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The general solution to a system of real split quaternion matrix equations

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Abstract. In this paper, we present a direct methodology for solving a novel system of split quaternion matrix equations. Leveraging the Moore-Penrose generalized inverse, the Kronecker product, the vec operator, and the real representation of split quaternion matrices, we offer a comprehensive toolkit. The primary aim of this paper is to establish the solvability conditions of a system over the split quaternions and provide a general solution expression when it is consistent. We also give an algorithm to find the approximate solution to this system when it is inconsistent. Finally, we give a numerical example to showcase the efficacy of our approach.

1. Introduction

Hamilton quaternions were discovered by Irish mathematician, William Rowan Hamilton in 1843 [26]. It is a significant discovery in terms of the mathematical history. Quaternions and quaternion matrices are not only used in mathematics but also have applications in numerous other fields, such as attitude control, computer graphics, robotics, control theory, physics, orbital mechanics, and signal processing (see, e.g.[18], [34], [52], [54]). The set of quaternions is denoted by $\mathbb{H}$ and defined as

$$\mathbb{H} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where $\mathbb{R}$ is the real number field, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

General characteristics of quaternions and quaternion matrices can be found in [63]. In 1849, six years after Hamilton discovered quaternions, the algebra of split quaternions or coquaternions was first presented

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by James Cockle [25]. The algebra of split quaternions is a four-dimensional real vector space with a specific multiplicative operation. The set of split quaternions is denoted by $\mathbb{H}_S$ and represented as

$$\mathbb{H}_S = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where

$$\mathbf{i}^2 = -1, \ \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1.$$

The main difference between split quaternions $\mathbb{H}_S$ and quaternions $\mathbb{H}$ is that $\mathbb{H}_S$ is not a skew field, and it contains many zero divisors and nilpotent elements [29]. Due to these complicated characteristics, studying split quaternions is more challenging than quaternions. As one of the emerging research topics, the split quaternions have also been applied in split quaternionic mechanics and some other fields, such as the model of public key cryptosystems, geometric theory, rotations in four-dimensional space E_4^2, and so on (see, e.g., [2],[19], [33], [35]). Many significant characteristics of split quaternions have been investigated in recent years, one may be found in [29].

Quaternion matrix equations find wide applications in various fields, including mathematics, engineering, system and control theory, data analysis, color image processing, and optimal control (see, e.g.,[7], [14], [63]). The problem of solving matrix equations holds significant practical value which is attracting considerable attention from scholars. As a result, numerous researchers (see, e.g., [1], [8–11], [26], [28], [36], [39], [43], [50], [56], [61], [64],[65]) have used various approaches to investigate the solutions of the matrix equations. As we know that Sylvester and Sylvester-type matrix equations are extensively applied in robust control [4], graph theory [12], output feedback control [42], neural networks [66] and other fields (see, e.g., [3], [8], [40]). Roth [39] derived the Sylvester-type matrix equation for the first time over the polynomial integral domian. Baksalary and Kala [1] gave the solvability conditions and established an expression of the general solution of Sylvester-type matrix equation. He and Wang ([15], [17], [47]) proposed the necessary and sufficient conditions for the solvability to the systems of one-sided coupled Sylvester-type quaternion matrix equations and derived the expressions of general solutions to these systems. Moreover, the general solutions of some systems of mixed type generalized Sylvester matrix equations were also studied in ([48], [49]). Wang et al. [51] derived the solvability conditions to the following two-sided coupled Sylvester-type matrix equations

$$\begin{cases} A_1X = E_1, \ \ XB_1 = E_2, \ \ C_1Y = E_3, \\ YD_1 = E_4, \ \ A_2XB_2 + C_2YD_2 = E_5, \end{cases} \tag{1}$$

and then provided the least squares solution with the least norm to the system (1) in [53]. In 2019 [54], the solvability conditions and the form of the general solution to the following system of matrix equations

$$\begin{cases} A_1X = C_1, \ \ A_2Y = C_2, \ \ A_3Z = C_3, \\ XB_1 = D_1, \ \ YB_2 = D_2, \ \ ZB_3 = D_3, \\ A_4XB_4 + C_4YD_4 = P, \ \ A_5ZB_5 + C_5YD_5 = Q. \end{cases} \tag{2}$$

were also investigated. Xie and Wang [55] gave the solvability conditions and the general solution to the system (2) over commutative quaternions. Some necessary and sufficient conditions for the solvability of the system of five quaternion matrix equations in terms of the ranks of matrices were derived in [57]. In 2022, Yuan and Wang [60] investigated a system of twelve matrix equations over quaternion algebra and established the solvability conditions and an expression of general solution when the system is consistent.

It is worth noting that η-Hermitian quaternion matrices and η-anti-Hermitian quaternion matrices have important applications in linear modeling and convergence analysis in statistical signal processing (see, e.g.,[44–46]). There are many results focusing on the η-Hermitian solution, the η-anti-Hermitian solution, and other solutions with special forms (see, e.g., [5], [16], [27],[37], [38],[67]). Recently, Kyrchei derived determinantal representations of the solutions to some systems of quaternion matrix equations and two-sided generalized Sylvester matrix equations (see, e.g.,[21–24]).

Now, we turn our attention to the solution of split quaternion matrix equations. A few studies have expanded the results of quaternion matrix equations to the split quaternion equations. Li et al. [26] used the real and complex representations of split quaternion matrices to examine the η-Hermitian solutions of the equation $(AXB, CXD) = (E, F)$. In [68], Zhang et al. studied the split quaternion least squares problem and provided two algebraic methods for finding solutions to the problems in split quaternionic mechanics. In order to explain the consistency of two types of split quaternion matrix equations $AX^* - XB = CY + D$ and $X - AX^*B = CY + D$, Liu and Zhang [29] derived some new real representations of split quaternion matrices. Yuan et al. [59] discussed the Hermitian solution of split quaternion matrix equation $AXB + CXD = E$ and established the necessary and sufficient conditions for the existence of the solutions. Yue et al. [58] investigated the bisymmetric and skew bisymmetric solutions of a split quaternion matrix equation and found the equivalent solvable conditions and general expressions of the (skew) bisymmetric solutions. Kyrchei [20] investigated Cramer's rules for left and right systems of linear equations with Hermitian split quaternion coefficient matrices. Liu and Zhang [30] derived the necessary and sufficient conditions and provided the expression of general solutions for the matrix equation $AXA^{\eta^*} = B$. Si and Wang [41] presented the general expression for solving a dual split quaternion matrix equation $AXB = C$. Gao et al. [13] established the necessary and sufficient conditions for the system of split quaternion matrix equations for the existence of η- anti-Hermitian solutions.

Motivated by the work mentioned above and keeping the interest in wide applications of split quaternion matrices, we in this paper consider the following problem which represents a significant extension of the previously considered equations. For the convenience, throughout this paper, we denote the sets of all $m \times n$ real matrices, complex matrices, quaternion matrices, and split quaternion matrices by $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, $\mathbb{H}^{m\times n}$, and $\mathbb{H}_S^{m\times n}$, respectively.

Problem 1. Let $A_1,\ C_1,\ E_1 \in \mathbb{H}_S^{m\times n}, A_2,\ C_2,\ E_2 \in \mathbb{H}_S^{n\times k}, E_3,\ E_4,\ E_5 \in \mathbb{H}_S^{m\times n}, F_3,\ F_4,\ F_5 \in \mathbb{H}_S^{n\times k}, B_1,\ D_1,\ F_1 \in \mathbb{H}_S^{m\times n}$, $B_2,\ D_2,\ F_2 \in \mathbb{H}_S^{n\times k}$, and $H \in \mathbb{H}_S^{m\times k}$. Find

$$\begin{aligned}\chi = \Big\{[X, Y, Z] \,\Big|\, & X, Y, Z \in \mathbb{H}_S^{n\times n}, A_1X = B_1,\ \ XA_2 = B_2,\ \ C_1Y = D_1, \\ & YC_2 = D_2,\ \ E_1Z = F_1,\ \ ZE_2 = F_2,\ \ E_3XF_3 + E_4YF_4 + E_5ZF_5 = H\Big\}.\end{aligned} \tag{3}$$

The remainder of this paper is outlined as follows. In section 2, we study the real representation of split quaternion matrix and also analyse the structure of vec(EXF) over split quaternions. In section 3, considering different methods mentioned in ([26], [55], [60]), we propose some necessary and sufficient conditions for the solvability of the system (3) and give an expression of the general solution to the system (3) when it is solvable. In section 4, we present an algorithm and a numerical example to illustrate the main results of this paper. Finally, we conclude this paper by giving some remarks in section 5.

2. Preliminary

In this section, we consider some definitions and lemmas that will be used in the following development of this paper.

For $A \in \mathbb{H}_S^{m\times n}$, A can be uniquely expressed as $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, where $A_0,\ A_1,\ A_2,\ A_3 \in \mathbb{R}^{m\times n}$. The conjugate matrix $\bar{A}$ is expressed as $\bar{A} = A_0 - A_1\mathbf{i} - A_2\mathbf{j} - A_3\mathbf{k}$, the transpose matrix $A^{\mathbf{T}}$ is defined as $A^{\mathbf{T}} = A_0^{\mathbf{T}} + A_1^{\mathbf{T}}\mathbf{i} + A_2^{\mathbf{T}}\mathbf{j} + A_3^{\mathbf{T}}\mathbf{k}$, and the conjugate transpose matrix A^* is represented as $A^* = A_0^{\mathbf{T}} - A_1^{\mathbf{T}}\mathbf{i} - A_2^{\mathbf{T}}\mathbf{j} - A_3^{\mathbf{T}}\mathbf{k}$. Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{s\times t}$, then the Kronecker product of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

For any given $a_i = (a_{1i}, a_{2i}, \cdots, a_{mi})$, we define $\text{vec}(A) = (a_1, a_2, \cdots, a_n)^{\mathbf{T}}$, where $a_i (i = 1, 2, \cdots, n)$ is the i^{th} column of A. The Moore-Penrose generalized inverse of $A \in \mathbb{C}^{m\times n}$ denoted by $A^{\dagger}$ is a unique matrix which satisfies the following equations,

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

For $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \in \mathbb{H}_S^{m\times n}$, $A_i \in \mathbb{R}^{m\times n}$, a real representation of A is given by

$$G(A) = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 \\ -A_1 & A_0 & -A_3 & A_2 \\ A_2 & -A_3 & A_0 & -A_1 \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix} \in \mathbb{R}^{4m\times 4n}.$$

It is easy to verify the following results.

Proposition 2.1 ([26], [30]). *For $A, B \in \mathbb{H}_S^{m\times n}$ and $k_1, k_2 \in \mathbb{R}$, we have the following:*
(1) *$A = B$ if and only if $G(A) = G(B)$;*
(2) *$G(AB) = G(A)G(B)$;*
(3) *$G(k_1A + k_2B) = k_1G(A) + k_2G(B)$;*
(4) *$G(I_n) = I_{4n}$, where I_n is an identity matrix with order n.*

For any $B = B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k} \in \mathbb{H}_S^{m\times n}$, we define $\phi_B = (B_0, B_1, B_2, B_3)$. Clearly,

$$||B|| = ||\phi_B|| = \sqrt{||B_0||^2 + ||B_1||^2 + ||B_2||^2 + ||B_3||^2},$$

and

$$B + C \cong \phi_B + \phi_C.$$

Thus

$$||\phi_{B+C}|| = ||\phi_B + \phi_C||.$$

Let $\overrightarrow{B} = (B_0, B_1, B_2, B_3)$ and then

$$\text{vec}(\overrightarrow{B}) = \text{vec}(\phi_B) = \begin{bmatrix} \text{vec}(B_0) \\ \text{vec}(B_1) \\ \text{vec}(B_2) \\ \text{vec}(B_3) \end{bmatrix}.$$

Moreover, we have

$$||\text{vec}(\overrightarrow{B})|| = ||\text{vec}(\phi_B)|| = \left\| \begin{bmatrix} \text{vec}(B_0) \\ \text{vec}(B_1) \\ \text{vec}(B_2) \\ \text{vec}(B_3) \end{bmatrix} \right\|.$$

Theorem 2.2. *If k is a real number and $A, B \in \mathbb{H}_S^{m\times n}$. Then*
(1) *$A = B$ if and only if $\phi_A = \phi_B$;*
(2) *$\phi_{A+B} = \phi_A + \phi_B$ and $\phi_{kA} = k\phi_A$;*
(3) *$\phi_{AB} = \phi_A G(B)$.*

Proof. (1) and (2) can be proved easily, here we only consider (3). The multiplication of two split quatrernion matrices A and B is expressed as

$$\begin{aligned}AB &= (A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})(B_0 + B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k})\\ &= (A_0B_0 - A_1B_1 + A_2B_2 + A_3B_3) + (A_0B_1 + A_1B_0 - A_2B_3 + A_3B_2)\mathbf{i}\\ &+ (A_0B_2 - A_1B_3 + A_2B_0 + A_3B_1)\mathbf{j} + (A_0B_3 + A_1B_2 - A_2B_1 + A_3B_0)\mathbf{k}.\end{aligned}$$

Thus

$$\phi_{AB} = (A_0, A_1, A_2, A_3)\begin{bmatrix} B_0 & B_1 & B_2 & B_3 \\ -B_1 & B_0 & -B_3 & B_2 \\ B_2 & -B_3 & B_0 & -B_1 \\ B_3 & B_2 & B_1 & B_0 \end{bmatrix} = \phi_A\, G(B).$$

□

By the definition of $A \otimes B$, it follows that

$$\mathrm{vec}(AXB) = (B^{\mathbf{T}} \otimes A)\mathrm{vec}(X).$$

However, it cannot hold in the split quaternion algebra for noncommutative multiplication of split quaternions. Thus, we have to study the structure of $\mathrm{vec}(\phi_{AXB})$.

Theorem 2.3. *Let* $E = E_0+E_1\mathbf{i}+E_2\mathbf{j}+E_3\mathbf{k} \in \mathbb{H}_S^{m\times n}$, $X = X_0+X_1\mathbf{i}+X_2\mathbf{j}+X_3\mathbf{k} \in \mathbb{H}_S^{n\times n}$, $F = F_0+F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k} \in \mathbb{H}_S^{n\times k}$. *Then*

$$\mathrm{vec}(\phi_{EXF}) = (G(F)^{\mathbf{T}} \otimes E_0,\ G(F)^{\mathbf{T}} \otimes E_1,\ G(F)^{\mathbf{T}} \otimes E_2,\ G(F)^{\mathbf{T}} \otimes E_3)\begin{bmatrix}\mathrm{vec}(\phi_X)\\ \mathrm{vec}(\phi_{iX})\\ \mathrm{vec}(\phi_{jX})\\ \mathrm{vec}(\phi_{kX})\end{bmatrix},$$

where

$$\mathrm{vec}(\phi_{iX}) = \begin{bmatrix}\mathrm{vec}(-X_1)\\ \mathrm{vec}(X_0)\\ \mathrm{vec}(-X_3)\\ \mathrm{vec}(X_2)\end{bmatrix},\ \mathrm{vec}(\phi_{jX}) = \begin{bmatrix}\mathrm{vec}(X_2)\\ \mathrm{vec}(-X_3)\\ \mathrm{vec}(X_0)\\ \mathrm{vec}(-X_1)\end{bmatrix},\ \mathrm{vec}(\phi_{kX}) = \begin{bmatrix}\mathrm{vec}(X_3)\\ \mathrm{vec}(X_2)\\ \mathrm{vec}(X_1)\\ \mathrm{vec}(X_0)\end{bmatrix}.$$

Proof. By Theorem 2.2, it follows that

$$\begin{aligned}\phi_{EXF} &= \phi_E G(XF) = \phi_E G(X)G(F)\\ &= (E_0, E_1, E_2, E_3)\begin{bmatrix} X_0 & X_1 & X_2 & X_3 \\ -X_1 & X_0 & -X_3 & X_2 \\ X_2 & -X_3 & X_0 & -X_1 \\ X_3 & X_2 & X_1 & X_0 \end{bmatrix}\begin{bmatrix} F_0 & F_1 & F_2 & F_3 \\ -F_1 & F_0 & -F_3 & F_2 \\ F_2 & -F_3 & F_0 & -F_1 \\ F_3 & F_2 & F_1 & F_0 \end{bmatrix}\\ &= [(E_0X_0F_0 - E_1X_1F_0 + E_2X_2F_0 + E_3X_3F_0 - E_0X_1F_1 - E_1X_0F_1 + E_2X_3F_1 - E_3X_2F_1\\ &+ E_0X_2F_2 - E_1X_3F_2 + E_2X_0F_2 + E_3X_1F_2 + E_0X_3F_3 + E_1X_2F_3 - E_2X_1F_3 + E_3X_0F_3),\\ &(E_0X_0F_1 - E_1X_1F_1 + E_2X_2F_1 + E_3X_3F_1 + E_0X_1F_0 + E_1X_0F_0 - E_2X_3F_0 + E_3X_2F_0\\ &- E_0X_2F_3 + E_1X_3F_3 - E_2X_0F_3 - E_3X_1F_3 + E_0X_3F_2 + E_1X_2F_2 - E_2X_1F_2 + E_3X_0F_2),\\ &(E_0X_0F_2 - E_1X_1F_2 + E_2X_2F_2 + E_3X_3F_2 - E_0X_1F_3 - E_1X_0F_3 + E_2X_3F_3 - E_3X_2F_3\\ &+ E_0X_2F_0 - E_1X_3F_0 + E_2X_0F_0 + E_3X_1F_0 + E_0X_3F_1 + E_1X_2F_1 - E_2X_1F_1 + E_3X_0F_1),\\ &(E_0X_0F_3 - E_1X_1F_3 + E_2X_2F_3 + E_3X_3F_3 + E_0X_1F_2 + E_1X_0F_2 - E_2X_3F_2 + E_3X_2F_2\\ &- E_0X_2F_1 + E_1X_3F_1 - E_2X_0F_1 - E_3X_1F_1 + E_0X_3F_0 + E_1X_2F_0 - E_2X_1F_0 + E_3X_0F_0)].\end{aligned}$$

Therefore,

$$\mathrm{vec}(\phi_{EXY}) = K_1 + K_2 + K_3 + K_4,$$

where

$$K_1 = \begin{bmatrix} (F_0^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_0) + (-F_1^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_1) + (F_2^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_2) + (F_3^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_3) \\ (F_1^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_0) + (F_0^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_1) + (-F_3^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_2) + (F_2^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_3) \\ (F_2^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_0) + (-F_3^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_1) + (F_0^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_2) + (F_1^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_3) \\ (F_3^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_0) + (F_2^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_1) + (-F_1^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_2) + (F_0^{\mathbf{T}} \otimes E_0)\mathrm{vec}(X_3) \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} F_0^{\mathbf{T}} & -F_1^{\mathbf{T}} & F_2^{\mathbf{T}} & F_3^{\mathbf{T}} \\ F_1^{\mathbf{T}} & F_0^{\mathbf{T}} & -F_3^{\mathbf{T}} & F_2^{\mathbf{T}} \\ F_2^{\mathbf{T}} & -F_3^{\mathbf{T}} & F_0^{\mathbf{T}} & F_1^{\mathbf{T}} \\ F_3^{\mathbf{T}} & F_2^{\mathbf{T}} & -F_1^{\mathbf{T}} & F_0^{\mathbf{T}} \end{bmatrix} \otimes E_0 \end{bmatrix} \begin{bmatrix} \mathrm{vec}(X_0) \\ \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(X_3) \end{bmatrix}$$

$$= (G(F)^{\mathbf{T}} \otimes E_0) \begin{bmatrix} \mathrm{vec}(X_0) \\ \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(X_3) \end{bmatrix}.$$

Similarly, we can prove that

$$K_2 = (G(F)^{\mathbf{T}} \otimes E_1) \begin{bmatrix} \mathrm{vec}(-X_1) \\ \mathrm{vec}(X_0) \\ \mathrm{vec}(-X_3) \\ \mathrm{vec}(X_2) \end{bmatrix}, \; K_3 = (G(F)^{\mathbf{T}} \otimes E_2) \begin{bmatrix} \mathrm{vec}(X_2) \\ \mathrm{vec}(-X_3) \\ \mathrm{vec}(X_0) \\ \mathrm{vec}(-X_1) \end{bmatrix}, \; K_4 = (G(F)^{\mathbf{T}} \otimes E_3) \begin{bmatrix} \mathrm{vec}(X_3) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(X_1) \\ \mathrm{vec}(X_0) \end{bmatrix}.$$

Hence,

$$\mathrm{vec}(\phi_{EXF}) = (G(F)^{\mathbf{T}} \otimes E_0, G(F)^{\mathbf{T}} \otimes E_1, G(F)^{\mathbf{T}} \otimes E_2, G(F)^{\mathbf{T}} \otimes E_3) \begin{bmatrix} \mathrm{vec}(\phi_X) \\ \mathrm{vec}(\phi_{iX}) \\ \mathrm{vec}(\phi_{jX}) \\ \mathrm{vec}(\phi_{kX}) \end{bmatrix}.$$

□

Note that the above results are important for solving a system of constrained two-sided coupled Sylvester-type matrix equations over the split quaternions.

Lemma 2.4. *Suppose that* $C = C_0 + C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k} \in \mathbb{H}_S^{n\times k}$, *then*

$$\begin{bmatrix} \mathrm{vec}(\phi_C) \\ \mathrm{vec}(\phi_{iC}) \\ \mathrm{vec}(\phi_{jC}) \\ \mathrm{vec}(\phi_{kC}) \end{bmatrix} = \zeta_{nk}\mathrm{vec}(\vec{C}),$$

where

$$\zeta_{nk} = \begin{bmatrix} \zeta_I \\ \zeta_{iI} \\ \zeta_{jI} \\ \zeta_{kI} \end{bmatrix}, \; \zeta_I = \begin{bmatrix} I_{nk} & 0 & 0 & 0 \\ 0 & I_{nk} & 0 & 0 \\ 0 & 0 & I_{nk} & 0 \\ 0 & 0 & 0 & I_{nk} \end{bmatrix}, \; \zeta_{iI} = \begin{bmatrix} 0 & -I_{nk} & 0 & 0 \\ I_{nk} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{nk} \\ 0 & 0 & I_{nk} & 0 \end{bmatrix},$$

$$\zeta_{jI} = \begin{bmatrix} 0 & 0 & I_{nk} & 0 \\ 0 & 0 & 0 & -I_{nk} \\ I_{nk} & 0 & 0 & 0 \\ 0 & -I_{nk} & 0 & 0 \end{bmatrix}, \; \zeta_{kI} = \begin{bmatrix} 0 & 0 & 0 & I_{nk} \\ 0 & 0 & I_{nk} & 0 \\ 0 & I_{nk} & 0 & 0 \\ I_{nk} & 0 & 0 & 0 \end{bmatrix}.$$

Proof.

$$\begin{bmatrix} \mathrm{vec}(\phi_C) \\ \mathrm{vec}(\phi_{iC}) \\ \mathrm{vec}(\phi_{jC}) \\ \mathrm{vec}(\phi_{kC}) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(C_0) + \mathrm{vec}(C_1)i + \mathrm{vec}(C_2)j + \mathrm{vec}(C_3)k \\ \mathrm{vec}(-C_1) + \mathrm{vec}(C_0)i + \mathrm{vec}(-C_3)j + \mathrm{vec}(C_2)k \\ \mathrm{vec}(C_2) + \mathrm{vec}(-C_3)i + \mathrm{vec}(C_0)j + \mathrm{vec}(-C_1)k \\ \mathrm{vec}(C_3) + \mathrm{vec}(C_2)i + \mathrm{vec}(C_1)j + \mathrm{vec}(C_0)k \end{bmatrix}$$

$$= \begin{bmatrix} \zeta_I \\ \zeta_{iI} \\ \zeta_{jI} \\ \zeta_{kI} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(C_0) \\ \mathrm{vec}(C_1) \\ \mathrm{vec}(C_2) \\ \mathrm{vec}(C_3) \end{bmatrix} = \zeta_{nk}\mathrm{vec}(\overrightarrow{C}).$$

□

Using Theorem 2.3 and Lemma 2.4, we can obtain the following result.

Corollary 2.5. *If $E = E_0+E_1\mathbf{i}+E_2\mathbf{j}+E_3\mathbf{k} \in \mathbb{H}_S^{m\times n}$, $X = X_0+X_1\mathbf{i}+X_2\mathbf{j}+X_3\mathbf{k} \in \mathbb{H}_S^{n\times n}$, and $F = F_0+F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k} \in \mathbb{H}_S^{n\times k}$. Then*

$$\mathrm{vec}(\phi_{EXF}) = (G(F)^{\mathbf{T}} \otimes E_0,\ G(F)^{\mathbf{T}} \otimes E_1,\ G(F)^{\mathbf{T}} \otimes E_2,\ G(F)^{\mathbf{T}} \otimes E_3)\ \zeta_{nk}\mathrm{vec}\ (\overrightarrow{X}).$$

To find the solution of system (3), we recall the following lemma.

Lemma 2.6 ([55]). *The matrix equation $Ax = b$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$, has a solution $x \in \mathbb{R}^n$ if and only if*

$$AA^{\dagger}b = b.$$

In this case, the general solution can be expressed as

$$x = A^{\dagger}b + (I_n - A^{\dagger}A)y,$$

where $y \in \mathbb{R}^n$ is an arbitrary vector. If $rank(A) = n$ the equation has a unique solution $x = A^{\dagger}b$.

3. The Solution of Problem 1

From the above discussions, we now pay attention to solving the system of split quaternion matrix equations (3). For convenience, we provide the following notations that will be used in the sequel. Let $A_1 = A_{10}+A_{11}\mathbf{i}+A_{12}\mathbf{j}+A_{13}\mathbf{k}, C_1 = C_{10}+C_{11}\mathbf{i}+C_{12}\mathbf{j}+C_{13}\mathbf{k}, E_1 = E_{10}+E_{11}\mathbf{i}+E_{12}\mathbf{j}+E_{13}\mathbf{k} \in \mathbb{H}_S^{m\times n}$, $A_2, C_2, E_2 \in \mathbb{H}_S^{n\times k}$, $E_t = E_{t0} + E_{t1}\mathbf{i} + E_{t2}\mathbf{j} + E_{t3}\mathbf{k} \in \mathbb{H}_S^{m\times n}$ $(t = \overline{3,5})$, $F_3, F_4, F_5 \in \mathbb{H}_S^{n\times k}$, $B_1, D_1, F_1 \in \mathbb{H}_S^{m\times n}$, $B_2, D_2, F_2 \in \mathbb{H}_S^{n\times k}$, and $H \in \mathbb{H}_S^{m\times k}$. Set

$$L = \begin{bmatrix} G(I)^{\mathbf{T}} \otimes A_{10} & G(I)^{\mathbf{T}} \otimes A_{11} & G(I)^{\mathbf{T}} \otimes A_{12} & G(I)^{\mathbf{T}} \otimes A_{13} \\ G(A_2)^{\mathbf{T}} \otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ G(F_3)^{\mathbf{T}} \otimes E_{30} & G(F_3)^{\mathbf{T}} \otimes E_{31} & G(F_3)^{\mathbf{T}} \otimes E_{32} & G(F_3)^{\mathbf{T}} \otimes E_{33} \end{bmatrix} \zeta_{n^2},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ G(I)^{\mathbf{T}} \otimes C_{10} & G(I)^{\mathbf{T}} \otimes C_{11} & G(I)^{\mathbf{T}} \otimes C_{12} & G(I)^{\mathbf{T}} \otimes C_{13} \\ G(C_2)^{\mathbf{T}} \otimes I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ G(F_4)^{\mathbf{T}} \otimes E_{40} & G(F_4)^{\mathbf{T}} \otimes E_{41} & G(F_4)^{\mathbf{T}} \otimes E_{42} & G(F_4)^{\mathbf{T}} \otimes E_{43} \end{bmatrix} \zeta_{n^2},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ G(I)^{\mathbf{T}} \otimes E_{10} & G(I)^{\mathbf{T}} \otimes E_{11} & G(I)^{\mathbf{T}} \otimes E_{12} & G(I)^{\mathbf{T}} \otimes E_{13} \\ G(E_2)^{\mathbf{T}} \otimes I & 0 & 0 & 0 \\ G(F_5)^{\mathbf{T}} \otimes E_{50} & G(F_5)^{\mathbf{T}} \otimes E_{51} & G(F_5)^{\mathbf{T}} \otimes E_{52} & G(F_5)^{\mathbf{T}} \otimes E_{53} \end{bmatrix} \zeta_{n^2}.$$

Let

$$V_j = [L_j,\ P_j,\ Q_j] \quad (j = \overline{0,3}), \tag{4}$$

$$T_1 = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix},\ \epsilon = \begin{bmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \end{bmatrix}, \tag{5}$$

where

$$E_j = \begin{bmatrix} \mathrm{vec}(\phi_{B_{1j}}) \\ \mathrm{vec}(\phi_{B_{2j}}) \\ \mathrm{vec}(\phi_{D_{1j}}) \\ \mathrm{vec}(\phi_{D_{2j}}) \\ \mathrm{vec}(\phi_{F_{1j}}) \\ \mathrm{vec}(\phi_{F_{2j}}) \\ \mathrm{vec}(\phi_{H_j}) \end{bmatrix} \quad (j = \overline{0,3}).$$

Theorem 3.1. *For $A_1,\ C_1,\ E_1 \in \mathbb{H}_S^{m\times n}$, $A_2,\ C_2,\ E_2 \in \mathbb{H}_S^{n\times k}$, $E_3,\ E_4,\ E_5 \in \mathbb{H}_S^{m\times n}$, $F_3,\ F_4,\ F_5 \in \mathbb{H}_S^{n\times k}$, $B_1,\ D_1,\ F_1 \in \mathbb{H}_S^{m\times n}$, $B_2,\ D_2,\ F_2 \in \mathbb{H}_S^{n\times k}$, and $H \in \mathbb{H}_S^{m\times k}$. Let $T_1,\ V_3$ and ϵ be defined in* (4) *and* (5)*. Then, Problem 1 has a solution $[X, Y, Z] \in \mathbb{H}_S^{n\times n}$ if and only if*

$$\begin{bmatrix} T_1 \\ V_3 \end{bmatrix} \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \epsilon = \epsilon. \tag{6}$$

In this case, the set of general solution can be expressed as

$$\chi = \left\{ [X, Y, Z] \,\middle|\, \begin{bmatrix} \mathrm{vec}(\vec{X}) \\ \mathrm{vec}(\vec{Y}) \\ \mathrm{vec}(\vec{Z}) \end{bmatrix} = \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \epsilon + \left[I_{12n^2} - \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \begin{bmatrix} T_1 \\ V_3 \end{bmatrix} \right] y \right\}, \tag{7}$$

where y is an arbitrary vector with appropriate order. Furthermore, if (6) *holds, then the system of split quaternion matrix equations* (3) *has a unique solution $[X, Y, Z] \in \chi$ if and only if*

$$rank \begin{bmatrix} T_1 \\ V_3 \end{bmatrix} = 12n^2. \tag{8}$$

In this case, we have

$$\chi = \left\{ [X, Y, Z] \,\middle|\, \begin{bmatrix} \mathrm{vec}(\vec{X}) \\ \mathrm{vec}(\vec{Y}) \\ \mathrm{vec}(\vec{Z}) \end{bmatrix} = \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \epsilon \right\}. \tag{9}$$

Proof. By Corollary 2.5 and Theorem 2.2, it follows that

$$
\begin{aligned}
(3) &\Leftrightarrow \begin{cases} \phi_{A_1X} = \phi_{B_1}, \ \ \phi_{XA_2} = \phi_{B_2}, \ \ \phi_{C_1Y} = \phi_{D_1}, \ \ \phi_{YC_2} = \phi_{D_2}, \\ \phi_{E_1Z} = \phi_{F_1}, \ \ \phi_{ZE_2} = \phi_{F_2}, \ \ \phi_{E_3XF_3} + \phi_{E_4YF_4} + \phi_{E_5ZF_5} = \phi_H, \end{cases} \\
&\Leftrightarrow \begin{cases} \mathrm{vec}(\phi_{A_1X}) = \mathrm{vec}(\phi_{B_1}), \ \ \mathrm{vec}(\phi_{XA_2}) = \mathrm{vec}(\phi_{B_2}), \ \ \mathrm{vec}(\phi_{C_1Y}) = \mathrm{vec}(\phi_{D_1}), \ \ \mathrm{vec}(\phi_{YC_2}) = \mathrm{vec}(\phi_{D_2}), \\ \mathrm{vec}(\phi_{E_1Z}) = \mathrm{vec}(\phi_{F_1}), \ \ \mathrm{vec}(\phi_{ZE_2}) = \mathrm{vec}(\phi_{F_2}), \ \ \mathrm{vec}(\phi_{E_3XF_3}) + \mathrm{vec}(\phi_{E_4YF_4}) + \mathrm{vec}(\phi_{E_5ZF_5}) = \mathrm{vec}(\phi_H), \end{cases} \\
&\Leftrightarrow L\mathrm{vec}(\overrightarrow{X}) + P\mathrm{vec}(\overrightarrow{Y}) + Q\mathrm{vec}(\overrightarrow{Z}) = \epsilon, \\
&\Leftrightarrow \begin{bmatrix} L_0 & P_0 & Q_0 \\ L_1 & P_1 & Q_1 \\ L_2 & P_2 & Q_2 \\ L_3 & P_3 & Q_3 \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \\ \mathrm{vec}(\overrightarrow{Z}) \end{bmatrix} = \epsilon, \\
&\Leftrightarrow \begin{bmatrix} T_1 \\ V_3 \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \\ \mathrm{vec}(\overrightarrow{Z}) \end{bmatrix} = \epsilon.
\end{aligned}
$$

By Lemma 2.6, Problem 1 has a solution $[X, Y, Z] \in \chi$ if and only if (6) holds. If this condition is satisfied, then

$$
\begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \\ \mathrm{vec}(\overrightarrow{Z}) \end{bmatrix} = \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \epsilon + \left[I_{12n^2} - \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \begin{bmatrix} T_1 \\ V_3 \end{bmatrix} \right] y,
$$

which implies (7) holds. Moreover, if (6) holds, Problem 1 has a unique solution $[X, Y, Z] \in \chi$ if and only if

$$
\begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \begin{bmatrix} T_1 \\ V_3 \end{bmatrix} = I_{12n^2}, \tag{10}
$$

that means that (8) holds. Thus we have (9). □

As mentioned in ([59], [64]), Theorem 3.1 is simple and convenient to solve the split quaternion matrix equations (3), especially when the known matrices are small in size. In order to deal with the Moore-Penrose generalized inverse of the block matrix $\begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger}$, we use the following results from [31] and derive them as follows. Let

$$
\begin{aligned}
s &= 12mn + 12kn + 4km, \\
R &= (I_{12n^2} - T_1^{\dagger} T_1) V_3^{\mathbf{T}}, \\
Z &= (I_s + (I_s - R^{\dagger} R) V_3 T_1^{\dagger} T_1^{\dagger \mathbf{T}} V_3^{\mathbf{T}} (I_s - R^{\dagger} R))^{-1}, \\
W &= R^{\dagger} + (I_s - R^{\dagger} R) Z V_3 T_1^{\dagger} T_1^{\dagger \mathbf{T}} (I_{12n^2} - V_3^{\mathbf{T}} R^{\dagger}), \\
\Theta_{11} &= I_{3s} - T_1 T_1^{\dagger} + T_1^{\dagger \mathbf{T}} V_3^{\mathbf{T}} Z (I_s - R^{\dagger} R) V_3 T_1^{\dagger}, \\
\Theta_{12} &= -T_1^{\dagger \mathbf{T}} V_3^{\mathbf{T}} (I_s - R^{\dagger} R) Z, \\
\Theta_{22} &= (I_s - R^{\dagger} R) Z.
\end{aligned}
$$

From the results in [31], we have

$$
\begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} = [T_1^{\dagger} - W^{\mathbf{T}} V_3 T_1^{\dagger}, W^{\mathbf{T}}], \ \ \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \begin{bmatrix} T_1 \\ V_3 \end{bmatrix} = T_1^{\dagger} T_1 + R R^{\dagger}, \tag{11}
$$

and

$$I_{4s} - \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \begin{bmatrix} T_1 \\ V_3 \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^{T} & \Theta_{22} \end{bmatrix}. \tag{12}$$

Theorem 3.2. *Problem* 1 *has a solution* $[X, Y, Z]$ *if and only if*

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^{\mathbf{T}} & \Theta_{22} \end{bmatrix} \epsilon = 0. \tag{13}$$

In this case, the set of general solution can be expressed as

$$\chi = \left\{ [X, Y, Z] \middle| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \\ \mathrm{vec}(\overrightarrow{Z}) \end{bmatrix} = [T_1^{\dagger} - W^{\mathbf{T}} V_3 T_1^{\dagger}, W^{\mathbf{T}}]\epsilon + (I_{12n^2} - T_1^{\dagger} T_1 - RR^{\dagger})y \right\}, \tag{14}$$

where y is an arbitrary vector with appropriate order. Furthermore, if (13) *holds, then the system of split quaternion matrix equations has a unique solution* $[X, Y, Z] \in \chi$ *if and only if* (8) *holds. In this case,*

$$\chi = \left\{ [X, Y, Z] \middle| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \\ \mathrm{vec}(\overrightarrow{Z}) \end{bmatrix} = [T_1^{\dagger} - W^{\mathbf{T}} V_3 T_1^{\dagger}, W^{\mathbf{T}}]\epsilon \right\}. \tag{15}$$

Proof. We have seen in Theorem 3.1 that (6) is the necessary and sufficient condition for the existence of solution $[X, Y, Z] \in \chi$. We can rewrite (6) as

$$\left(I_{4s} - \begin{bmatrix} T_1 \\ V_3 \end{bmatrix} \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} \right) \epsilon = 0.$$

From (12), system (3) has a solution $[X, Y, Z] \in \chi$ if and only if (13) holds. From Theorem 3.1, and (11), (7) implies (14). Furthermore, if both (8) and (13) hold then we can write (9) in the form of (15). □

Corollary 3.3. *Let the condition be satisfied in Theorem* 3.2*. Then the optimization problem*

$$\min_{[X,Y,Z]\in\chi} (\|\phi_X\|^2 + \|\phi_Y\|^2 + \|\phi_Z\|^2)$$

has a unique minimizer $[X_l, Y_l, Z_l]$ *which satisfies*

$$\begin{bmatrix} \mathrm{vec}(\overrightarrow{X_l}) \\ \mathrm{vec}(\overrightarrow{Y_l}) \\ \mathrm{vec}(\overrightarrow{Z_l}) \end{bmatrix} = [T_1^{\dagger} - W^{\mathbf{T}} V_3 T_1^{\dagger}, W^{\mathbf{T}}]\epsilon. \tag{16}$$

Proof. From (14), we can see that the solution set χ is a nonempty closed convex set. Hence,

$$\begin{aligned} \min_{[X,Y,Z]\in\chi} (\|\phi_X\|^2 + \|\phi_Y\|^2 + \|\phi_Z\|^2) &= \min_{[X,Y,Z]\in\chi} (\|\overrightarrow{X}\|^2 + \|\overrightarrow{Y}\|^2 + \|\overrightarrow{Z}\|^2) \\ &= \min_{[X,Y,Z]\in\chi} (\|\mathrm{vec}(\overrightarrow{X})\|^2 + \|\mathrm{vec}(\overrightarrow{Y})\|^2 + \|\mathrm{vec}(\overrightarrow{Z})\|^2) \\ &= \min_{[X,Y,Z]\in\chi} \left\| \begin{bmatrix} \mathrm{vec}(\overrightarrow{X}) \\ \mathrm{vec}(\overrightarrow{Y}) \\ \mathrm{vec}(\overrightarrow{Z}) \end{bmatrix} \right\|^2 . \end{aligned}$$

By Theorem 3.2, we have $\begin{bmatrix} \mathrm{vec}(\vec{X_l}) \\ \mathrm{vec}(\vec{Y_l}) \\ \mathrm{vec}(\vec{Z_l}) \end{bmatrix}$ is in the form of (16). □

We have two theorems to solve Problem 1 so far. We have derived the necessary and sufficent conditions for the existence of a solution of the system of equations (3) and provide the general solution formulas, respectively. Obviously, Theorem 3.2 is based on Theorem 3.1. The only difference between them is from the viewpoint of calculations. Theorem 3.2 takes advantages of the results of equations (11) and (12) to solve Problem 1. Consequently, Theorem 3.2 is more general than Theorem 3.1.

4. Numerical Verification

Based on the discussions in sections 2 and 3, we provide an algorithm and a numerical example for finding the solution of Problem 1. The algorithm is based on Theorem 3.2 and Corollary 3.3. If the condition (13) for the system of matrix equations (3) holds, the following algorithm give the numerical solution of Problem 1 for $[X, Y, Z] \in \mathbb{H}_S^{n\times n}$.

Algorithm 1

1. Input the matrix: Input $A_1 = A_{10} + A_{11}\mathbf{i} + A_{12}\mathbf{j} + A_{13}\mathbf{k}$, $C_1 = C_{10} + C_{11}\mathbf{i} + C_{12}\mathbf{j} + C_{13}\mathbf{k}$, $E_1 = E_{10} + E_{11}\mathbf{i} + E_{12}\mathbf{j} + E_{13}\mathbf{k} \in \mathbb{H}_S^{m\times n}$, $A_2,\ C_2,\ E_2 \in \mathbb{H}_S^{n\times k}$, $E_t = E_{t0} + E_{t1}\mathbf{i} + E_{t2}\mathbf{j} + E_{t3}\mathbf{k} \in \mathbb{H}_S^{m\times n}$ $(t = \overline{3,5})$, $F_3, F_4, F_5 \in \mathbb{H}_S^{n\times k}$, $B_1,\ D_1,\ F_1 \in \mathbb{H}_S^{m\times n}$, $B_2,\ D_2,\ F_2\ \in \mathbb{H}_S^{n\times k}$ and $H \in \mathbb{H}_S^{m\times k}$.
2. Compute T_1, V_3, R, Z, W, Θ_{11}, Θ_{12}, Θ_{22} and ϵ.
3. If both (8) and (13) hold, then calculate $[X_l,\ Y_l,\ Z_l] \in \chi$ according to (15).
4. If only (13) holds, then calculate $[X_l,\ Y_l,\ Z_l] \in \chi$ according to (14). Or else, go to next step.
5. Calculate $[X_l,\ Y_l,\ Z_l] \in \chi$ according to (16).

If the system (3) is consistent, then

$$N_1 = \left\|\begin{bmatrix} T_1 \\ V_3 \end{bmatrix}\begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger}\epsilon - \epsilon\right\|, \quad N_2 = \left\|\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}\epsilon\right\|$$

and

$$N_3 = \left\|I - \begin{bmatrix} T_1 \\ V_3 \end{bmatrix}\begin{bmatrix} T_1 \\ V_3 \end{bmatrix}^{\dagger} - \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}\right\|$$

must be small.

Example 4.1. *Let* $m = 3, n = 2, k = 3$, *and* $A_u = A_{u0} + A_{u1}\boldsymbol{i} + A_{u2}\boldsymbol{j} + A_{u3}\boldsymbol{k}$, $C_u = C_{u0} + C_{u1}\boldsymbol{i} + C_{u2}\boldsymbol{j} + C_{u3}\boldsymbol{k}$, $u = 1, 2$ $E_v = E_{v0} + E_{v1}\boldsymbol{i} + E_{v2}\boldsymbol{j} + E_{v3}\boldsymbol{k}$, $(v = \overline{1,5})$, $F_w = F_{w0} + F_{w1}\boldsymbol{i} + F_{w2}\boldsymbol{j} + F_{w3}\boldsymbol{k}$, $(w = \overline{3,5})$, $\widehat{X} = X_{10} + X_{11}\boldsymbol{i} + X_{12}\boldsymbol{j} + X_{13}\boldsymbol{k}$, $\widehat{Y} = Y_{10} + Y_{11}\boldsymbol{i} + Y_{12}\boldsymbol{j} + Y_{13}\boldsymbol{k}$, $\widehat{Z} = Z_{10} + Z_{11}\boldsymbol{i} + Z_{12}\boldsymbol{j} + Z_{13}\boldsymbol{k}$. *We take*

$$A_{10} = \begin{bmatrix} 0.15 & 0 \\ 0.38 & 0.87 \\ 0.16 & 0.35 \end{bmatrix},\ A_{11} = \begin{bmatrix} 0 & -2 \\ -1 & 1 \\ 2 & 2 \end{bmatrix},\ A_{12} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \\ -2 & 0 \end{bmatrix},\ A_{13} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ -2 & 2 \end{bmatrix},$$

$$A_{20} = \begin{bmatrix} 0.65 & 0.94 & 0.24 \\ 0.96 & 0.46 & 0.76 \end{bmatrix},\ A_{21} = \begin{bmatrix} 1 & 0.25 & 2 \\ 0 & 1 & -1 \end{bmatrix},\ A_{22} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -2 & 1 \end{bmatrix},$$

$$A_{23} = \begin{bmatrix} 0 & -2 & -2 \\ 2 & -1 & 0 \end{bmatrix},\ C_{10} = \begin{bmatrix} 0.66 & 0.75 \\ 0.04 & 0 \\ 0.81 & 0.53 \end{bmatrix},\ C_{11} = C_{12} = C_{13} = zeros(3, 2),$$

$$C_{20}=\begin{bmatrix}0.79&0.44&0.75\\0.58&0.26&0.23\end{bmatrix},\ C_{21}=\begin{bmatrix}2&0.5&0\\1&0&1\end{bmatrix},\ C_{22}=\begin{bmatrix}0&-1&0\\1&0&2\end{bmatrix},\ C_{23}=\begin{bmatrix}1&2&-1\\0&2&0\end{bmatrix},$$

$$E_{10}=\begin{bmatrix}0.5&0.4\\0.6&0\\0.9&0.8\end{bmatrix},\ E_{11}=\begin{bmatrix}2&-1\\0&-1\\2&0\end{bmatrix},\ E_{12}=\begin{bmatrix}-1&2\\1&2\\1&2\end{bmatrix},\ E_{13}=\begin{bmatrix}0&2\\2&1\\1&0\end{bmatrix},$$

$$E_{20}=\begin{bmatrix}-0.2&0.2&0.8\\0.9&0&0.6\end{bmatrix},\ E_{21}=\begin{bmatrix}2&-1&0\\0&-2&2\end{bmatrix},\ E_{22}=\begin{bmatrix}0&-2&-1\\0&0&-1\end{bmatrix},\ E_{23}=\begin{bmatrix}1&-2&1\\0&2&-1\end{bmatrix},$$

$$E_{30}=\begin{bmatrix}0.38&0.53\\0.62&0.26\\0.58&0.25\end{bmatrix},\ E_{31}=\begin{bmatrix}-2&-1\\0&1\\2&-2\end{bmatrix},\ E_{32}=\begin{bmatrix}0&1\\2&-2\\-1&-1\end{bmatrix},\ E_{33}=\begin{bmatrix}0&1\\-1&-2\\-1&0\end{bmatrix},$$

$$E_{40}=\begin{bmatrix}0.14&0.04\\0.22&0.11\\0.18&0.62\end{bmatrix},\ E_{41}=\begin{bmatrix}1&-2\\2&0\\-1&0\end{bmatrix},\ E_{42}=\begin{bmatrix}-1&2\\0.25&0\\0&2\end{bmatrix},\ E_{43}=\begin{bmatrix}-2&1\\0&0\\-1&-2\end{bmatrix},$$

$$E_{50}=\begin{bmatrix}0.09&0\\0.04&0.31\\0.56&0.18\end{bmatrix},\ E_{51}=zeros(3,2), E_{52}=\begin{bmatrix}-1&-2\\1&0\\-2&-2\end{bmatrix},\ E_{53}=\begin{bmatrix}0&-1\\-1&-2\\2&2\end{bmatrix},$$

$$F_{30}=\begin{bmatrix}0.55&0.51&0\\0.58&0.08&0.99\end{bmatrix},\ F_{31}=\begin{bmatrix}-2&-1&2\\-1&2&2\end{bmatrix},\ F_{32}=\begin{bmatrix}0&1&-1\\0.5&0&2\end{bmatrix},\ F_{33}=\begin{bmatrix}-1&0&0\\0&-0.3&0.25\end{bmatrix},$$

$$F_{40}=\begin{bmatrix}0.70&0.22&0.67\\0.73&0.27&0.48\end{bmatrix},\ F_{41}=\begin{bmatrix}1&0.5&0\\1&0&1\end{bmatrix},\ F_{42}=zeros(2,3),\ F_{43}=\begin{bmatrix}1&2&-1\\-1&2&-1\end{bmatrix},$$

$$F_{50}=\begin{bmatrix}0.65&0.39&0.84\\0.83&0.75&0.32\end{bmatrix},\ F_{51}=\begin{bmatrix}2.5&1&1\\2&0.25&0\end{bmatrix},\ F_{52}=\begin{bmatrix}zeros(1,3)\\ones(1,3)\end{bmatrix},\ F_{53}=\begin{bmatrix}-1&1&1\\-2&-2&1\end{bmatrix},$$

$$X_{10}=\begin{bmatrix}1&-1\\1&-1\end{bmatrix},\ X_{11}=\begin{bmatrix}0.97&0.84\\0.33&0.73\end{bmatrix},\ X_{12}=ones(2,2),\ X_{13}=\begin{bmatrix}1&-0.25\\-1&0\end{bmatrix},$$

$$Y_{10}=\begin{bmatrix}1&0.1\\-1&1\end{bmatrix},\ Y_{11}=\begin{bmatrix}0.25&0.5\\-1&0.89\end{bmatrix},\ Y_{12}=zeros(2,2),\ Y_{13}=\begin{bmatrix}1&0\\-1&0\end{bmatrix},$$

$$Z_{10}=\begin{bmatrix}1&1\\-1&-1\end{bmatrix},\ Z_{11}=\begin{bmatrix}-1&1\\1&-1\end{bmatrix},\ Z_{12}=\begin{bmatrix}1&1\\0.5&1\end{bmatrix},\ Z_{13}=\begin{bmatrix}0.01&-0.05\\-0.1&-0.02\end{bmatrix},$$

where $zeros(n,k)$ is an $n\times k$ matrix whose all elements are zero and $ones(n,k)$ is an $n\times k$ matrix with all one elements. Let

$$\phi_{B_1}=\phi_{A_1}G(\widehat{X}),\ \phi_{B_2}=\phi_{\widehat{X}}G(A_2),\ \phi_{D_1}=\phi_{C_1}G(\widehat{Y}),$$
$$\phi_{D_2}=\phi_{\widehat{Y}}G(C_2),\ \phi_{F_1}=\phi_{E_1}G(\widehat{Z}),\ \phi_{F_2}=\phi_{\widehat{Z}}G(E_2),$$
$$\phi_H=\phi_{E_3}G(\widehat{X})G(F_3)+\phi_{E_4}G(\widehat{Y})G(F_4)+\phi_{E_5}G(\widehat{Z})G(F_5).$$

Using **Matlab** and **Algorithm 1**, we obtain

$$\operatorname{rank}\begin{bmatrix}T_1\\V_3\end{bmatrix}=48,\quad N_2=9.6181\times10^{-14}.$$

Therefore, we can see that the system of equations (3) *is consistent. Additionally, we can compute* $N_1=9.904\times10^{-14}$ *and* $N_3=9.7774\times10^{-14}$. *Thus, Problem 1 has a unique solution* $[X,Y,Z]=[X_l,Y_l,Z_l]\in\chi$ *and we can get* $\|\phi_{[X_l,Y_l,Z_l]}-\phi_{[\widehat{X},\widehat{Y},\widehat{Z}]}\|=1.1036\times10^{-14}$.

5. Conclusion

In this paper, we have discused how to find the solution of the system of split quaternion matrix equations (3) by using the Kronecker product, the Moore-Penrose generalized inverse, the vec operator, and the real representation of split quaternion matrix. We have proposed necessary and sufficient conditions for the solvability of the system (3) and established the expression of the general solution of the system (3). If the system (3) is inconsistent, then we have given an algorithm to find its approximate solution. Moreover, a numerical example has also been designed to illustrate the results of this paper. Inspired by ([32], [62]), we plan to explore solving the system (3) of tensor equations over the split quaternion algebra in future work.

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