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Asymptotics of solutions to a first-order partial differential equation with a power-law boundary layer

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Abstract. In the article, a regularized asymptotic of any order of a mixed problem for a first-order partial differential equation is constructed, when the limit equation has a regular singularity. The constructed asymptotic contains boundary-layer functions of two types: power, exponential, and angular functions. The asymptotic of the solution is constructed by a special class of function corresponding to the structure of the fundamental system of solutions. The asymptotic character of the constructed solution is established.

1. Introduction

So far, such problems have not been studied from the standpoint by the method of singularly perturbed problems. In works [1-4] the asymptotic of the boundary layer type were constructed. The asymptotic constructed there has a complex structure and the process of constructing the solution consists of several stages. In the work [1], for a system of equations in partial derivatives of the first order, the asymptotic of the inner transitional layer was constructed. In the works [2-4], the Cauchy problems for systems of singularly perturbed partial differential equations of the first order are studied, when the matrix at the desired function has one zero eigenvalue, and an asymptotic of any order of the boundary layer type is constructed. Our approach greatly simplifies the process of constructing the asymptotic. The numerical solution of singularly perturbed problems is the subject of works whose bibliography is given in [11-12]. Russian-language works not cited in [13-15]. The works [10-15] are devoted to the construction of difference schemes for singularly perturbed ordinary differential equations. In [13], singularly perturbed problems were studied on a piecewise uniform grid. The works [17-19] use the decomposition of the grid solution into regular and singular components, which are solutions of grid sub problems on piecewise uniform grids. This article proposes a new approach for the numerical solution of singularly perturbed ordinary differential equations, which is based on the synthesis of S.A. Lomov's regularization method [5] and known numerical methods (finite elements, finite differences, direct lines). The idea of the method is to regularize a singularly perturbed problem, by introducing an additional regularizing independent variable, the original problem is expanded into a space of higher dimension. The extended problem obtained in this case will be regular in a small parameter, then the resulting regular problem is decomposed, the resulting equations for

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the components are applied, one of the known numerical methods is applied. Previously, this method was applied in [20-24] to various singularly perturbed ordinary differential equations, and in [25] to a parabolic equation. The method of lines in [20] solved the initial problem for a differential equation with a small first-order parameter. Singularly perturbed ordinary differential equations with one boundary layer function and two boundary layer functions, based on the finite difference method, were studied respectively in [21, 22]. The finite element method was applied in [23, 24] to solve singularly perturbed ordinary differential equations. In [25], the finite difference method is used to solve a singularly perturbed heat equation.

2. Asymptotic solution

Consider the problem

$$(\varepsilon + t)\partial_t u + \varepsilon a(x)\partial_x u + b(x,t)u = f(x,t), \quad (x,t) \in \Omega$$

$$u(x,t,\varepsilon)|_{t=0} = u^0(x), \quad u(x,t,\varepsilon)|_{x=0} = u^1(t) \tag{1}$$

here $\varepsilon > 0$ is a small parameter, $a(x) \in C^\infty[0,1]$, $b(t) \in C^\infty(\Omega)$, $f(x,t) \in C^\infty(\Omega)$, $\Omega = \{0 < x < 1, 0 < t \le T\}$.

The problem is studied at $b(x,0) > 0$, $\forall t \in [0,T]$. The degenerate ($\varepsilon = 0$) equation has a singularity at $t = 0$, which leads to the appearance of a power-law boundary layer. The power-law boundary layer [1] is described by the function

$$\Pi(t,\varepsilon) = \left(\frac{\varepsilon}{t+\varepsilon}\right)^\lambda, \quad \lambda > 0,$$

in addition, the problem (1) along the characteristic has a gap.

We have constructed a continuous asymptotic solution that contains regular, power and angular boundary layer functions. Previously, the problem solutions of which contain power-law boundary-layer functions were studied in [5-9]. Thus, in [5, 6] ordinary differential equations are studied, [7-9] are devoted to the construction of an asymptotic solution of parabolic equations.

2.1. *Regularization of the problem*

Let's regularize [5] the problem (1), for which we introduce the regularizing functions

$$\xi_1 = \varphi_1(x,t,\varepsilon), \quad \xi_2 = \varphi_2(x,t,\varepsilon), \quad \varphi_1(x,0,\varepsilon) = 0, \quad \varphi_2(0,t,\varepsilon) = 0 \tag{2}$$

and the extended function

$$\tilde{u}(x,t,\xi,\varepsilon)|_{\xi=\varphi(x,t,\varepsilon)} u(x,t,\varepsilon), \quad \xi = (\xi_1,\xi_2), \quad \varphi = (\varphi_1,\varphi_2). \tag{3}$$

From (3), based on (2), we find the derivatives of $\partial_t u$, $\partial_x u$ and choose the regularizing functions as solutions of the equations

$$\varepsilon a(x)\partial_x\varphi_2(x,t,\varepsilon) + (\varepsilon + t)\partial_t\varphi_2(x,t,\varepsilon) = b(x,0), \qquad (\varepsilon + t)\partial_t\varphi_1(x,t,\varepsilon) + \varepsilon a(x)\partial_x\varphi_1(x,t,\varepsilon) = b(x,0) \tag{4}$$

Then the extended task for $\tilde{u}(x,t,\xi,\varepsilon)$ will be written

$$b(x,0)\partial_\tau\tilde{u} + b(x,0)\partial_\xi\tilde{u} + b(x,0)\tilde{u} + [b(x,t) - b(x,0)]\tilde{u} + t\partial_t\tilde{u} = -\varepsilon\partial_t\tilde{u} - \varepsilon a(x)\partial_x\tilde{u} + f(x,t), \quad (x,t,\xi) \in Q, \tag{5}$$

$$\tilde{u}|_{t=\tau=0} = u^0(x), \quad \tilde{u}|_{x=\xi=0} = u^1(t), \quad Q = \Omega \times (0,\infty) \times (0,\infty)$$

Solving problems (4), (2) will be written:

$$\varphi_1(x,t,\varepsilon) = \int_0^\tau b(A^{-1}(\varepsilon\eta - \tau + s), 0)ds, \quad \tau = b(x,0)ln\left(\frac{t+\varepsilon}{\varepsilon}\right), \tag{6}$$

$$\varphi_2(x,t,\varepsilon) = \int_0^\eta b(A^{-1}(\varepsilon s), 0)ds, \quad \eta = \frac{1}{\varepsilon}\int_0^x \frac{ds}{a(s)} \equiv \frac{1}{\varepsilon}A(x)$$

The solution of problem (5) will be defined as a series

$$\tilde{u}(x,t,\xi,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k u_k(M),\ \ M=(x,t,\xi,\tau),$$

then for the coefficients we get the following iterative problems:

$$\begin{gathered}Tu_0\equiv b(x,0)[\partial_\tau+\partial_\xi+1]u_0+t\partial_t u_0+[b(x,t)-b(x,0)]u_0=f(x,t),\\ Tu_k=-\partial_t u_{k-1}-a(x)\partial_x u_{k-1},\\ u_0|_{t=\tau=0}=u^0(x),\quad u_0|_{x=\xi=0}=u^1(t),\\ u_k|_{t=\tau=0}=u_k|_{x=\xi=0}=0.\end{gathered}\tag{7}$$

2.2. Solvability of iterative problems

Iterative problems (7) will be solved in the class of functions

$$\begin{gathered}U=\Big\{u(M):u(M)=c_1(x,t)e^{-\tau}+d_1(x,t)e^{-\xi}+e_1(x,t)\Phi(\tau-\xi)e^{-\xi}+\\ f_1(x,t)\Phi(\xi-\tau)e^{-\tau}+v(x,t),\\ c_1(x,t),d_1(x,t),e_1(x,t),f_1(x,t),v(x,t)\in C^\infty(\bar{\Omega})\}\\ \Phi(\xi)=\begin{cases}0, & \xi<0,\\ e^{-\xi}, & \xi\geq 0.\end{cases}\end{gathered}$$

where the term $c_1(x,t)e^{-\tau}=c_1(x,t)(\frac{\varepsilon}{t+\varepsilon})^{b(x,0)}$ describes a power boundary layer along $t=0$, $d_1(x,t)e^{-\xi}$ describes an exponential boundary layer along $x=0$; the remaining two terms describe an angular boundary layer in the vicinity of point $(0,0)$.

Calculate the action of the operator T on the function $u(M)\in U$:

$$\begin{gathered}Tu=b(x,0)\Big[-c_1(x,t)e^{-\tau}-d_1(x,t)e^{-\xi}-e_1(x,t)\Phi(\tau-\xi)e^{-\xi}-\\ f_1(x,t)\Phi(\xi-\tau)e^{-\tau}+e_1(x,t)\Phi^{'}(\tau-\xi)e^{-\xi}-e_1(x,t)\Phi^{'}(\tau-\xi)e^{-\xi}+\\ f_1(x,t)\Phi^{'}(\xi-\tau)e^{-\tau}-f_1(x,t)\Phi^{'}(\xi-\tau)e^{-\tau}+\\ c_1(x,t)e^{-\tau}+d_1(x,t)e^{-\xi}+e_1(x,t)\Phi(\tau-\xi)e^{-\xi}+\\ f_1(x,t)\Phi(\xi-\tau)e^{-\tau}\Big]+t\Big[\partial_t c_1(x,t)e^{-\tau}+\partial_t d_1(x,t)e^{-\xi}+\\ \partial_t e_1(x,t)\Phi(\tau-\xi)e^{-\xi}+\partial_t f_1(x,t)\Phi(\xi-\tau)e^{-\tau}\Big]+\\ [b(x,t)-b(x,0)]\Big[c_1(x,t)e^{-t}+d_1(x,t)e^{-\xi}+e_1(x,t)\Phi(\tau-\xi)e^{-\xi}+\\ f_1(x,t)\Phi(\xi-\tau)e^{-\tau}\Big]=[t[\partial_t c_1(x,t)+(b(x,t)-b(x,0))c_1(x,t)]e^{-\tau}+\\ [t\partial_t d_1(x,t)+(b(x,t)-b(x,0))d_1(x,t)]e^{-\xi}+\\ [t\partial_t e_1(x,t)+(b(x,t)-b(x,0))e_1(x,t)]\times\Phi(\tau-\xi)e^{-\xi}+\\ [t\partial_t f_1(x,t)-(b(x,t)-b(x,0))f_1(x,t)]\Phi(\xi-\tau)e^{-\tau}+\\ b(x,t)v(x,t)+t\partial_t v(x,t),\end{gathered}\tag{8}$$

here

$$c_1(x,t)=c(x,t)+P_1(x),$$

$$d_1(x,t) = d(x,t) + P_2(x),$$
$$f_1(x,t) = f(x,t) + P_4(x),$$
$$e_1(x,t) = e(x,t) + P_3(x).$$

From the boundary conditions (7) of the function $u(M) \in U$ we find

$$c_1(x,t) = -v(x,0) - P_1(x), \quad f_1(x,t) = -d_1(x,t) - P_4(x),$$
$$d_1(0,t) = -v(0,t) - P_2(0), \quad e_1(0,t) = -c_1(0,t) - P_3(0). \tag{9}$$

Satisfying the function $u_0(M) \in U$ to equation (7) for $k = 0$, based on calculations (8), we obtain

$$t\partial_t c^0(x,t) + [b(x,t) - b(x,0)]\left[c^0(x,t) - P_1^0(x)\right] = 0,$$
$$t\partial_t d^0(x,t) + [b(x,t) - b(x,0)]\left[d^0(x,t) - P_2^0(x)\right] = 0, \tag{10}$$
$$t\partial_t e^0(x,t) + [b(x,t) - b(x,0)]\left[e^0(x,t) - P_3^0(x)\right] = 0,$$
$$t\partial_t f^0(x,t) + [b(x,t) - b(x,0)]\left[f^0(x,t) - P_4^0(x)\right] = 0.$$

These equations, under initial conditions (9), have smooth solutions. The functions $P_i^0(x), \ \ i = 1,2,3,4$ included here and in the initial condition will be defined in the next iteration step. In the next iteration step, the right side of the equations will include

$$F_1(M) = -\partial_t u_0 - a(x)\partial_x u_0.$$

We substitute only the term $(c^0(x,t) + P_1^0(x))e^{-\tau}$, into it, the other terms of the function $u_0(M)$ are transformed in the same way:

$$F_2(M) = -\left[\partial_t c^0(x,t) + a(x)\left(\partial_x c^0(x,t) + \frac{dP_1^0}{dx}\right)\right]e^{-\tau}.$$

To ensure the solvability of the equation with respect to $c^1(x,t)$ from (10), we assume

$$\frac{dP_1^0(x)}{dx} = -\left(\partial_t c^0(x,t) + a(x)\partial_x c^0(x,t)\right)\frac{1}{a(x)}|_{t=0}.$$

Substitute here the value of $c^0(x,t)$ found as the solution of the problem (10), (9), with respect to $P_1^0(x)$ we obtain the equation. The resulting equation is solved under an arbitrary initial condition for $x = 0$.

Then the process repeats. The asymptotic character of the constructed solution is proved.

Theorem 2.1. *The given functions satisfy the conditions: $a(x) \in C^\infty([0,1]), b(x,t), f(x,t) \in C^\infty(\bar{\Omega})$ and initial conditions. Then, for sufficiently small $\varepsilon > 0$, problem (1) has a smooth asymptotic solution, i.e. there is an estimate*

$$\left|u(x,t,\varepsilon) - u_{\varepsilon_n}(x,t,\xi,\varepsilon)\right| < c\varepsilon^{n+1}, \quad \forall n \geq 0.$$

Proof. Let's rewrite problem (1)

$$\partial_t u + \frac{\varepsilon}{\varepsilon + t}a(x)\partial_x u + \frac{1}{\varepsilon + t}b(x,t)u = \frac{1}{\varepsilon + t}f(x,t).$$

Here, the expression $(t + \varepsilon)$ for sufficiently small ε does not affect the properties of the function $a(x), b(x,t)$ for which the conditions of the maximum principle theorem are valid [26]. Therefore, on the basis of this theorem, it is not difficult to establish an estimate. □

3. Numerical solution

Our method is based on the method of S. A. Lomov. First, the singularly perturbed equation under study is reduced by these methods to a regularly perturbed equation, then the resulting equation is decomposed. The equations obtained after decomposition with initial conditions are solved by a well-known numerical method. In the numerical solution in equation (4), $b(x,-\varepsilon)$ is taken instead of $b(x,0)$.

In this work, the finite difference method is used. The solution of problem (5) will be defined as

$$\tilde{u}(x,t,\xi,\tau,\varepsilon) = c^1(x,t)e^{-\tau} + c^2(x,t)e^{-\xi} + \Phi(\xi-\tau)c^3(x,t)e^{-\tau} + c^4(x,t)\Phi(\tau-\xi)e^{-\xi} + v(x,t), \tag{11}$$

for the coefficients we obtain the problem

$$(\varepsilon+t)\partial_t c^l(x,t) + \varepsilon a(x)\partial_x c^l(x,t) + [b(x,t) - b(x,-\varepsilon)]\, c^l(x,t) = 0,\ l = 1,2,3,4, \tag{12}$$

$$(\varepsilon+t)\partial_t v(x,t) + \varepsilon a(x)\partial_x v(x,t) + b(x,t)v(x,t) = f(x,t) \tag{13}$$

Equation (13) is solved without the initial condition, and for equations (12) the initial conditions are given in the form

$$c^1(x,0) = u^0(x) - v(x,0),\ \ c^3(x,0) = -c^2(x,0),\ \ c^2(0,t) = u^1(t) - v(0,t),\ \ c^4(0,t) = -c^1(0,t). \tag{14}$$

Difference equations equivalent to these problems can be written as

$$(\varepsilon+t_j)\frac{c^l_{i,j+1} - c^l_{i,j}}{k} + \varepsilon a_i \frac{c^l_{i+1,j} - c^l_{i,j}}{h} + \left[b_{ij} - b_i\right] c^l_{ij} = O(k+h),\ \ l = \overline{1,4} \tag{15}$$

$$(\varepsilon+t_j)\frac{v_{i,j+1} - v_{i,j}}{k} + \varepsilon a_i \frac{v_{i+1,j} - v_{i,j}}{h} + b_{ij}v_{ij} = f_{ij} + O(k+h),\ \ i = \overline{1,n},\ j = \overline{1,m} \tag{16}$$

$$h = \frac{1}{n},\ k = \frac{1}{m},\ \ c^1_{i,0} = u^0_i - v_{i,0},\ \ c^3_{i,0} = -c^2_i,\ \ c^2_{0,j} = u^1_j - v_{0,j},\ \ c^4_{o,j} = -c^1_{0,j},\ \ b_{ij} = b(ih,jl),\ b_i = b(ih,-\varepsilon).$$

Equation (15) with $l = 1,3$ we write

$$c^l_{i,j+1} = q_{ij}c^l_{ij} - p_{ij}c^l_{i+1,j} + O(k+h),\ \ c^1_{i,0} = u^0_i - v_{i,0},\ \ c^3_{i,0} = -c^2_{i,0}, \tag{17}$$

$$q_{ij} = 1 + \frac{\varepsilon a_i r - (b_{ij} - b_i)k}{\varepsilon + t_j},\ \ r = \frac{k}{h},\ \ p_{ij} = \frac{\varepsilon a_i}{t_j + \varepsilon} r,$$

for $l = 2,4$ we write

$$c^l_{i+1,j} = q^1_{ij}c^l_{ij} - r_1 p^1_{ij} c^l_{i,j+1} + O(k+h),\ \ c^2_{0,j} = u^1_j - v_{0,j},\ \ c^4_{0,j} = -c^1_{oj}, \tag{18}$$

$$q^1_{ij} = 1 + \frac{r_1(\varepsilon + t_j) - (b_{ij} - b_i)h}{\varepsilon a_i},\ \ p^1_{ij} = \frac{r_1(t_j + \varepsilon)}{\varepsilon a_i},\ \ r_1 = \frac{h}{k}.$$

To determine $v_i j$, we have the equation

$$v_{i,j+1} = q^2_{ij}v_{ij} - p^2_{ij}v_{i+1,j} + \frac{f_{ij}}{t_j + \varepsilon} + O(k+h) \tag{19}$$

$$q^2_{ij} = 1 + \frac{\varepsilon a_i r - b_{ij}k}{\varepsilon + t_j},\ \ p^2_{ij} = \frac{\varepsilon a_i r}{t_j + \varepsilon}$$

From (17) we have the estimate

$$\left|c^l_{i,j+1}\right| \le \left|1 - \frac{[b_{ij} - b_i]k}{t_j + \varepsilon}\right| \left|c^l_{ij}\right| + O(k+h),$$

for sufficiently small $\varepsilon > 0$, for (18) we obtain

$$\left|c^l_{i+1,j}\right| \leq \left|1 - \frac{[b_{ij} - b_i]h}{\varepsilon a_i}\right| \left|c^l_{ij}\right| + O(k+h). \tag{20}$$

From (19)

$$\left|v_{i,j+1}\right| \leq \left|1 - \frac{b_{ij}k}{t_j + \varepsilon}\right| \left|v_{ij}\right| + \frac{f_{ij}}{t_j + \varepsilon} + O(k+h). \tag{21}$$

These estimates, according to the assumptions made at the beginning, imply the stability and convergence of schemes (17), (18), (19) at a rate of $O(k+h)$. Solving problems (17) - (19) we find $c_i j^l,\ v_i j,\ i, j = \bar{1,n}$, using them we make a narrowing in (11) setting $x_i = (i-1)h,\ \ t_j = (j-1)k,\ \ i, j = \bar{1,n}$:

$$\tau = \tau_{i,j} = \varphi^j_{1,i} = \int_0^{t_j} \frac{1}{s+\varepsilon} b(A^{-1}(\varepsilon(z_{ij} + ln(s+\varepsilon)), 0)ds$$

$$\xi = \xi_{ij} = \varphi^j_{2,i} = \frac{1}{\varepsilon} \int_0^{x_i} \frac{b(s,0)}{a(s,0)} ds$$

define the solution of the original problem

$$u_{uv} = u_{ij} = v_{ij} + c^1_{i,j} e^{-\tau_{ij}} + c^2_{ij} e^{-\xi_{ij}} + \Phi(\xi_{ij} - \tau_{ij}) c^3_{ij} e^{-\tau_{ij}} + \Phi(\tau_{ij} - \xi_{ij}) c^4_{ij} e^{-\xi_{ij}}.$$

Following the methodology of [17], [18] and based on estimates (20), (21), we obtain the estimate

$$|u(x,t,\varepsilon) - u_{uv}| < c(k+h).$$

Theorem 3.1. *Let the given functions satisfy the above conditions. Then the solution constructed by the methods described above converges ε – uniformly at a rate of $O(k+h)$.*

4. Conclusion

To construct an asymptotic solution of the problem posed with respect to regularizing functions, a first-order partial differential equation is obtained, the solution of which is described along the characteristic. One regularizing variable is used to describe the power-law boundary layer ($\xi_1 = \varphi_1(x,t,\varepsilon)$), the second regularizing variable allows describing the exponential boundary layer along the straight line $x = 0(\xi_2 = \varphi_2(x,t,\varepsilon))$. To describe the corner boundary layer, an additional function $\Phi(\xi)$ is introduced, which allows describing the named boundary layer.

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