection i is un-U-Dunford-Pettis (because c_0 is an atomic order continuous Banach lattice), but T is not un-U-Dunford-Pettis.

We recall from [11] that a subset A of a Banach lattice E is said to be un-compact (respectively, sequentially un-compact), if every net (x_α) (respectively, every sequence (x_n)) in A has a subnet (respectively, subsequence) which is un-norm convergent.

Theorem 3.7. *Let T be an operator from a Banach lattice E into a Banach lattice F. Then T is un-U-Dunford-Pettis if and only if T sends order bounded weakly relatively compact subset of E to relatively sequentially un-compact subset of F.*

Proof. $\Longrightarrow$) Let A be an order bounded weakly relatively compact subset of E and let (x_n) be a sequence of A, then (x_n) has a convergent subsequence (x_{n_k}) to an element $x \in E$, on the other hand (x_{n_k}) is order bounded. Since T is un-U-Dunford-Pettis, then $(T(x_{n_k}))$ is un-convergent to $T(x)$ and hence T is relatively sequentially un-compact subset of F.
$\Longleftarrow$) Let (x_n) be an order bounded weakly null sequence of E. The set $A = \{(x_n), n \in \mathbb{N}\} \bigcup \{0\}$ is an order bounded weakly relatively compact subset of E, then $T(A)$ is a relatively sequentially un-compact subset of F and hence there exists $T(x_{n_k})$ a subsequence of $(T(x_n))$ is un-convergent to $y \in F$; the uniqueness of the un-limit implies that $y = 0$, so $T(x_{n_k}) \xrightarrow{un} 0$. This implies that $T(x_n) \xrightarrow{un} 0$ and shows that T is un-U-Dunford-Pettis. □

We recall from [14] that an operator $T : E \longrightarrow F$ in said to be AM-unbounded norm-compact (AM-un-compact, for short) whenever $T[0, x]$ is an un-compact subset of F for each $x \in E^+$.
We note by:

$$unAM_c(E, F) = \{T \in L(E, F) \ : T \text{ is } AM - un - compact \text{ operator } \}$$

Proposition 3.8.
1. *If $S : E \longrightarrow F$ is an interval preserving operator and $R : F \longrightarrow G$ is an AM-un-comacpt operator, then the product operator $R \circ S$ is un-U-Dunford-Pettis.*
2. *If $S : E \longrightarrow F$ is a un-U-Dunford-Pettis operator and $R : F \longrightarrow G$ is an M-weakly compact operator, then the product operator $R \circ S$ is U-Dunford-Pettis.*

Proof.
1. Let A be an order bounded weakly compact set in E, then there exist $x \in E^+$ such that $A \subset [-x, x]$, as S is an interval preserving operator then $S[-x, x] = [S(-x), S(x)]$ which implies that $R \circ S(A) \subset R([S(-x), S(x)])$. Since R is AM-un-compact, then $R \circ S([-x, x])$ is relatively un-comapct, and hence $R \circ S$ in un-U-Dunford-Pettis.
2. Let A be an order bounded weakly compact set in E and S is a un-U-Dunford-Pettis operator, then $S(A)$ is relatively un-comapct, and since R is M-weakly compact then it follows from [10, Proposition 3.8] that $R \circ S(A)$ is relatively compact which means that $R \circ S$ is U-Dunford-Pettis operator.
□

Theorem 3.9. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. If every positive operator T from E into F is un-U-Dunford-Pettis, then E or F is order continuous.*

Proof. Assume that neither E nor F is not order continuous, then by [12, Theorem 2.4.2] there exists a disjoint order bounded sequence (x_n) of E^+ which is not norm null, and it follows from [6, Lemma 3.4] that there exists a bounded positive disjoint sequence (g_n) in E' such that $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for all

$m \neq n$. So, we can define the following positive operator R from E into ℓ^∞ by $R(x) = (g_n(x))_n$. On the other hand, since F is not order continuous then by [12, Theorem 2.4.2] there exists a sequence (y_n) of F with $\|y_n\| = 1$ and $0 \leq y_n \leq y$ for some $y \in F^+$. Since F is σ-Dedekind complete, then we can consider the positive operator $S : \ell^\infty \longrightarrow F$ defined by $S(\lambda_n) = \sum_{n=1}^{\infty} \lambda_n y_n$.
Now, we consider the positive operator $T = S \circ R$. We note that (x_n) is a weakly null sequence, and we have

$$\begin{aligned} \||T(x_n)| \wedge y\| &= \|y_n \wedge y\| \\ &= \|y_n\| = 1 \end{aligned}$$

so that $T(x_n)$ is not un-null. That is T is not un-U-Dunford Pettis. $\square$

As a consequence of the above Theorem, we have the following result.

Corollary 3.10. *Let E and F be two Banach lattices such that F is σ-Dedekind complete and E has o-un-UDPP (see the definition 3.13). Then, the following statements are equivalent.*

1. *Every positive operator T from E into F is un-U-Dunford-Pettis.*
2. *E or F is order continuous.*

3.2. *Relationship between un-U-Dunford-Pettis and other operators*

We recall from [2] that an operator $T : E \longrightarrow X$ defined from a Banach lattice to a Banach space is said to be order weakly compact whenever $T[0, x]$ is a relatively weakly compact subset of X for each $x \in E^+$, equivalently, $\|T(x_n)\| \longrightarrow 0$ for each order bounded sequence (x_n) of E^+. We note by

$$L_{owc}(E, F) = \{T \in L(E, F) \ : T \text{ is order weakly compact } \}.$$

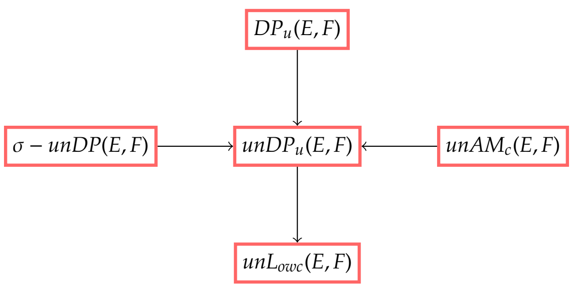
First, we consider the following class of operators.

Definition 3.11. *An operator $T : E \longrightarrow F$ between two Banach lattices E and F is un-order weakly compact if $T(x_n) \xrightarrow{un} 0$ for every order bounded weakly null sequence (x_n) of E^+.*

We note by:

$$unL_{owc}(E, F) = \{T \in L(E, F) \ : T \text{ is un-order weakly compact } \}$$

It is clear that every order weakly compact operator is un-order weakly compact, but the converse is not true in general. In fact, in [10, Remark 3.4] we have T is un-order weakly compact but it not o-weakly compact. On the other hand, every un-U-Dunford-Pettis operator is un-order weakly compact, but the converse is not true in general. In fact, $Id_{L_2[0,1]}$ is un-order weakly compact but it is not un-U-Dunford-Pettis.
We propose the following schema.



Now we will focus on the reciprocal inclusions of the above schema.

We note that a un-U-Dunford-Pettis is not in general U-Dunford-Pettis. In fact, every operator $T : \ell^\infty \longrightarrow c_0$ is un-U-Dunford-Pettis (Proposition 3.2), but it follows from the Proposition 3.3 that there exists an operator $T : \ell^\infty \longrightarrow c_0$ which is not U-Dunford-Pettis.
In the following results, we give sufficient conditions under which un-U-Dunford-Pettis operators are U-Dunford-Pettis.

Theorem 3.12. *Let $T : E \longrightarrow F$ be an operator between two Banach lattices. Then, every un-U-Dunford-Pettis operator T from E into F is U-Dunford–Pettis if one of the following assertions is valid:*

1. *F has an order unit.*
2. *T is an order bounded operator.*

Proof. 1. Let (x_n) be an order bounded sequence in E such that $x_n \xrightarrow{w} 0$. As T is un-U-Dunford-Pettis, then $T(x_n) \xrightarrow{un} 0$, and since F has an order unit then by [11, Theorem 2.3] we infer that $Tx_n \xrightarrow{\|\cdot\|} 0$, as desired.

2. Let (x_n) be an order bounded sequence in E such that $x_n \xrightarrow{w} 0$. As T is un-U-Dunford-Pettis then $T(x_n) \xrightarrow{un} 0$, and since T is order bounded then it follows from [9, Proposition 2.4] $Tx_n \xrightarrow{\|\cdot\|} 0$.

□

We note that a weakly compact (resp, order weakly compact) operator is not necessarily un-U-Dunford-Pettis and conversely a un-U-Dunford-Pettis operator is not weakly compact (resp, order weakly compact) in general. Indeed, the identity operator of $L_2[0,1]$ is weakly compact but it is not un-U-Dunford-Pettis, and the identity operator of c_0 is un-U-Dunford-Pettis but it is not weakly compact. On the other hand, there exists a un-U-Dunford-Pettis operator which is not order weakly compact (see assertion (2) of Remark 3.6). We recall from [1] that a Banach lattice E is said to have order U-Dunford-Pettis property (abb; o-UDPP), if each order weakly compact operator T from E into a Banach space Y is U-Dunford-Pettis. On the other hand, a Banach lattice E is said to have U-Dunford-Pettis property (abb; UDPP), if each weakly compact operator T from E into a Banach space Y is U-Dunford-Pettis. A Banach lattice E is said to have the reciprocal U-Dunford-Pettis property (abb; RUDPP), if each U-Dunford-Pettis operator T from E into a Banach space Y is weakly compact.

Now, we present a generalization of the order U-Dunford-Pettis property, by introducing a new property using the unbounded norm convergence in Banach lattices.

Definition 3.13.
1. *A Banach lattice E is said to have the order un-U-Dunford-Pettis property (abb. o-un-UDPP), if each order weakly compact operator T from E into F is un-U-Dunford-Pettis for each Banach lattice F.*
2. *A Banach lattice E is said to have the reciprocal order un-U-Dunford-Pettis property (abb. R-o-un-UDPP), if each un-U-Dunford-Pettis operator T from E into F is order weakly compact for each Banach lattice F.*
3. *A Banach lattice E is said to have the un-U-Dunford-Pettis property (abb. un-UDPP), if each weakly compact operator T from E into F is un-U-Dunford-Pettis for each Banach lattice F.*
4. *A Banach lattice E is said to have the reciprocal un-U-Dunford-Pettis property (abb. R-un-UDPP), if each un-U-Dunford-Pettis operator T from E into F is weakly compact for each Banach lattice F.*

The spaces ℓ^1, ℓ^2, c_0 have o-un-UDPP, but the spaces $L^1[0,1]$ and $L^2[0,1]$ does not have the o-un-UDPP. The space ℓ^2 has un-UDPP, but $L^2[0,1]$ does not have the un-UDPP. On the other hand, c_0 has the R-un-UDPP.

As a consequence of the above theorem, we can check the following corollaries:

Corollary 3.14. *Let $T : E \longrightarrow F$ be an operator from a Banach lattice E with o-un-UDPP into a Banach lattice F. If T is an order bounded operator, then the following statements are equivalent:*

1. *T is un-U-Dunford-Pettis.*
2. *T is U-Dunford-Pettis.*
3. *T is order weakly compact.*

Corollary 3.15.
1. *If E is order continuous and E has the o-un-UDPP, then each operator T from E into a Banach lattice F is un-U-Dunford-Pettis .*
2. *If E is order continuous, then the following statements are equivalent:*
   (a) *E is atomic.*
   (b) *The lattice operations of E are weakly sequentially continuous.*

(c) *E has the o-UDPP.*
(d) *E has the o-un-UDPP.*

Corollary 3.16. 1. *If each un-U-Dunford-Pettis operator $T : E \longrightarrow F$ is U-Dunford-Pettis then E has the R-o-un-UDPP.*

2. *If a Banach lattice E has the R-un-UDPP then E′ is order continuous.*

Proof. 1. Obvious.

2. Let E be a Banach lattice with the R-un-UDPP and $T : E \longrightarrow \ell^1$ a positive operator. It follows from the Proposition 3.2 that T is a un-U-DP operator, and by our hypothesis T is weakly compact. So, it follows from [2, Theorem 5.29] that E' is order continuous.
□

We recall from ([13]) that an operator $T : E \longrightarrow F$ is said σ-u-Dunford-Pettis (abb. σ-u-DP) if for every sequence (x_n) in E, $x_n \xrightarrow{uaw} 0$ implies $T(x_n) \xrightarrow{un} 0$.
We note by:

$$uDP(E, F) = \{T \in L(E, F) : T \text{ is } \sigma\text{-u-Dunford-Pettis}\}.$$

A σ-u-Dunford-Pettis operator is not in general un-U-Dunford-Pettis. In fact, the identity operators $id_{L^2[0,1]}$ of the Banach lattice $L^2[0, 1]$ is σ-u-DP ([16, Theorem 4]), but it is not un-U-DP (see Proposition 3.3).

In the following result, we give a sufficient condition under which σ-u-DP operators are un-U-Dunford-Pettis.

Theorem 3.17. *Let E and F be two Banach lattices such that the lattice operations of E are weakly sequentially continuous. Then, every σ-u-DP operator T from E into F is un-U-Dunford-Pettis.*

Proof. Let (x_n) be an order bounded sequence in E such that $x_n \xrightarrow{w} 0$. Since the lattice operations of E are weakly sequentially continuous, then $|x_n| \xrightarrow{w} 0$ implies $x_n \xrightarrow{uaw} 0$. As T is σ-u-DP, then $T(x_n) \xrightarrow{un} 0$, so T is a un-U-DP operator. □

The converse is not true in generale. Indeed, let T be an operator defined from ℓ^1 into $\mathbb{R}$ by $T((x_n)) = \sum_{i=1}^{\infty} x_n$ for every $(x_n) \in \ell^1$. It is clear that T is compact and then it is un-U-Dunford-Pettis, but T can not be a σ-u-DP operator. In fact, we consider the standard basis (e_n) of ℓ^1 which is a disjoint sequence of ℓ^1. By [16, Lemma 2] we have that $e_n \xrightarrow{uaw} 0$, but $\|T(e_n)\| = 1$.
If an addition E has order unit, then every un-U-DP operator T defined on E is σ-u-DP.

We note that a σ-un-DP operator is not necessarily U-Dunford-Pettis and conversely an U-Dunford-Pettis operator is not σ-un-Dunford-Pettis in general. Indeed, there exists an operator $T : \ell^\infty \longrightarrow c_0$ which is not U-Dunford-Pettis (Proposition 3.3). However, every operator $T : \ell^\infty \longrightarrow c_0$ is σ-un-Dunford-Pettis (Proposition 3.2). On the other hand, there exists an operator $T : c_0 \longrightarrow \ell^\infty$ which is not σ-un-DP (Proposition 3.2). However, every operator $T : c_0 \longrightarrow \ell^\infty$ is U-Dunford-Pettis.

The next results, tells us when every σ-un-Dunford-Pettis operator is U-Dunford-Pettis.

Theorem 3.18. *Let E and F be two Banach lattices. We have the following assertions:*

1. *If F has an order unit, then every σ-un-Dunford-Pettis operator T defined from E into F is U-Dunford–Pettis.*
2. *Every σ-un-Dunford–Pettis order bounded operator T defined from E into F is U-Dunford–Pettis.*

Proof. 1) Let (x_n) be an order bounded sequence in E such that $x_n \xrightarrow{w} 0$. Since T is a σ-un-Dunford-Pettis operator, then $T(x_n) \xrightarrow{un} 0$ and since F has an order unit then $T(x_n) \xrightarrow{\|.\|} 0$ ([11, Theorem 2.3]), so T is a U-Dunford-Pettis operator.

2) Let (x_n) be an order bounded sequence in E such that $x_n \xrightarrow{w} 0$. Since T is σ-un-Dunford–Pettis operator then $T(x_n) \xrightarrow{un} 0$ and by assumption T is an order bounded operator implies that $T(x_n) \xrightarrow{\|.\|} 0$, as desired. □

We continue with study the relation between U-Dunford-Pettis and σ-u-Dunford-Pettis. We have the canonical injection i from c_0 into ℓ_∞ is U-Dunford-Pettis but it is not σ-u-Dunford-Pettis. Conversely, the identity operator $id_{L^2[0,1]}$ of $L^2[0,1]$ is σ-u-Dunford-Pettis but it is not U-Dunford-Pettis.
In the next results, we present a sufficient condition under which σ-u-Dunford-Pettis operators are U-Dunford–Pettis, and conversely.

Theorem 3.19. *Let E and F be two Banach lattices. Then we have the following assertions:*

1. *If the lattice operations of E are weakly sequentially continuous and F has an order unit, then every σ-u-Dunford-Pettis operator T from E into F is U-Dunford–Pettis.*
2. *If E has an order unit, then every U-Dunford-Pettis operator T from E into F is σ-u-Dunford–Pettis.*

Proof. 1) Let T be a σ-u-Dunford-Pettis operator, since the lattice operations of E are weakly sequentially continuous then it follows from the Theorem 3.17 that T is un-U-Dunford-Pettis. As F has an order unit, we infer that T is an U-Dunford-Pettis operator.

2) Let T be an U-Dunford-Pettis operator, then T is un-U-Dunford-Pettis, and since E has order unit then it follows from [16, Theorem 7] that T is σ-u-Dunford–Pettis operator. □

As a consequence of the preceding results we have;

Corollary 3.20. *Let E and F be two Banach lattices such that E has an order unit and T be an order bounded operator from E into F. The following statements are equivalent:*

1. *T is Dunford-Pettis.*
2. *T is σ-un-Dunford-Pettis.*
3. *T is un-U-Dunford-Pettis.*
4. *T is σ-u-Dunford-Pettis*
5. *T is U-Dunford-Pettis.*
6. *T is un-order weakly compact.*
7. *T is order weakly compact.*
8. *T is weakly compact.*

In this position, we study the relation between un-U-Dunford-Pettis and un-order weakly compact operators.

Theorem 3.21. *Let E and F be two Banach lattices such that the lattice operations of E are weakly sequentially continuous. Then each un-order weakly compact operator $T : E \longrightarrow F$ is un-U-Dunford–Pettis.*

Proof. Let (x_n) be an order bounded sequence in E such that $x_n \xrightarrow{w} 0$. Since the lattice operations of E are weakly sequentially continuous, then $x_n^+ \xrightarrow{w} 0$ and $x_n^- \xrightarrow{w} 0$. As T is un-order weakly compact, then $T(x_n^+) \xrightarrow{un} 0$ and $T(x_n^-) \xrightarrow{un} 0$ implies that $T(x_n) \xrightarrow{un} 0$, and hence T is un-U-Dunford-Pettis. □

Now, we are in position to study the relation between un-U-DP and AM-un-compact operators. It is easy to show that every AM-un-compact operator is un-U-DP. And the converse is not true in general. Indeed, we consider the Banach lattice ℓ^∞. Since the norm of ℓ^∞ is not order continuous and the Banach lattice $(\ell^\infty)'$ is not discrete, then it follows from [7, Theorem 2.5] that there exist two positive operators S, T from ℓ^∞ into ℓ^∞ with $0 \leq S \leq T$ and T is AM-compact but S is not one; as S is order bounded then S is not un-AM-compact. On the other hand, since T is a U-DP operator and ℓ^∞ has o-U-DPP, it follows from [1, Theorem 3.1] that S is U-DP and hence S is un-U-Dunford-Pettis.

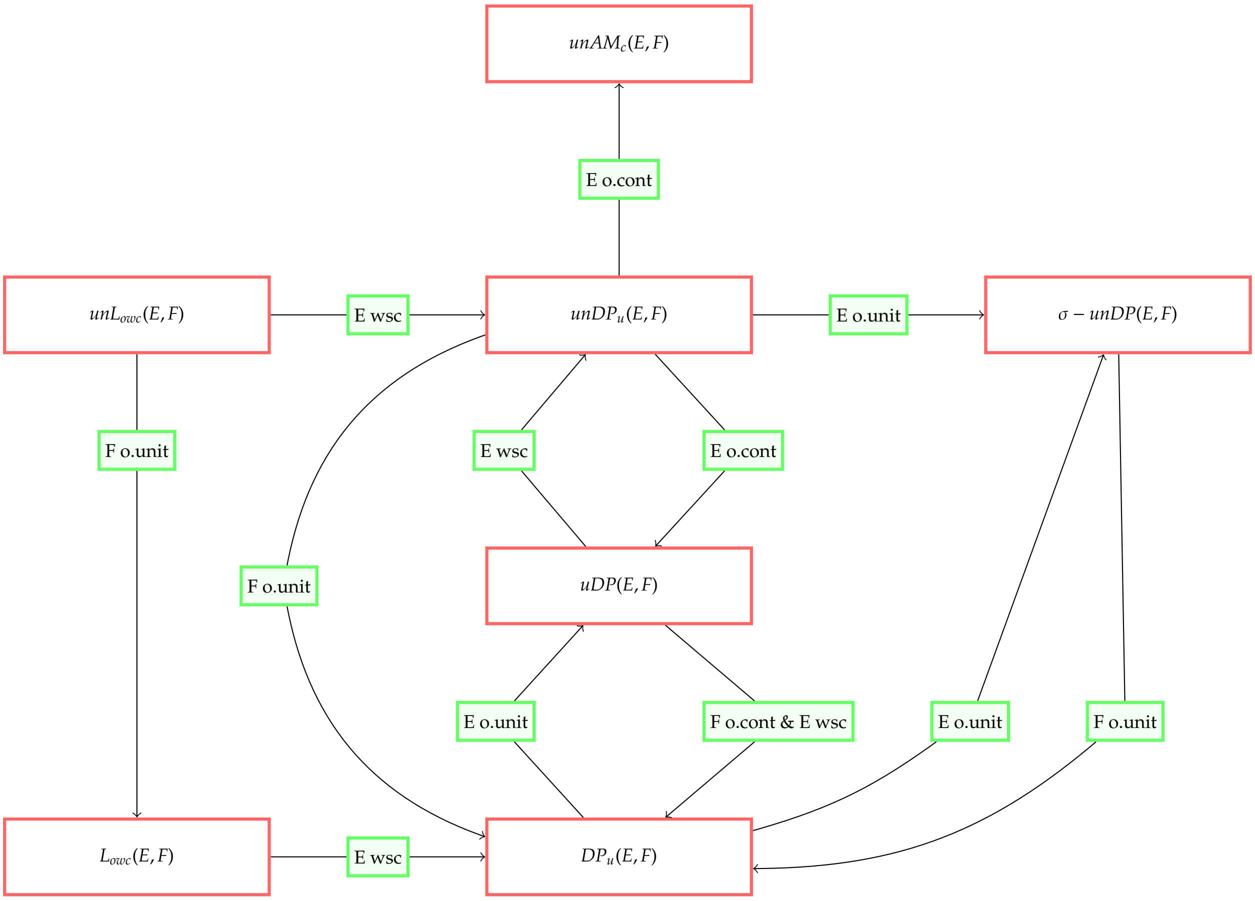
Theorem 3.22. *Let E and F be two Banach lattices. If E is order continuous, then each un-U-Dunford-Pettis operator T form E into F is AM-un-compact.*

Proof. Let x in E^+. Since E is order continuous, then $[-x, x]$ is a weakly compact subset of E. As T is un-U-DP, then $T[-x, x]$ is un-compact in F. So, T is an AM-un-compact operator. □

Now, we will show that the product of an order weakly compact operator and a un-U-Dunford-Pettis operator is AM-un-compact.

Proposition 3.23. *Let E and F be two Banach lattices and Y be a Banach space and let* $S : E \longrightarrow F$ *and* $T : F \longrightarrow Y$ *be two operators. If T is un-U-Dunofrd-Pettis and S is order bounded and order weakly compact, then* $T \circ S$ *is AM-un-compact.*

Proof. Since S is order weakly compact and order bounded, then $S[-x, x]$ order bounded weakly compact, for each x in E^+. As T is un-U-DP then $T \circ S[-x, x]$ is un-compact, and so $T \circ S$ AM-un-compact. □



3.3. *On the duality problem for un-U-DP operators*

In the class of un-U-Dunford-Pettis operators there exists a un-U-Dunford-Pettis operator whose adjoint is not un-U-Dunford-Pettis. In fact, the identity operator Id_{ℓ^1} is un-U-Dunford-Pettis but its adjoint which is the identity operator Id_{ℓ^∞} is not un-U-Dunford-Pettis.

In the following result, we give sufficient and necessary conditions under which an operator is un-U-Dunford-Pettis whenever its adjoint operator is un-U-Dunford-Pettis.

Theorem 3.24. *Let E and F be two Banach lattices such that the lattice operations of F′ are weakly sequentially continuous. Then, the following statements are equivalent:*

1. *For each operator* $T : E \longrightarrow F$, $T' : F' \longrightarrow E'$ *is un-U-Dunford-Pettis.*
2. *The adjoint of each un-U-Dunford-Pettis operator T from E into F is un-U-Dunford-Pettis.*
3. *One of the following conditions is valid:*

(a) *E' is order continuous;*
(b) *F' is order continuous.*

Proof. 1) ⇒ 2) Obvious.

2) ⇒ 3) Assume that E' is not order continuous and F' is not order continuous. Then, by [12, Theorem 2.4.2] there exists an order bounded disjoint sequence $(g_n) \in (E')^+$ such that $\|g_n\| = 1$ for all $n \in \mathbb{N}$ and $|g_n| \leq g$ for some $g \in (E')^+$, and there exists a positive order bounded disjoint sequence (f_n) of F' satisfying $\|f_n\| = 1$ for all n. So, we can find $(y_n) \in B_F$ such that $|f_n(y_n)| \geq 1/2$ for each $n \in N$.

We consider the following operators,

$$\begin{gathered} S_2 : E \rightarrow \ell^1 \\ x \rightarrow (g_n(x)) \\ S_1 : \ell^1 \rightarrow F \\ (\lambda_n)_n \rightarrow \sum_{n=1}^{\infty} \lambda_n y_n \end{gathered}$$

S_1 and S_2 are well-defined operators, and the product operator $T = S_1 \circ S_2$ is un-U-Dunford-Pettis, but its adjoint operator is not un-U-Dunford-Pettis. Indeed, we note that $f_n \xrightarrow{w} 0$ and we put $u = \frac{1}{2} g \in (E')^+$; we have

$$\begin{aligned} |T'(f_n)| \wedge u &= |\sum_{n=1}^{\infty} f_n(y_n) g_n| \wedge u \\ &= | \vee_{n=1}^{\infty} f_n(y_n) g_n| \wedge u \\ &\geq (|f_n(y_n)| g_n) \wedge u \\ &\geq \frac{1}{2} g_n \wedge \frac{1}{2} g \\ &\geq \frac{1}{2} g_n. \end{aligned}$$

Hence, $\| |T'(f_n)| \wedge u\| \geq \frac{1}{2} \|g_n\| = \frac{1}{2}$, and so (f_n) is a weakly null sequence but $T'(f_n)$ is not un-null. That is T' is not un-U-Dunford-Pettis.

3-a)⇒ 1) Since E' is order continuous (E' is a KB-space), then it follows from [6, Theorem 3.3] that T' is an order weakly compact operator, and since the lattice operations of F' are weakly sequentially continuous then T' is an U-DP operator and hence T' is a un-U-DP operator.

$3-b)$ ⇒ 1) Since F' is order continuous and the lattice operations of F' are weakly sequentially continuous then F' is atomic and order continuous and hence it follows from the Proposition 3.3 that $T' : F' \longrightarrow E'$ is a un-U-DP operator. □

As consequences of the Theorem 3.24 and the Proposition 3.3, we have the following results.

Corollary 3.25. *Let E be a Banach lattice. Then, the following statements are equivalent:*

1. *The adjoint of each un-U-Dunford-Pettis operator $T : E \longrightarrow F$ is un-U-Dunford-Pettis, for each Banach lattice F.*
2. *E' is order continuous.*

Corollary 3.26. *Let F be a Banach lattice. Then, the following statements are equivalent:*

1. *The adjoint of each un-U-Dunford-Pettis operator $T : E \longrightarrow F$ is un-U-Dunford-Pettis, for each Banach lattice E.*
2. *F' is order continuous.*

Also, the Reciprocal duality is not satisfied in the class of un-U-Dunford-Pettis operators (it suffice to consider the identity operator of the Banach lattice c).

Theorem 3.27. *Let E and F be two Banach lattices such that the lattice operations of E are weakly sequentially continuous and F a σ-Dedekind complet with the (b)-property. Then, the following statements are equivalent:*

1. *Each operator $T : E \longrightarrow F$ is un-U-Dunford-Pettis whenever its adjoint operator $T' : F' \longrightarrow E'$ is un-U-Dunford-Pettis.*
2. *One of the following conditions is valid:*
   (a) *E is order continuous;*
   (b) *F is order continuous.*

Proof. $2 - a) \Rightarrow 1)$ Since E is order continuous, then the operator $T : E \longrightarrow F$ is order weakly compact, and since the lattice operations of E are weakly sequentially continuous then, the operator T is U-Dunford-Pettis and hence T is un-U-Dunford-Pettis.

$2 - b) \Rightarrow 1)$ Since F is order continuous and has the (b)-property, then F is a KB-space, and hence it follows from [6, Corollary 2.3] that T is an order weakly compact operator. As the lattice operations of E are weakly sequentially continuous, then the operator T is un-U-Dunford-Pettis.

$1) \Rightarrow 2)$ Assume that neither F is order continuous nor E is order continuous. Then, by [12, Theorem 2.4.2] there exists a disjoint sequence (y_n) of F^+ such that $\|y_n\| = 1$ and $0 \leq y_n \leq y$ for all n and some $0 \leq y \in F$; on the other hand, there exists a positive order bounded disjoint sequence (x_n) of E satisfying $\|x_n\| = 1$ for all n; and by [6, Lemma 3.4] there exists a disjoint sequence $(g_n) \in (E')^+$ with $\|g_n\| \leq 1$ such that $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.

We consider the following operators which are mentioned in the Remark 3.4,

$$\begin{array}{rcl} S : E & \rightarrow & c_0 \\ x & \mapsto & (g_n(x)) \end{array} \qquad \begin{array}{rcl} R : c_0 & \rightarrow & F \\ (\lambda_n)_n & \mapsto & \sum_{n=1}^{\infty} \lambda_n y_n. \end{array}$$

We consider the product operator $T = R \circ S$ which is not un-U-Dunford-Pettis, and we note that $T' = S' \circ R'$ is un-U-Dunford-Pettis (because ℓ^1 is an atomic order continuous Banach lattice). □

As consequences of the Theorem 3.27, the Proposition 3.3 and [15, Theorem 6.1] we have the following results.

Corollary 3.28. *Let E be a Banach lattice such that the lattice operations of E are weakly sequentially continuous. Then, the following statements are equivalent:*

1. *Each operator $T : E \longrightarrow F$ is un-U-Dunford-Pettis whenever its adjoint is un-U-Dunford-Pettis, for each Banach lattice F.*
2. *E is σ-Dedekind complete.*

Corollary 3.29. *Let F be a σ-Dedekind complete Banach lattice. Then, each operator $T : E \longrightarrow F$ is un-U-Dunford-Pettis whenever its adjoint is un-U-Dunford-Pettis, for each Banach lattice E.*

Corollary 3.30. *Let F be a σ-Dedekind complete with the (b)-property. If each operator $T : E \longrightarrow F$ is un-U-Dunford-Pettis whenever its adjoint is un-U-Dunford-Pettis, for each Banach lattice E, then F is order continuous.*

References

[1] O. Aboutafail, K. El Fahri, J. H'Michane, and L. Zraoula, *On the Class of U-Dunford-Pettis Operators*, Quaestiones Mathematicae, DOI: 10.2989/16073606.2021.1887390 (2021).

[2] C.D. Aliprantis, and O. Burkinshaw, *Positive operators*. Reprint of the 1985 original. Springer, Dordrecht, 2006.

[3] B. Aqzzouz, and J. H'michane, *AM-Compactness of some classes of operators*, Comment.Math.Univ.Carolin. 53,4 (2012) 509–518. https://dml.cz/handle/10338.dmlcz/143185.

[4] B. Aqzzouz, and A. Elbour, *Characterization of the order weak compactness of semi-compact operators*. Journal of Mathematical Analysis and Applications, Volume 355, Issue 2, 2009, Pages 541-547, ISSN 0022-247X, https://doi.org/10.1016/j.jmaa.2009.01.072.

[5] B. Aqzzouz, and J. H'michane, *The duality problem for the class of order weakly compact operators*. Glasgow Math. J. 51 (2009), 101–108. doi:10.1017/S0017089508004576.

[6] B. Aqzzouz, A. Elbour, and J. H'michane, *The duality problem for the Class of b-weakly compact operators*, Positivity 13 (2009), 683–692. DOI 10.1007/s11117-008-2288-6.

[7] B. Aqzzouz, R. Nouira, and L. Zraoula, *Compactness of AM-compact operators on Banach lattices*. Ann. Sci. Math. Québec 31 (2007), no 1, 1–11.

[8] K.H. Azara, M. Matina, R. Alavizadeha, *Unbounded Order-Norm Continuous and Unbounded Norm Continuous Operator*, Filomat 35:13 (2021), 4417–4426.https: doi.org/10.2298/FIL2113417A.

[9] Y. Deng, M. O'Brien, and V. G. Troitsky, *Unbounded norm convergence in Banach lattices*, Positivity., 21(3) (2017), pp. 963–974. DOI 10.1007/s11117-016-0446-9.

[10] N. Hafidi, and J. H'michane, *Some results on the class of σ-unbounded Dunford-Pettis operators*, Commentationes Mathematicae Universitatis Carolinae (2021), Volume: 62, Issue: 4, page 431-443, ISSN: 0010-2628.

[11] M. Kandic, M.A.A. Marabeh, and V. G. Troitsky, *Unbounded norm topology in Banach lattices*, J. Math. Anal.Appl., 451(1)(2017), 259–279. DOI:10.1016/J.JMAA.2017.01.041.

[12] P. Meyer-Nieberg, *Banach lattices*, Universitext, Springer-Verlag, Berlin, 1991.

[13] M. Ouyang, Z. Chen, J. Chen, and Z. Wang, *σ-Unbounded Dunford-Pettis operators on Banach lattices*. https://arxiv.org/pdf/1909.07681.pdf

[14] Z. Wang, Z. Chen, and J. Chen, *AM-unbounded norm compact operators on Banach lattices*. https://arxiv.org/pdf/1909.13222.

[15] W. Wnuk, *Banach lattices with order continuous norms*, Warsaw: Polish Scientific Publishers (1999).

[16] O. Zabeti, *Unbounded absolute weak convergence in Banach lattices*, Positivity. DOI 10.1007/s11117-017-0524-7.